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THE TIME DEPENDENT SOLUTION OF $M^{[x]}/G/1$ QUEUEING SYSTEM WITH A STANDBY SERVER

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Abstract

In this paper, we have obtained time dependent solution of a queueing system which is equipped with a standby server. When either the main server takes vacation or it is under repair process, the stand-by server starts to serve the customers. The service time, vacation time and repair time are assumed to follow general (arbitrary) distribution while the stand by server provides service according to exponential distribution. The steady state results have been obtained in the form of probability generating functions for the number of customers and the average waiting time in the queueing explicitly.

Introduction

The research study on queueing models with break down and repair have been extensively increased due to their wide applications in productions, communication systems. A queueing system might suddenly break down and hence the server will not be able to continue providing service unless the system is repaired. Avi-Itzhak, B. and Naor [1], Takine and Sengupta [11],

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Federgruen [3], Vinck and Bruneel [15] have studied different queueing systems subject to random breakdowns. Jayawardene and Kella [10] have studied $M/G/q$ queues with altering renewal breakdowns. Kulkarni and Choi [9] and Wang et al. [13] have studied retrial queues with system breakdowns and repairs. Madan and Maraghi [20] have studied batch arrival queue with break down and repair.

In this paper we consider queueing system in which the main server is equipped with a standby server. This paper considers both vacations and main server breakdowns including the assumptions of deployment of a stand by during the vacation periods and repair periods. Customers arrive at the system in batches of variable size in a compound Poisson process. The service is provided one-by-one on a first come-first served. In this model, the service time, vacation time and repair time are followed by general (arbitrary) distribution where as the stand-by server's service distribution is followed by exponential distribution. The rest of the paper is organized as follows. The mathematical description of our model is in Section 2 and equations governing the model are given in Section 3. The time dependent solution have been obtained in Section 4 and the corresponding steady state results have been derived explicitly in Section 5. Mean queue length and mean waiting time are computed in Section 6 respectively.

Mathematical Definition of the Queueing Model

We assume the following assumptions to describe the queueing model of our study.

* Customers arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i \Delta t$, $i = 1, 2, 3, \ldots$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + \Delta t)$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches. The customers are served one-by-one on a "first come-first" served basis.

* The service times of the main server follow a general (arbitrary) distribution with distribution function $B(v)$ and density function $b(v)$. Let $\mu(x)$ be the conditional probability density of service completion during the

interval $(x, x + dx]$, given that the elapsed service time is x, so that

$$\mu(x) = \frac{b(x)}{1 - B(x)} \tag{1}$$

and therefore

$$b(v) = \mu(v)e - \int_0^v \mu(x)dx \tag{2}$$

* On completion of a service, the server may go for a vacation of random length with probability p, or may stay in the system for providing service with probability 1-p, where $0 \leq p \leq 1$.

* The vacation time of the server follow a general (arbitrary) distribution with distribution function $V(s)$ and the density function $v(s)$. Let $\beta(x)$ be the conditional probability of a completion of a vacation during the interval $(x, x + dx)$ given that the elapsed vacation time is x

$$\beta(x) = \frac{v(x)}{1 - V(x)} \tag{3}$$

and hence

$$v(s) = \beta(s)e - \int_0^s \beta(x)dx. \tag{4}$$

* The system may have breakdowns at random, and breakdowns are assumed to occur according to Poisson stream with mean breakdown rate $\alpha > 0$. Further we assume that once the system breaks down, the customer whose service is interrupted comes back to the head of the queue but it is immediately taken for by the stand-by server.

* Once the system breaks down, its repair start immediately and the duration of repairs follows a general (arbitrary) distribution with distribution function $F(r)$ and the density function $f(r)$. Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, X + dx)$ given that the elapsed vacation time is x.

$$\gamma(x) = \frac{f(x)}{1 - F(x)} \tag{5}$$

and therefore

$$f(r) = \gamma(r)e - \int_0^s \gamma(x)dx. \tag{6}$$

* The stand-by server starts serving the customers as soon as the main server breaks down or as soon as the mains server leaves for a vacation after completing a service. The stand-by service follow an exponential distribution with stand-by service rate $\delta > 0$ and mean stand-by service $\frac{1}{\delta}$.

