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$[j, k]$-SET DOMINATION OF n-BARBELL GRAPH, COMPLETE-BIPARTITE GRAPH AND n-CROWN GRAPH

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Abstract

Domination is an important theoretic concept in graph theory. Various types of domination have been studied. By $[j, k]$-set domination, a set $D \subseteq V$ in a graph $G = (V, E)$ is a $[j, k]$-set, if every vertex $v \in V - D$, $j \leq |N(v) \cap D| \leq k$ for non-negative integer j and k, that is every vertex $v \in V - D$ is adjacent to at least j but not more than k vertices in D. The domination number of G is denoted by $\gamma_{[j, k]}(G)$, which is the minimum cardinality of a dominating set of graph G. in this paper $[j, k]$ domination number of n-Barbell graph, complete-bipartite graph and n-crown graphs have been studied.

1. Introduction

The origin of graph theory can be traced back to Euler's work on the Konigsberg bridges problem (1735). The father of graph theory was the great Swiss mathematician Leonhard Euler, Oystein Ore [5] introduced the terms "Dominating set and Domination number" of a graph. Cockayne and Steve Hedetnien [1] first started to study dominating sets in the graph in 1975. In 1988 E. Sampath Kumar [6] introduced the concept of $[1, K]$-dominating set of a graph. Murugesan et al. [3] introduced the $[1, 2]$-domination in the

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graph. Basic concepts of graph were discussed by D.B. West [7], fundamentals of domination in graphs were explained by T.N. Haynes et al. [2], the concepts of $[1, 2]$-sets were explained by Mustapha Chellai et al. [4] and various theorems on $[1, 2]$-domination in graphs have been explained by Xiaojing Yang [8].

2. Preliminaries

Let $G = (V, E)$ be a simple connected graph of order $n = |V|$ and size $m = |E|$. The open neighborhood of a vertex $v \in V$ is the set $N(v) = \{u \in V \mid uv \in E\}$ of vertices adjacent to v. The closed neighborhood of a vertex $v \in V$ is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is $\deg(v) = |N(v)|$. The open neighborhood of a set $D \subseteq V$ of vertices is $N(D) = U_{v \in D} N(v)$ while the closed neighborhood of a set D is the set $N[D] = U_{v \in D} N[v]$. The minimum degree of G is $\delta(G) = \min_{v \in v(G)}\{d(v)\}$ and the maximum degree is $\Delta(G) = \max_{v \in v(G)}\{d(v)\}$. In this, we have studied $[1, 1]$-dominating sets and $[1, 2]$ dominating sets for the given graphs.

Definition 2.1. A set $D \subseteq V$ of vertices in a graph $G = (V, E)$ is a dominating set if and only if every vertex $v \in V - D$ is adjacent to at least one vertex in D.

Definition 2.2. A set D of G is called efficient, if $|N(v) \cap D| = 1$ for every vertex $v \in V(G)$, that is a dominating set D is efficient if and only if every vertex is dominated exactly once.

Definition 2.3. For a graph $G = (V, E)$, a set $D \subseteq V$ is independent in no two vertices in D are adjacent. The dominating number is denoted by $i(G)$.

3. $[j\ K]$ – Set Domination of n-Barbell Graph, Complete-Bipartite Graph, and n-Crown Graph

Definition 3.1. In the mathematical discipline of graph theory, the n-Barbell graph is a special type of undirected graph consisting of two nonoverlapping n-vertex cliques together with a single edge that has an endpoint in each clique.

Properties:

1. the n-Barbell graph is connected graph
2. Number of vertices - $2n$
3. Number of edges - $n(n-1)+1$
4. The Barbell graph has a bridge between two complete graphs.

Theorem 3.2. *The number of* $[1, 1]$ - *dominating sets of the n-Barbell graph when* $n = k + 2,\ k = 1, 2, 3, \ldots$ *is*

$$[D_{[1,\,1]}(B_{n,\,n})] = \{(n-1)^2 + 1, \quad n = 3, 4, 5, \ldots\}$$

Proof. Let $v_1, v_2, v_3, \cdots, v_n, v_{n+1}, v_{n+2}, \cdots, v_{2n}$ are the vertices of the n-Barbell graph. Here $\delta(v_i) = n-1, \quad i = 1, 2, 3, \cdots, n-1, n+2, n+3, \cdots, 2n$ $\Delta(v_j) = n,\ j = n,\ n+1$. So vertices v_i dominates $n-1$ vertices and vertices v_j dominates n vertices.

Since in a complete graph, every vertex is adjacent to neighboring vertices so every vertex dominates the adjacent vertices.

