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CONVERGENCE OF PICARD-S HYBRID ITERATION PROCESS FOR GENERALIZED α-NONEXPANSIVE MAPPINGS

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Abstract

In this paper, we prove some convergence results for generalized α-nonexpansive mappings, using the Picard-S hybrid iteration process in the context of uniformly convex Banach space.

1. Introduction

Fixed point theory is a very interesting research area of nonlinear analysis. This theory is applied to a wide class of problems arising in different branches of mathematics, such as: variational inequalities, equilibrium problems, optimization, etc. Approximation of fixed points for nonlinear mappings using the different iterative methods is one of the goals of fixed-point theory. In the last decades, many iteration processes have been developed in this direction. Let C be a nonempty subset of a real Banach space X and $T : C \to C$ be a mapping with the fixed point set $F(T)$, i.e., $F(T) = \{p \in C : Tp = p\}$. Now, we consider some well-known iteration processes. The Picard iteration process is defined by

$$x_{n+1} = Tx_n, \tag{1.1}$$

for all $n \geq 0$, (for example, see [14]). The Mann iteration process is defined by

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$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1.2}$$

for all $n \geq 0$, and $\alpha_n \in (0, 1)$ (for example, see [9]). Also, the Ishikawa iteration process is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n,$$
$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n \tag{1.3}$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0, 1)$ (for example, see [7]). The Noor iteration process is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n,$$
$$y_n = (1 - \beta_n)x_n + \beta_n Tz_n,$$
$$z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n \tag{1.4}$$

for all $n \geq 0$, where $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ (for example, see [10]). In 2007, Agarwal et al. [2] defined their iteration process by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n Ty_n,$$
$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n \tag{1.5}$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0, 1)$ (for example, see [2]). In 2014, Abbas and Nazir [1] defined their iteration process by

$$x_{n+1} = (1 - \alpha_n)Ty_n + \alpha_n Tz_n,$$
$$y_n = (1 - \beta_n)Tx_n + \beta_n Tz_n,$$
$$z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n \tag{1.6}$$

for all $n \geq 0$, where $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ (for example, see [1]).

In 2014, Gursoy and Karakaya [6] defined a new iteration method called Picard-S hybrid iteration as follows:

$$x_{n+1} = Ty_n$$
$$y_n = (1 - \alpha_n)Tx_n + \alpha_n Tz_n$$
$$z_n = (1 - \beta_n)x_n + \beta_n Tx_n, \tag{1.7}$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0, 1)$ (for example, see [6]). They used the Picard-S hybrid iteration process to approximate the fixed points of contraction mappings. Also, they showed that the Picard-S hybrid iteration method converges faster than all Picard, Mann, Ishikawa, Noor, and some other iteration methods.

In the last years, many researchers study the class of nonexpansive mappings. The mapping T is called nonexpansive if

$$\| Tx - Ty \| \leq \| x - y \|, \forall x, y \in C$$

and T is called quasi-nonexpansive if

$$\| Tx - p \| \leq \| x - p \|, \forall x \in C \text{ and } p \in F(T).$$

In 2008, Suzuki [17] introduced an interesting extension of nonexpansive mappings as follows:

Definition 1.1[17]**.** A self-mapping T on a nonempty subset C of a Banach space is said to satisfy condition $(\mathcal{C})$ if for each two elements $x, y \in C$

$$\frac{1}{2} \| x - Tx \| \leq \| x - y \| \Rightarrow \| Tx - Ty \| \leq \| x - y \|. \qquad (\mathcal{C})$$

Suzuki showed that every nonexpansive mapping satisfied condition $(\mathcal{C})$. Also he proved that a mapping which satisfies this condition and has a fixed point is quasi-nonexpansive.

Example 1.2[17]**.** Let $C = [0, 3]$ be a subset of $\mathbb{R}$. Define a mapping $T : C \to C$ by

$$T(x) = \begin{cases} 0, & \text{if } x \neq 3 \\ 1, & \text{if } x = 3. \end{cases}$$

It is easy to prove that T satisfies condition $(\mathcal{C})$, but T is not nonexpansive. Later, Phuengrattana [13] proved fixed point results for mappings which satisfies condition $(\mathcal{C})$ using the Ishikawa iteration process. In 2017, Pant and Shukla [12] introduced the class of generalized α-nonexpansive mappings as follows:

Definition 1.3[12]**.** A self-mapping T on a nonempty subset C of a Banach space is said to be generalized α-nonexpansive mapping if one can find a real number $\alpha \in [0, 1)$ such that for each two elements $x, y \in C$,

$$\frac{1}{2}\| x - Tx \| \leq \| x - y \|$$

$$\Rightarrow \| Tx - Ty \| \leq \alpha \| Tx - y \| + \alpha \| Ty - x \| + (1 - 2\alpha) \| x - y \|.$$

It is obviously, when $\alpha = 0$ a generalized α-nonexpansive mapping reduces to a mapping which satisfying condition ($\mathcal{C}$).

Example 1.4. Let $C = [0, 4]$ be a closed convex subset of a Banach space $X = \mathbb{R}$. Define $T : C \to C$ by

$$T(x) = \begin{cases} 0, & \text{if } x \neq 4 \\ 2, & \text{if } x = 4. \end{cases}$$

Then T is a generalized α-nonexpansive mapping with $\alpha \geq \frac{1}{2}$, but T does not satisfy Suzuki's condition ($\mathcal{C}$). Recently, fixed point theorems for generalized α-nonexpansive mappings have been studied by many authors, see e.g. [3, 8, 18] and references therein.

Motived and inspired by the above, we prove some strong and weak convergence results using the Picard-S hybrid iteration process for generalized α-nonexpansive mappings in uniformly convex Banach spaces.