* Further we assume that the main server joins the system immediately after the completion of its vacation or completion of its repairs, and the customer being served by the stand-by server is immediately transferred to the main server to start a service afresh.

* Customers are arriving in batches, but they are served one by one.

* All stochastic process involved in the system are independent of each others.

Notations

We let,

(i) $P_n(x, t)$ represent probability that at time t, there are $n \geq 0$ customers in the queue excluding one customer in the service served by the main server, and the elapsed service time of this customer is x. Accordingly, $P_n(t) = \int_0^\infty P_n(x, t)dx$ denotes the probability that there are $n \geq 1$ customers in the queue excluding one customer in service irrespective of the value of x.

(ii) $V_n(x, t)$ represent Probability that at time t, there are $n \geq 0$ customers in the queue (and one customer is being served by the stand-by server), and the main server is on vacation with elapsed vacation time x. Accordingly, $V_n(t) = \int_0^\infty V_n(x, t)dx$ denotes the probability that at time t, there are $n \geq 1$ customers in the queue and the server is on vacation irrespective of the value of x. As soon as the vacation starts the stand-by server starts serving the customers in the system.

(iii) $R_n(x, t)$ follows Probability that at the time t, there are $n \geq 0$ customers in the queue (and one customer is being served by stand -by server) while the system is under repair with elapsed repair time x. Accordingly, $R_n(t) = \int_0^\infty R_n(x, t)dx$ denotes the probability that at time t, there are $n \geq 1$ customers in the queue and the server is under repair irrespective of the value of x.

(iv) $Q(t)$ denotes probability that at time t, there are no customers in the system and the server is idle but available in the system.

Definitions and Equations Governing the System

Having the above assumptions for our model, we obtain the following differential-difference equations.

$$\frac{\partial}{\partial t} P_n(x, t) + \frac{\partial}{\partial x} P_n(x, t) + (\lambda + \mu(x) + \alpha)P_n(x, t) = \lambda \sum_{i=1}^{n-1} c_i P_{n-i}(x, t);\ n \geq 1. \quad (7)$$

$$\frac{\partial}{\partial t} P_0(x, t) + \frac{\partial}{\partial x} P_0(x, t) + (\lambda + \mu(x) + \alpha)P_0(x, t) = 0 \quad (8)$$

$$\frac{\partial}{\partial t} V_n(x, t) + \frac{\partial}{\partial x} V_n(x, t) + (\lambda + \beta(x) + \delta)V_n(x, t)$$

$$= \lambda \sum_{i=1}^{n} c_i V_{n-i}(x, t) + \delta V_{n+1}(x, t);\ n \geq 1. \quad (9)$$

$$\frac{\partial}{\partial t} V_0(x, t) + \frac{\partial}{\partial x} V_0(x, t) + (\lambda + \beta(x) + \delta)V_0(x, t) = \delta V_1(x, t). \quad (10)$$

$$\frac{\partial}{\partial t} R_n(x, t) + \frac{\partial}{\partial x} R_n(x, t) + (\lambda + \gamma(x) + \delta)R_n(x, t)$$

$$= \lambda \sum_{i=1}^{n} c_i R_{n-i}(x, t) + \delta R_{n+1}(x, t);\ n \geq 1. \quad (11)$$

$$\frac{\partial}{\partial t} R_0(x, t) + \frac{\partial}{\partial x} R_0(x, t) + (\lambda + \gamma(x) + \delta) R_0(x, t) = \delta R_1(x, t). \tag{12}$$

$$\frac{d}{dt} \lambda Q(t) = -\lambda Q(t) + \int_0^\infty R_0(x, t)\gamma(x)dx + (1-p)$$

$$\int_0^\infty P_0(x, t)\mu(x)dx + (1-r)\int_0^\infty V_0(x, t)\beta(x)dx \tag{13}$$