We can find $(n-1)^2 + 1$ dominating set and these dominating sets are given below

$$\{v_1, v_{n+2}\}, \{v_2, v_{n+2}\}, \{v_3, v_{n+2}\} \ldots \ldots \ldots \{v_{n-1}, v_{n+2}\}$$

$$\{v_1, v_{n+3}\}, \{v_2, v_{n+3}\}, \{v_3, v_{n+3}\} \ldots \ldots \ldots \{v_{n-1}, v_{n+3}\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\{v_1, v_{2n}\}, \{v_2, v_{2n}\}, \{v_3, v_{2n}\} \ldots \ldots \ldots \{v_{n-1}, v_{2n}\}$$

$$\text{and } \{v_n, v_{n+1}\}$$

The generalization of $[1, 1]$ - dominating sets of the n-Barbell graph is

$$D_{[1,1]}[B_{n,n}] = \begin{cases} \{v_{k1}, v_{k2}\} \\ \text{when } K1 = 1,\ K2 = n+2,\ n+3 \ldots \ldots 2n \\ K1 = 2,\ K2 = n+2,\ n+3 \ldots \ldots 2n \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ K1 = n-1,\quad K2 = n+2,\ n+3 \ldots \ldots 2n \\ \{v_n, v_{n+1}\}, \qquad n = 3, 4, 5 \ldots \\ \text{Hence } \left| D_{[1,1]}(B_{n,n}) \right| = (n-1)^2 + 1 \\ \text{and } \gamma_{[1,1]}(B_{n,n}) = 2 \end{cases}$$

Theorem 3.3. *The number of* $[1, 2]$*-dominating sets of the n-Barbell graph when* $n = k+2,\ k = 1, 2, 3 \ldots$ *is*

$$\left| D_{[1,2]}(B_{n,n}) \right| = \{2(n-1),\ n = 3, 4, 5 \ldots\}$$

Proof. Let $v_1, v_2, v_3 \ldots v_n, v_{n+1}, v_{n+2} \cdots v_{2n}$ are the vertices of the n-Barbell graph. Here $\delta(v_i) = n-1$, $i = 1, 2, 3 \cdots n, n+2 \ldots 2n$ and $\Delta(v_j) = n$, $j = n, n+1$. So vertices v_i in D dominates $n-1$ vertices $V - D$ and vertices v_j in D dominates n vertices in $V - D$.

Since in a complete graph, every vertex is adjacent to neighboring vertices, so every vertex dominates the adjacent vertices. We can find $2(n-1)$ $[1, 2]$-dominating sets and these dominating sets are given below.

There are two sets of $(n-1)$ number of dominating vertices, they are

$$\{v_1, v_{n+1}\}, \{v_2, v_{n+1}\}, \{v_3, v_{n+1}\} \cdots \{v_{n-1}, v_{n+1}\}$$

and

$$\{v_n, v_{n+2}\}, \{v_n, v_{n+3}\}, \quad \{v_n, v_{n+4}\} \cdots \{v_n, v_{2n}\}$$

The generalization of $[1, 2]$-dominating sets of the n-Barbell graph is

$$D_{[1,2]}[B_{n,n}] = \left[\begin{array}{l} \{v_{k1}, v_{k2}\} \\ \text{when } k1 = 1, 2, 3 \cdots n-1 \\ k2 = n+2 \\ \{v_{k1}, v_{k2}\} \\ \text{when } k1 = n, \\ k2 = n+2, n+3 \ldots 2n \end{array} \right.$$

Hence $|D_{[1,\ 2]}[B_{n,\ n}]| = 2(n-1)$ and $\gamma_{[1,\ 2]}[B_{n,\ n}] = 2.$

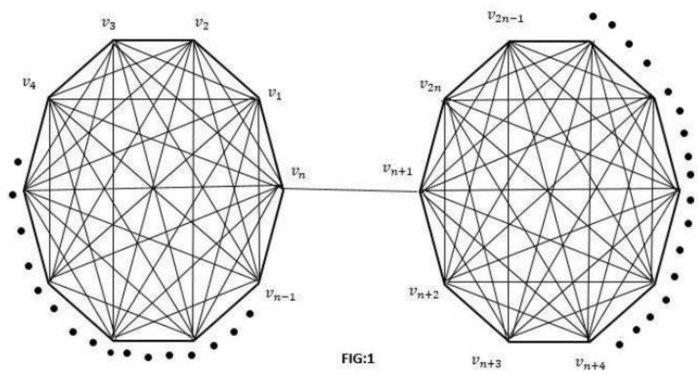

FIG:1

Definition 3.3. A Graph $G = (V,\ E)$ is called a complete bipartite graph if its vertices V can be partitioned into two subsets V_1 and V_2 such that each vertex of V_1 is connected to each vertex V_2. The number of edges in a complete bipartite graph is mn as each of the m vertices is connected to each of the n vertices.

