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COVER PEBBLING NUMBER OF LADDER GRAPH TRIANGULAR GRID AND SHADOW GRAPH OF A CYCLE

R. PRABHA and B. SANDHIYA

Department of Mathematics
Ethiraj College for Women
Chennai-600 008, Tamil Nadu, India
E-mail: prabha75@gmail.com
sandhiyabasskar14@gmail.com

Abstract

A pebbling move on a connected graph G with a distribution of p pebbles, involves the removal of two pebbles from a vertex and an addition of one pebble to an adjacent vertex. The minimum number of pebbles that are required to place at least one pebble on each vertex of the given graph G regardless of the initial distribution of pebbles is called the cover pebbling number and it is denoted as $\gamma(G)$. The complexity of the problem of computing the cover pebbling number is NP-complete [7]. In this paper we compute the cover pebbling number of Ladder graphs, Triangular grid and Shadow Graph of a Cycle.

1. Introduction

The concept of Graph pebbling has gained a lot of attention, in recent days. This concept was introduced by Lagarias and Saks and later introduced into literature by Chung [3]. The cop and the robber's problem, transportation problem that involves consumable resources like petrol, gas, transmission of information in Ad hoc network are some of the applications of Graph Pebbling concept. Pebbling was initially played on directed graphs along the directed edges and this seems to be solved completely by Eriksson [4].

The cover pebbling concept is applied in situation where an information must be communicated simultaneously to various nodes of a network or soldiers need to be deployed across an area modeled by a graph. Consider a

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simple, connected and undirected graph G with a distribution of p-pebbles on its vertices. Let some vertex say v_i, be the target vertex. Our aim is to place at least one pebble on v_i, after a sequence of pebbling moves. The minimum number of pebbles that are required to reach any such target vertex regardless of the initial distribution of pebbles is known as the pebbling number of G and it is denoted as $\pi(G)$. There are various categories of pebbling numbers such as Optimal pebbling number [2], Cover pebbling number, Generalized pebbling number [1], etc.

The weighted cover pebbling number or w-cover pebbling number is the minimum number of pebbles that ensures after a sequence of pebbling moves, $w(v)$ pebbles are placed at every vertex of G Here, w is a weight function which assigns a non-negative integer value, $w(v) > 0$ to every vertex v of the given graph G. If $w(v)$ takes the value one, then one pebble will be placed at every vertex after a sequence of pebbling moves. Thus, cover pebbling number or 1-cover pebbling number is the minimum number of pebbles that are required to place one pebble at each vertex of the given connected graph G, irrespective of the initial configuration and it is denoted as $\gamma(G)$.

2. Preliminaries

The graphs that are considered in this article are simple, connected and undirected graphs. We refer to Bondy and Murty [5] for basic definitions.

Definition 2.1. Given a connected graph G, we denote the vertex set and the edge set by $V(G)$ and $E(G)$ respectively. The distance $d(u, v)$ is the length of the shortest path between the two vertices u and v of the graph G.

Definition 2.2. The distance of a vertex v is the sum of its distance from each vertex of $V(G)$, that is,

$$d(v) = \sum_{v \in V(G)} d(u, v)$$

for all $u \in V(G)$, $u \neq v$. A vertex $v \in V(G)$ is called a key vertex if $d(v)$ is maximum. For the sake of convenience, we determine the cover pebbling number using a key vertex [8].

Definition 2.3. An initial distribution is said to be simple if all the pebbles are placed at a single vertex in a graph G.

Definition 2.4. Given a graph G with n vertices, let $d(v_i)$ denote the number of pebbles on the vertex v_i, $i = 1, 2, \ldots, n$. We say that a vertex v is a

i D-vertex if $d(v_i) < 0$

ii N-vertex if $d(v_i) = 0$

iii S-vertex if $d(v_i) > 0$

where D, N and S stands for Demand, Neutral and Supply respectively.

We always consider only simple initial distribution of pebbles while computing the cover pebbling number because of the following theorem called the cover pebbling theorem.

Theorem 1. [6]. *Let w be a positive goal distribution. To determine the w-cover pebbling number of a (directed or undirected) connected graph, it is sufficient to consider a simple initial distribution. In fact, for any initial distribution that admits no cover pebbling, all pebbles may be concentrated to one of the supply vertex with cover pebbling still not possible. The cover pebbling number of certain fundamental graphs computed in* [8] *are given below*:

- For a path P_n, $\gamma(P_n) = 2^n - 1$, $n \geq 1$.
- For a complete graph K_n, $\gamma(K_n) = 2n - 1$, $n \geq 1$.
- For a wheel graph W_n, $\gamma(W_n) = 4n - 5$, $n \geq 3$.
- For a fan graph F_n, $\gamma(F_n) = 4n - 3$, $n \geq 3$.