From the equations (7)-(13) are to be solved subject to the boundary conditions at $x = 0$

$$P_n(0, t) = (1-p)\int_0^\infty P_{n+1}(0, t)\mu(x)dx + \int_0^\infty V_{n+1}(0, t)\beta(x)dx$$

$$+ \int_0^\infty R_{n+1}(0, t)\gamma(x)dx + \lambda C_{n+1} Q(t). \tag{14}$$

$$V_n(0, t) = p\int_0^\infty P_n(x, t)\mu(x)dx;\ n \geq 0. \tag{15}$$

$$R_n(0, t) = \alpha \int_0^\infty P_{n-1}(x, t)dx;\ n \geq 1. \tag{16}$$

$$R_0(0, t) = 0. \tag{17}$$

We assume that initially there is no customer in the system and the server is idle, but available in the system. So the initial conditions are $P_0(0) = 0$, $V_0(0) = 0$ and $Q(0) = 1$.

Generating Functions of the Queue Length the time Dependent Solution

$$P_q = (x, z, t) = \sum_{n=0}^{\infty} z^n P_n(x, t). \tag{18}$$

$$P_q = (x, z) = \sum_{n=0}^{\infty} z^n P_n(t). \tag{19}$$

$$V_q = (x, z, t) = \sum_{n=0}^{\infty} z^n V_n(x, t). \tag{20}$$

$$V_q = (x, z) = \sum_{n=0}^{\infty} z^n V_n(t). \tag{21}$$

$$R_q = (x, z, t) = \sum_{n=0}^{\infty} z^n R_n(x, t). \tag{22}$$

$$R_q = (x, z) = \sum_{n=0}^{\infty} z^n R_n(t). \tag{23}$$

Which are convergent inside the circle given by $|z| \leq 1$.

Define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^{\infty} f(t) e^{-st} dt. \tag{24}$$

Taking Laplace transforms of the equations from (7) to (17) by using

$$\frac{\partial}{\partial x} \overline{P}_n(x, s) + (s + \lambda + \mu(x) + \alpha)\overline{P}_n(x, s) = \lambda \sum_{i=1}^{n-1} c_i \overline{P}_{n-i}(x, s);\ n \geq 1. \tag{25}$$

$$\frac{\partial}{\partial x} \overline{P}_0(x, s) + (s + \lambda + \mu(x) + \alpha)\overline{P}_0(x, s) = 0. \tag{26}$$

$$\frac{\partial}{\partial x} \overline{V}_n(x, s) + (s + \lambda + \beta(x) + \delta)\overline{V}_n(x, s) = \lambda \sum_{i=1}^{n} c_i \overline{V}_{n-i}(x, s);\ n \geq 1. \tag{27}$$

$$\frac{\partial}{\partial x} \overline{V}_0(x, s) + (s + \lambda + \beta(x) + \delta)\overline{V}_0(x, s) = \delta \overline{V}_1(x, s). \tag{28}$$

$$\frac{\partial}{\partial x} \overline{R}_n(x, s) + (s + \lambda + \gamma(x) + \delta)\overline{R}_n(x, s) = \lambda \sum_{i=1}^{n} c_i \overline{R}_{n-i}(x, s);\ n \geq 1. \tag{29}$$

$$\frac{\partial}{\partial x} \overline{R}_0(x, s) + (s + \lambda + \gamma(x) + \delta)\overline{R}_0(x, s) = \delta \overline{R}_1(x, s). \tag{30}$$

$$(s+\lambda)\overline{Q}(s) = 1 + \int_0^\infty \overline{R}_0(x, s)\gamma(x)dx + (1-p)\int_0^\infty \overline{P}_0(x, s)\mu(x)dx + (1-r)$$

$$\int_0^\infty \overline{V}_0(x, s)\beta(x)dx. \tag{31}$$

$$\overline{P}_n(0, s) = \int_0^\infty \overline{R}_{n+1}(x, s)\gamma(x)dx + (1-p)\int_0^\infty \overline{P}_{n+1}(x, s)\mu(x)dx + (1-r)$$

$$\int_0^\infty \overline{V}_{n+1}(x, s)\beta(x)dx + \lambda C_{n+1}\overline{Q}(s). \tag{32}$$

$$\overline{V}_n(0, s) = p\int_0^\infty \overline{P}_n(x, s)\mu(x)dx;\ n \geq 0. \tag{33}$$

$$R_n(0, s) = \alpha\int_0^\infty \overline{P}_{n-1}(x, s)dx;\ n \geq 1. \tag{34}$$

$$\overline{R}_0(0, s) = 0. \tag{35}$$

Now multiplying the equation (25) by z^n from 0 to ∞ and taking summation over from 1 to ∞, adding (26), and then simplifying using (20)

$$\frac{\partial}{\partial x}\overline{P}_q(x, z, s) + (s + \lambda - \lambda C(z) + \mu(x) + \alpha)\overline{P}_q(x, z, s) = 0. \tag{36}$$