Theorem 3.4. *The number of* $[1, 1]$ *- dominating sets of the complete-bipartite graph* $k_{m,\ n}$ *is*

$$|\ D_{[1,\ 1]}(k_{m,\ n})\ | = \begin{cases} mn, & if\ \ m > n \\ mn, & if\ \ m < n \end{cases}$$

Proof. Let $u_1, u_2, u_3 \cdots u_m, v_1 v_2 \cdots v_n$ are the vertices of the complete-bipartite graph. Vertices $u_i,\ i = 1,\ 2,\ 3 \cdots m$ in $V - D$ is adjacent to the vertices $v_j,\ j = 1,\ 2,\ 3 \cdots n$ in D, and also vertices $v_j,\ j = 1,\ 2 \cdots n$ in $V - D$ is adjacent to the vertices $u_i,\ i = 1,\ 2,\ 3 \cdots m$ in D, so u_i dominates v_i and v_j dominates u_i.

Case (i) if $m > n$

In this case $\delta(u_i) = n,\ i = 1,\ 2 \cdots m,\ j = 1,\ 2 \cdots n$ and $\Delta(v_j) = m$, $j = 1,\ 2,\ 3 \cdots n$ vertices u_i in $V - D,\ i = 1,\ 2,\ 3 \cdots \cdots m$ is adjacent to the vertices v_j in $D,\ j = 1,\ 2 \cdots n$ and also vertices v_j in $V - D$,

$j = 1, 2, 3 \cdots n$ is adjacent to the vertices u_i in D, $i = 1, 2, 3 \cdots m$. So v_j dominates and u_i dominates v_j. We can find mn $[1, 1]$-dominating set and these dominating sets are given below

$$\{u_1, v_1\}, \{u_2, v_1\}, \{u_3, v_1\} \cdots \{u_m, v_1\}$$

$$\{u_1, v_2\}, \{u_2, v_2\}, \{u_3, v_2\} \cdots \{u_m, v_2\}$$

$$\{u_1, v_3\}, \{u_2, v_3\}, \{u_3, v_3\} \cdots \{u_m, v_3\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\{u_1, v_n\}, \{u_2, v_n\}, \{u_3, v_n\} \cdots \{u_m, v_n\}$$

Therefore the generalization of $[1, 1]$-dominating sets of the complete-bipartite graph is given by

$$D_{[1,1]}[K_{m,n}] = \begin{cases} & \{u_{k1}, v_{k2}\} & \\ \text{when} & k1 = 1, \quad k2 = 1, 2, 3 \cdots n & m > n \\ & k1 = 2, \quad k2 = 1, 2, 3 \cdots n & \\ & k1 = 3, \quad k2 = 1, 2, 3 \cdots n & \\ & \cdots\cdots\cdots\cdots\cdots\cdots & \\ & \cdots\cdots\cdots\cdots\cdots\cdots & \\ & k1 = m, \quad k2 = 1, 2, 3 \cdots n & \end{cases}$$

Hence

$$|D_{[1,1]}[k_{m,n}]| = mn \text{ and } \gamma_{[1,1]}(k_{m,n}) = 2$$

Case (ii) If $m < n$

In the case $\delta(v_j) = m$, $j = 1, 2, \ldots n$ and $\Delta(u_i) = n$, $i = 1, 2 \cdots m$

Vertices v_j in $v - D$ is adjacent to the vertices u_i in D and also the vertices in $V - D$ is adjacent to the vertices v_j in D. So vertices u_i dominates the vertices v_j and vertices v_j dominates the vertices u_i. We can find mn $[1, 1]$-dominating sets and these dominating sets are given below

$$\{u_1, v_1\}, \{u_2, v_1\}, \{u_3, v_1\} \cdots \{u_m, v_1\}$$

$$\{u_1, v_2\}, \{u_2, v_2\}, \{u_3, v_2\} \cdots \{u_m, v_2\}$$

$$\{u_1, v_3\}, \{u_2, v_3\}, \{u_3, v_3\} \cdots \{u_m, v_3\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\{u_1, v_n\}, \{u_2, v_n\}, \{u_3, v_n\} \cdots \{u_m, v_n\}$$