In sections 3, 4 and 5 we compute the cover pebbling number of Ladder graph, Triangular Grid and Shadow graph of a cycle respectively.

3. Cover Pebbling Number of Ladder Graph

Definition 3.1. The n-ladder graph is defined as $L_n = P_2 \times P_n$ where P_n is a path graph with n vertices. It has $2n$ vertices and $3n - 2$ edges. We observe that any vertex of degree two is a key vertex (refer figure 1).

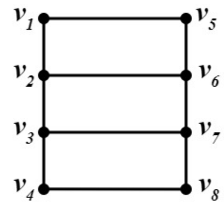

Figure 1. Ladder graph L_4.

Theorem 3.1. *If L_n denotes a ladder graph with $2n$ vertices then,*

$$\gamma(L_n) = 1 + \sum_{d=1}^{n-1} 2^d.2 + 2^n,\ n \geq 2.$$

Proof: Let L_n be a ladder graph with $2n$ vertices. We denote the vertices as $v_1, v_2, \ldots, v_n$ refer figure 1. By the cover pebbling theorem, it is enough to consider a simple initial distribution. Choose a vertex of degree two as a key vertex, say v_1. Now place $(1 + \sum_{d=1}^{n-1} 2^d.2 + 2^n) - 1$ pebbles on v_1. We require 2^d pebbles to cover each of $n-1$ other vertices which is at a distance d from v_1. Then no pebbles remain to cover v_1. Hence, we have $\gamma(L_n) \geq 1 + \sum_{d=1}^{n-1} 2^d \cdot 2 + 2^n$.

Consider an initial distribution of $1 + \sum_{d=1}^{n-1} 2^d \cdot 2 + 2^n$ pebbles that admits no cover pebbling, then there exists at least one vertex, say x with no pebbles on it. Here $1 + \sum_{d=1}^{n-1} 2^d.2 + 2^n$, is the number of pebbles used to cover all the vertices of L_n from the key vertex. By pigeonhole principle there exists at least one S-vertex. We know that the vertices of the given ladder graph are either of degree two or three. If any one of the adjacent vertices of x is a S-vertex, say u then we can cover x with two pebbles from u If none of the adjacent vertex is an S-vertex then we choose an immediate nearby S-vertex, say v which is at a distance d and cover x by using 2^d pebbles from it.

Suppose the number of pebbles on v is less than 2^d then we make pebbling moves from v until it becomes an N-vertex and stop. Now choose the next nearby S-vertex and continue the same process till x is covered. Continuing this way, we cover all the uncovered vertices with $\gamma(L_n) \le 1 + \sum_{d=1}^{n-1} 2^d.2 + 2^n$. Hence the proof.

4. Cover Pebbling Number of Triangular Grid

Definition 4.1. A grid that is formed by tiling the plane regularly with equilateral triangles is called Triangular Grid and it is denoted as T_n, where n denotes the number of rows in the triangular grid. We observe that any vertex of T_n, which is of degree two is a key vertex (refer figure 2).

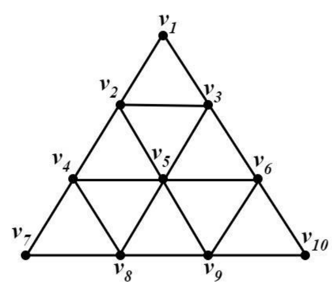

Figure 2. Triangular Grid T_3.

Theorem 4.1. *If* T_n *denotes a triangular grid with n rows, then* $\gamma(T_n) = 5 + \sum_{d=2}^{n} 2^d (d+1)$, $n \ge 2$.

Proof: Let T_n be a triangular grid with n rows and m vertices. We denote the vertices as $u_1, u_2, \ldots, u_m$. By the cover pebbling theorem, it is enough to consider a simple initial distribution. Choose a vertex of degree two as a key vertex, say u_1. Now place $(5 + \sum_{d=2}^{n} 2^d (d+1)) - 1$ pebbles on u_1. We require 2^d pebbles to cover each of $m-1$ other vertices which is at a distance d from u_1. Then no pebbles remain to cover u_1. Hence, we have $\gamma(T_n) \ge 5 + \sum_{d=2}^{n} 2^d (d+1)$.