Performing similar operations on (27), (28), (29) and (30) we get respectively

$$\frac{\partial}{\partial x}\overline{V}_q(x, z, s) + \left(s + \lambda - \lambda C(z) + \beta(x) + \delta - \frac{\delta}{2}\right)\overline{V}_q(x, z, s) = 0. \tag{37}$$

$$\frac{\partial}{\partial x}\overline{R}_q(x, z, s) + \left(s + \lambda - \lambda C(z) + \gamma(x) + \delta - \frac{\delta}{2}\right)\overline{R}_q(x, z, s) = 0. \tag{38}$$

similarly multiplying (32) by z^{n+1}, summing over n from 0 to ∞ and by using (20)

$$z\overline{P}_q(0, z, s) = \int_0^\infty \overline{R}_q(x, z, s)\gamma(x)dx + (1-p)\int_0^\infty \overline{P}_q(x, z, s)\mu(x)dx$$

$$+ \int_0^\infty \overline{V}_q(x, z, s)\beta(x)dx + \lambda C(z)\overline{Q}(s)$$

$$-\left[\int_0^\infty \overline{R}_0(x, s)\gamma(x)dx + (1-p)\int_0^\infty \overline{P}_0(x, s)\mu(x)dx + (1-r)\right.$$

$$\left.\int_0^\infty \overline{V}_0(x, s)\beta(x)dx\right]. \tag{39}$$

From the equation (31)

$$z\overline{P}_q(0, z, s) = (1-p)\int_0^\infty \overline{P}_q(x, z, s)\mu(x)dx + \int_0^\infty \overline{V}_q(x, z, s)\beta(x)dx$$

$$+\int_0^\infty \overline{R}_q(x, z, s)\gamma(x)dx + [1 - s\overline{Q}(s)] + \lambda\overline{Q}(s)[C(z)-1]. \tag{40}$$

Multiplying (33) by z^n and summing over n from 0 to ∞, we get

$$\overline{V}_q(0, z, s) = p\int_0^\infty \overline{P}_n(x, z, s)\mu(x)dx. \tag{41}$$

Similarly from the equation (16)

$$\overline{R}_q(0, z, s) = \alpha z\int_0^\infty \overline{P}_q(x, z, s)dx \geq 1$$

$$(ie)\,\overline{R}_q(0, z, s) = \alpha zP(\overline{z}, s). \tag{42}$$

Integrating (31) from 0 to x yields

$$\overline{P}_q(x, z, s) = \overline{P}_q(0, z, s)e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^x \mu(t)dt}. \tag{43}$$

where $\overline{P}_q(0, z, s)$ is given by (40)

Integrate (43) by parts with respect to x, yields

$$\overline{P}(z, s)\overline{P}_q(0, z, s)\frac{1-\overline{B}[s+\lambda-\lambda C(z)+\alpha]}{[s+\lambda-\lambda C(z)+\alpha]}. \tag{44}$$

where $\overline{B}[s+\lambda-\lambda C(z)+\alpha] = \int_0 e^{-(s+\lambda-\lambda C(z)+\alpha)x}dB(x)$ is the Laplace Stieltjes transform of the service time $B(x)$. Now multiplying both sides of (43) by $\mu(x)$ and integrating over x, we get

$$\int_0^\infty \overline{P}_q(x, z, s)\mu(x)dx = \overline{P}_q(0, z, s)\overline{B}[s+\lambda-\lambda C(z)+\alpha]. \tag{45}$$

Using (45) and from (41)

$$\overline{V}_q(0, z, s) = p\overline{P}_q(0, z, s)\overline{B}[s+\lambda-\lambda C(z)+\alpha]. \tag{46}$$