Therefore the generalization of $[1, 1]$-dominating sets of the complete-bipartite graph is given by as

$$D_{[1,1]}[K_{m,n}] = \begin{cases} \quad\quad\quad \{u_{k1}, v_{k2}\} & \\ \text{when} \quad k1 = 1, \quad k2 = 1, 2, 3 \cdots n & m > n \\ \quad\quad\quad k1 = 2, \quad k2 = 1, 2, 3 \cdots n & \\ \quad\quad\quad k1 = 3, \quad k2 = 1, 2, 3 \cdots n & \\ \quad\quad\quad \cdots\cdots\cdots\cdots\cdots & \\ \quad\quad\quad \cdots\cdots\cdots\cdots\cdots & \\ \quad\quad\quad k1 = m, \quad k2 = 1, 2, 3 \cdots n & \end{cases}$$

Hence

$$|D_{[1,1]}(k_{m,n})| = mn \quad \text{and} \quad \gamma_{[1,1]}(k_{m,n}) = 2$$

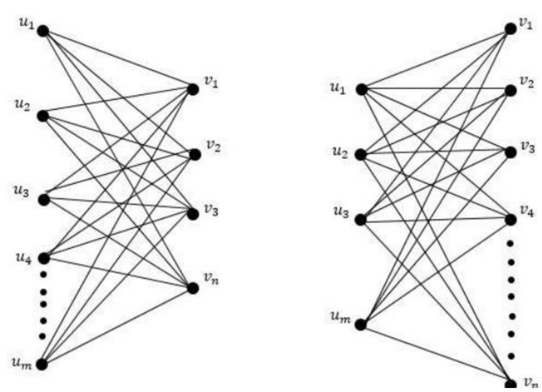

Figure 2. Complete-bipartite graph.

Definition 3.5. In graph theory, an n-crown graph on $2n$ vertices is an undirected graph with two $\{u_1, u_2 \cdots u_n\}$ and $\{v_1, v_2 \cdots v_n\}$ and with an edge from u_i to v_j whenever $i \neq j$.

Properties:

Number of vertices - $2n$

Number of edges - $n(n-1)$

Theorem 3.6. *The number of* $[1, 1]$ - *dominating sets of the n-crown graph is*

$$|D_{[1, 1]}(S_n^0)| = \{n \quad when \quad n = 2, 3, 4, \cdots \cdots\}$$

Proof. Let $u_1, u_2, u_3 \cdots u_n, v_1, v_2 \cdots v_n$ are the vertices of the n-crown graph. Here $\Delta(u_i) = \delta(u_i) = n-1$, $i = 1, 2 \cdots n$ and $\Delta(v_j) = \delta(v_j) = n-1$, $j = 1, 2 \cdots n$. The vertices v_j in $V - D$ is adjacent to the vertices u_i in D whenever $i \neq j$, $i = 1, 2, 3 \cdots n$ and $j = 1, 2, 3 \cdots n$.

The vertices v_j in $V - D$ is adjacent to vertices u_i in D.

So the vertices u_i in D dominates the vertices v_j in $V - D$ and vertices u_i in $V - D$ is adjacent to the vertices v_j in D. So the vertices v_j in D dominates in vertices u_i in $V - D$.

Required dominating sets are given below

$$\{u_1, v_1\}, \{u_2, v_2\}, \{u_3, v_3\} \cdots \{u_n, v_n\}$$

Generalization of the $[1, 1]$ - dominating sets of the n-crown graph is

$$D_{[1, 1]}(S_n^0) = \begin{cases} & \{u_{k1}, v_{k2}\} \\ & k1 = 1, 2, 3 \cdots n \\ \text{when} & k2 = 1, 2, 3 \cdots n \end{cases}$$

Thus $|D_{[1, 1]}(S_n^0)| = n$ and $\gamma_{[1, 1]}(S_n^0) = 2$.

4. Applications

There are many applications in graphs used in land surveying, Electrical networks, networking, Routing problems, coding theory, and computer communication.

The theft item can be found easily using the complete-bipartite graph. The barcode is one of the best systems which is used in anti-theft networking, railways, shopping malls, textiles, and the also in the defense department so that valuable things can be protected.

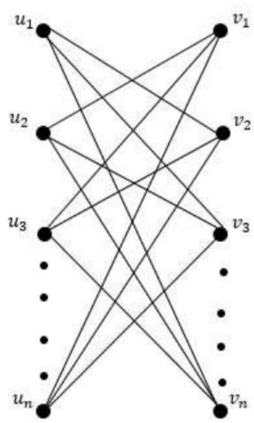

Figure 3. n-Crown graph.

5. Conclusion

In this paper, we have generalized the $[1, 1]$-set domination $[1, 2]$-set domination of the n-Barbell graph. $[1, 1]$-set domination of the complete-bipartite graph and $[1, 1]$-set domination of the n-crown graph.

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