Consider an initial distribution of $[5 + \sum_{d=2}^{n} 2^d (d+1)]$ pebbles that admits no cover pebbling, then there exists at least one vertex with no pebble on it, say x. Here $[5 + \sum_{d=2}^{n} 2^d (d+1)]$ is the number of pebbles that are used to cover all the vertices of the triangular grid from the key vertex. By pigeonhole principle there exists at least one S-vertex. The vertices of the triangular grid are of degree either two, four or six. If anyone among these adjacent vertices of x is an S-vertex, then we can cover x with two pebbles from it. If none of the adjacent vertex is an S-vertex, then we choose a next immediate nearby S-vertex, say u which is at a distance d from x and make pebbling moves to cover x by using 2^d pebbles from it. If suppose u has less than 2^d pebbles on it then we make pebbling moves till u becomes an N-vertex and stop. Choose the next nearby S-vertex and repeat the process until x is covered. Continuing this way, we cover all the uncovered vertices with $\gamma(T_n) \leq 5 + \sum_{d=2}^{n} 2^d (d+1)$. Hence the proof.

5. Cover Pebbling Number of Shadow Graph of a Cycle

Definition 5.1. Let G be a cycle with n vertices. The shadow graph of G denoted as $D_2(G)$ is obtained by taking two copies of G say G' and G'' and joining each vertex v_0 in G' to the neighbors of corresponding vertex v_0' in G''. We observe that all the vertices of $D_2(G)$ are of same degree, and hence each of $v \in V(D_2(G))$ is a key vertex (refer figure 3).

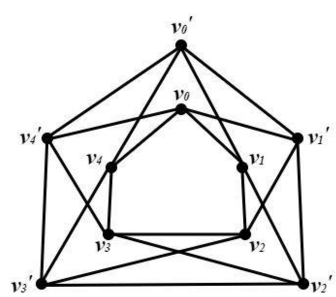

Figure 3. Shadow graph of C_5.

Theorem 5.1. If $D_2(C_n)$ *denotes the shadow graph of the cycle C with n vertices, then for* $n > 5$,

$$\gamma(D_2(C_n)) = \begin{cases} 29 + 2^{\left\lfloor \frac{n}{2} \right\rfloor} \times 2, \ n \text{ is even} \\ 29 + 2^{\left\lfloor \frac{n}{2} \right\rfloor} \times 4, \ n \text{ is odd} \end{cases}$$

Proof: Let G and G' be two copies of the cycle C_n, n is odd. We denote the vertices of G as $v_0, v_1, \ldots, v_n$ and G' as $v_0', v_1', \ldots v_n'$. By the cover pebbling theorem, it is enough to consider a simple initial distribution. Without loss of generality, we place $(29 + 2^{\left\lfloor \frac{n}{2} \right\rfloor} \times 4) - 1$ pebbles at v_0' (key vertex). Then 2^d pebbles will be used to cover each of $(2n - 1)$ other vertices which is at a distance d from v_0' leaving no pebble to cover v_0'. Hence, $\gamma(D_2(C_n)) \geq (29 + 2^{\left\lfloor \frac{n}{2} \right\rfloor} \times 4)$.

Consider an initial distribution of $29 + 2^{\left\lfloor \frac{n}{2} \right\rfloor} \times 4$ pebbles that admits no cover pebbling, then there exists at least one vertex that has no pebble on it, say x Here, $29 + 2^{\left\lfloor \frac{n}{2} \right\rfloor} \times 4$ is the number of pebbles that are used to cover all the vertices of $D_2(C)$ from the key vertex. Since $\deg(v_i) = \deg(v_i')$ each vertex of $D_2(C)$ is a key vertex. Also, every vertex of $D_2(C)$ has four adjacent vertices. By pigeonhole principle there exists at least one S-vertex. If any one of the adjacent vertices is an S-vertex then we can cover x with two pebbles else, we choose an immediate nearby S-vertex, say v which is at a distance d and cover x by using 2^d pebbles from it. Suppose the number of pebbles on v is less than 2^d then we make pebbling moves from v until it becomes an N-vertex and stop. Now choose the next nearby S-vertex and continue the same process till x is covered. Proceeding this way, we cover all the uncovered vertices with $\gamma(D_2(C_n)) \leq (29 + 2^{\left\lfloor \frac{n}{2} \right\rfloor} \times 4)$. In a similar way, we can prove for the even cycle. Hence the proof.

6. Conclusion

The Cover Pebbling problem is an interesting NP-complete problem in Graph Theory. In this paper, we have computed the cover pebbling number of Ladder graph, Triangular grid and Shadow graph of a Cycle.

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