Similarly integrating (37) from 0 to x yields

$$\overline{V}_q(x, z, s) = \overline{V}_q(0, z, s)e^{-\left(s+\lambda-\lambda C(z)+\delta-\frac{\delta}{z}\right)x-\int_0^x \beta(t)dt}. \tag{47}$$

substituting the value of $\overline{V}_q(0, z, s)$ from (46) in (47)

$$\overline{V}_q(x, z, s) = p\overline{P}_q(0, z, s)\overline{B}[s+\lambda-\lambda C(z)+\alpha]e^{-\left(s+\lambda-\lambda C(z)+\delta-\frac{\delta}{z}\right)x-\int_0^x \beta(t)dt}. \tag{48}$$

Integrating (48) by parts with respect to x, we get

$$\overline{V}_q(z, s) = \overline{V}_q(0, z, s)\overline{B}[s+\lambda-\lambda C(z)+\alpha]\frac{1-\overline{V}\left[s+\lambda-\lambda C(z)+\delta-\frac{\delta}{z}\right]}{\left[s+\lambda-\lambda C(z)+\delta-\frac{\delta}{z}\right]}. \tag{49}$$

where $\overline{V}\left[s+\lambda-\lambda C(z)+\delta\frac{\delta}{z}\right] = \int_0 e^{-}\left(s+\lambda-\lambda C(z)+\delta\frac{\delta}{z}\right)xdV(x)$ is the Laplace Stieltjes transform of the vacation time $V(x)$. Now multiplying both sides of (48) by $\beta(x)$ and integrating over x, we get

$$\int_0^\infty \overline{V}_q(x, z, s)\beta(x)dx = p\overline{P}_q(0, z, s)\overline{B}[s+\lambda-\lambda C(z)+\alpha] + \overline{V}\left[s+\lambda-\lambda C(z)+\delta-\frac{\delta}{z}\right]. \tag{50}$$

Now integrating (33) from 0 to x, yields

$$\overline{R}_q(x, z, s) = \overline{R}_q(0, z, s)e^{-\left(s+\lambda-\lambda C(z)+\delta-\frac{\delta}{z}\right)x-\int_0^x \gamma(t)dt}. \tag{51}$$

substituting the value of $\overline{R}_q(0, z, s)$ from (42) and using (44) we get

$$\overline{R}_q(x, z, s) = \alpha z\overline{P}_q(0, z, s)\frac{1-\overline{B}[s+\lambda-\lambda C(z)+\alpha]}{[\lambda-\lambda C(z)+\alpha]}e^{-}\left(\lambda-\lambda C(z)+\delta-\frac{\delta}{Z}\right)$$

$$x - \int_0^x \gamma(t)dt. \tag{52}$$

Integrating (52) by parts with respect to x we get

$$\overline{R}_q(z, s) = \alpha z \overline{P}_q(0, z, s) \frac{[1 - \overline{B}[s + \lambda - \lambda C(z) + \alpha]][1 - \overline{R}\left[s + \lambda - \lambda C(z) + \delta - \frac{\delta}{z}\right]}{[s + \lambda - \lambda C(z) + \alpha]\left[s + \lambda - \lambda C(z) + \delta - \frac{\delta}{z}\right]}. \tag{53}$$

where $\overline{R}\left[s + \lambda - \lambda C(z) + \delta - \frac{\delta}{z}\right] = \int_0^\infty e^{-}\left(\lambda - \lambda C(z) + \delta - \frac{\delta}{z}\right) x dR(x)$ is the Laplace Stieltjes transform of the repair time $R(x)$.

Now multiplying (46) by $\gamma(x)$ and integrating over x we get

$$\int_0^\infty \overline{R}_q(x, z, s)\gamma(x)dx = \alpha z \overline{P}_q(0, z, s)\left(\frac{1 - \overline{B}[s + \lambda - \lambda C(z) + \alpha]}{[s + \lambda - \lambda C(z) + \alpha]}\right)$$
$$\overline{R}\left[s + \lambda - \lambda C(z) + \delta - \frac{\delta}{z}\right]. \tag{54}$$

Now using (46), (50) and (54), hence the equation (41) becomes

$$\overline{P}_q(z, s) = \frac{-f_1(z)[s\overline{Q}(s) - 1] + \lambda \overline{Q}(s)[1 - C(z)]}{f_1(z)[z - \overline{B}(f_1(z))][1 - p + p\overline{V}(f_2(z))] - \alpha z[1 - \overline{B} f_1(z)]\overline{R}(f_2(z))} s \tag{55}$$

where $f_1(z) = s + \lambda - \lambda C(z) + \alpha$ and $f_2(z) = s + \lambda - \lambda C(z) + \delta - \frac{\delta}{z}$.

Now Using (55), (45), (50) and (53) becomes respectively

$$\overline{P}_q(z, s) = \frac{-[(sQ(\bar{s}) - 1] + \lambda \overline{Q}(s)(1 - C(z))[1 - \overline{B}[f_1(z)]]}{f_1(z)[z - \overline{B}(f_1(z))(1 - p + p\overline{V}(f_2(z))) - \alpha z[(1 - \overline{B}(f_1(z)))]\overline{R}(f_2(z))}. \tag{56}$$

$$\overline{V}_q(z, s) = \frac{-f_1(z)[(sQ(\bar{s}) - 1] + \lambda \overline{Q}(s)[1 - C(z)][1 - \overline{V} f_2(z)]\overline{B}(f_1(z))}{f_1(z) f_2(z)[z - \overline{B}(f_1(z))](1 - p + p\overline{V}(f_2(z)) - \alpha z f_1(z) [1 - \overline{B}(f_1(z))]\overline{R}(f_2(z))}. \tag{57}$$

$$\overline{R}_q(z, s) = \frac{-\alpha z[(sQ(\bar{s})-1] + \lambda\overline{Q}(s)(1-C(z))[1-\overline{V}(f_2(z))][1-\overline{R}(f_2(z))]}{f_1(z)f_2(z)[z-\overline{B}(f_1(z))][1-p+p\overline{V}(f_2(z))] - \alpha z f_2(z) f_2(z)[1-\overline{B}f_1(z)]\overline{R}(b)}. \tag{58}$$

Thus $\overline{P}_q(z, s)$, $\overline{V}_q(z, s)$ and $\overline{R}_q(z, s)$ can be determined from (56), (57) and (58).

The Steady State Analysis

In this section, we have obtained the steady state probability distributions for our queueing model. To define the state probabilities suppress the argument 't' wherever it appears in the time dependent analysis.

By using the well "Tauberian property"

$$Lt_{s\to 0} s\bar{f}(s) = Lt_{t\to\infty} f(t). \tag{59}$$

Multiplying both sides equations (56), (57) and (58), applying (59) and simplifying, we get

$$P_q(z) = \frac{-\lambda Q(1-C(z))\left[1-\overline{B}(a)\right]}{a[(z-\overline{B}(a))][(1-p+p\overline{V}(b))] - \alpha z[(1-\overline{B}(a))]\overline{R}(b)}. \tag{60}$$

$$V_q(z) = \frac{-a\lambda Q(1-C(z))\left[1-\overline{V}(a)\right]\overline{B}(a)}{ab[(z-\overline{B}(a))(1-p+p\overline{V}(b))] - \alpha z b[(1-\overline{B}(a))]\overline{R}(b)}. \tag{61}$$

$$R_q(z) = \frac{-\alpha z\lambda Q(1-C(z))[1-\overline{V}(b)][1-\overline{R}(b)]}{ab[(z-\overline{B}(a))(1-p+p\overline{V}(b))] - \alpha z b[(1-\overline{B}(a))]\overline{R}(b)}. \tag{62}$$

Let $W_q(z)$ be denote the probability generating function of the queue size irrespective of the state of the system.

(i.e) $W_q(z) = P_q(z) + V_q(z) + R_q(z)$.

Adding (58), (59) and (60) yields

$$W_q(z) = \frac{-Q\lambda(1-C(z))(1-\overline{B}(a))[b+\alpha z(1-\overline{R}(b))] - a\lambda(1-C(z))Qp\overline{B}(a)(1-\overline{V}(b))}{ab[(z-\overline{B}(a))(1-p+p\overline{V}(b))] - \alpha z b[(1-\overline{B}(a))]\overline{R}(b)}. \tag{63}$$

In order to find Q, Using normalization condition

$$W_q(1) + Q = 1. \tag{64}$$

Note that if $z = 1$ then $W_q(1)$ is indeterminate of $0/0$ form. There fore, applying L' Hospitals rule twice on (63), we get

$$W_q(1) = Lt_{z \to 1} \frac{N''(z)}{D''(z)}. \tag{65}$$

Where $N(z)$ and $D(z)$ are the numerator and denominator of the right sides of (63) respectively. Double primes in (65) denote the second derivative at $z = 1$.

$$N''(1) = -2Q\lambda E(I)(\lambda E(I) - \delta)[[(1 - \overline{B}(\alpha))[1 + \alpha E(R)] + \alpha p\overline{B}(a)E(V)]]. \tag{66}$$

$$\begin{aligned} D''(1) = {} & 2(\lambda E(I) - \delta)[(1 - \overline{B}(a))\lambda E(I) + \alpha[1 + (\lambda E(I) - \delta)E(R)]] \\ & - \alpha(1 - p(\lambda E(I) - \delta)\overline{B}(a)E(V)). \end{aligned} \tag{67}$$

where $C(1) = 1$, $C'(1) = E(I)$ is the mean batch size of the arriving customers. $\overline{V}(0) = 1$ and $\overline{V}'(0) = -E(V)$ is the mean vacation time, and $\overline{R}(0) = 1$, $\overline{R}'(0) = -E(R)$ is the mean repair time.

Then from the equation (63)

$$Q = \frac{\begin{array}{l}[\alpha(1 - p(\lambda E(I) - \delta)\overline{B}(a)E(V)) - (1 - \overline{B}(a))\lambda E(I) \\ + \alpha[1 + (\lambda E(I) - \delta)E(R)]\end{array}}{\alpha[\delta E(R)(1 - \overline{B}(a)) + \overline{B}(a)(1 + p\delta(V))]}. \tag{68}$$

From (68) the utilization factor ρ can be found where $\rho = 1 - Q$.

The Average Queue Size and The Average Waiting Time

Let L_q be denote the mean number of customers in the queue under the steady state. Then

$$L_q \frac{d}{dz} W_q(z)|_{z=1} . \tag{69}$$

Since (69) is indeterminate of $0/0$ form. There fore, applying L' Hospitals rule four times on (69), we get

$$L_q - Lt_{z \to 1} \frac{D''(z)N'''(z) - D'''(z)N''(z)}{3[D''(z)]^2}. \tag{70}$$

where $N''(1)$ and $D''(1)$ are given in (66)and (67) respectively

$$N'''(1) = -3Qny(1-\overline{B}(\alpha))(1+\alpha E(R)) - 6Qym^2\overline{B}'(\alpha)[1+\alpha E(R)]$$
$$-3Qm(1-\overline{B}(a))[x(1+\alpha E(R)) + ay^2E(R^2)]$$
$$+3pQE(V)y\overline{B}(a)[2m^2 - \alpha x] + 6\alpha pQm^2yE(V)$$
$$\overline{B}'(a) - 3\alpha pQm\overline{B}(\alpha)[y^2E(V^2) + xE(V)]. \tag{71}$$

$$D'''(1) = 3[(1-\overline{B}(a))pE(V)y][2my - ax] + 6m^2y\overline{B}'(\alpha) + 3(1-\overline{B}(a))(ny + mx)$$
$$-3\alpha y[2\overline{B}'(a)myE(V) - \overline{B}(a)E(V^2)y^2 + xE(V)]$$
$$+ 6\alpha y^2(1-\overline{B}(a))E(R) + 3\alpha[1 + yE(R)](x)(1-\overline{B}(a)) + 2my\overline{B}'(\alpha)$$
$$+ 3\alpha y(1-\overline{B}(a))[y^2E(R^2) + xE(R)]. \tag{72}$$

Conclusion

In this paper, a time dependent solution of a queueing system has been analyzed with stand-by server considering both server and customers relation aspects like main server takes vacation or ii is under repair process, the stand-by server starts to serve customer. Both transient and steady state solution an obtained for the model accompanied with queue characteristics.

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