2. Preliminaries

Definition 2.1[4]**.** A Banach space X is called uniformly convex if, for any $\varepsilon \in [0, 1)$, one can find a real number $\delta \in (0, \infty)$ such that $\| x + y \|/2 \leq (1 - \delta)$, whenever $\| x \| \leq 1, \| y \| \leq 1$ and $\| x - y \| \geq \varepsilon$ for each $x, y \in X$. X is called strictly convex if, for any $x, y \in X$ satisfying $\| x \| = \| y \| = 1$ and $x \neq y$, it follows that $\| x + y \| < 2$.

Definition 2.2[11]**.** A Banach space X is said to satisfy Opial's condition if, for every weakly convergent sequence $\{x_n\}$ to $x \in X$, it follows that

$$\liminf_{n\to\infty} \| x_n - x \| < \liminf_{n\to\infty} \| x_n - y \|$$

for all $y \in X$, with $y \neq x$.

Definition 2.3[5]. Let C be a nonempty subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, let

– asymptotic radius of $\{x_n\}$ at x by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \| x_n - x \|;$$

– asymptotic radius of $\{x_n\}$ with respect to C by

$$r(C, \{x_n\}) = \inf \{r(x, \{x_n\}) : x \in C\};$$

– asymptotic center of $\{x_n\}$ with respect to C by

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

If space X is uniformly convex, then set $A(C, \{x_n\})$ is a singleton.

Lemma 2.4[15]. *Let X be a real uniformly convex Banach space and $0 < a \le t_n \le b < 1$, for all $n \in N$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\limsup_{n\to\infty} \| x_n \| \le r$, $\limsup_{n\to\infty} \| y_n \| \le r$ and $\lim_{n\to\infty} \| (1 - t_n)x_n + t_n y_n \| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty} \| x_n - y_n \| = 0$.*

In the following, we prove some key lemma that will be used in our presentation.

Lemma 2.5. *Let T be a self-mapping on a nonempty subset C of a Banach space. If T is generalized α-nonexpansive mapping with a fixed point p, then T is quasi-nonexpansive.*

Proof. Let $p \in F(T)$. Since $(1/2)\| p - Tp \| = 0 \le \| x - p \|$, we get

$$\| Tx - Tp \| \le \alpha \| Tx - p \| + \alpha \| Tp - x \| + (1 - 2\alpha) \| x - p \|$$

$$= \alpha \| Tx - p \| + (1 - \alpha) \| x - p \|$$

It follows that

$$(1 - \alpha) \| Tx - p \| \le (1 - \alpha) \| x - p \|.$$

Since $(1-\alpha) > 0$, we obtain our result. □

Lemma 2.6. *Let T be a self-mapping on a nonempty subset C of a Banach space. If T is generalized α-nonexpansive mapping, then for all* $x, y \in C$:

(a) $\| Tx - T^2x \| \leq \| x - Tx \|$;

(b) *Either* $(1/2)\| x - Tx \| \leq \| x - y \|$ *or* $(1/2)\| Tx - T^2x \| \leq \| Tx - y \|$ *holds*;

(c) *Either* $\| Tx - Ty \| \leq \alpha\| Tx - y \| + \alpha\| x - Ty \| + (1-2\alpha)\| x - y \|$ *or* $\| T^2x - Ty \| \leq \alpha\| Tx - Ty \| + \alpha \| T^2x - y \| + (1-2\alpha)\| Tx - y \|$ *holds*.

Proof. (a) Since $(1/2)\| x - Tx \| \leq \| x - Tx \|$, we have

$$\begin{aligned} \| Tx - T^2x \| &\leq \alpha\| T^2x - x \| + (1-2\alpha)\| x - Tx \| \\ &\leq \alpha(\| x - Tx \| + \| Tx - T^2x \|) + (1-2\alpha)\| x - Tx \|. \end{aligned}$$

It follows that

$$(1-\alpha)\| Tx - T^2x \| \leq (1-\alpha)\| x - Tx \|.$$

Since $(1-\alpha) > 0$, we get our result. The condition (c) follows from (b). Let us prove (b). We suppose the contrary, i.e., $(1/2)\| x - Tx \| > \| x - y \|$ and $(1/2)\| Tx - T^2x \| > \| Tx - y \|$. Using ($a$), we have

$$\begin{aligned} \| x - Tx \| &\leq \| x - y \| + \| y - Tx \| \\ &< (1/2)\| x - Tx \| + (1/2)\| Tx - T^2x \| \\ &< (1/2)\| x - Tx \| + (1/2)\| Tx - Tx \| \\ &= \| x - Tx \| \end{aligned}$$

this is a contradiction. So, we obtain the desired result. □

Lemma 2.7. *Let T be a self-mapping on a nonempty subset C of a Banach space. If T is generalized* α*-nonexpansive mapping, then for all* $x, y \in C$, *we have*

$$\| x - Ty \| \leq \left(\frac{3+\alpha}{1-\alpha}\right) \| Tx - x \| + \| x - y \|.$$

Proof. By Lemma 2.6 (c), either

$$\| Tx - Ty \| \leq \alpha \| Tx - y \| + \alpha \| x - Ty \| + (1 - 2\alpha) \| x - y \|,$$

or

$$\| T^2x - Ty \| \leq \alpha \| Tx - Ty \| + \alpha \| T^2x - y \| + (1 - 2\alpha) \| Tx - y \|.$$

holds. In the first case, we get

$$\begin{aligned} \| x - Ty \| &\leq \| x - Tx \| + \| Tx - Ty \| \\ &\leq \| x - Tx \| + \alpha \| Tx - y \| + \alpha \| x - Ty \| + (1 - 2\alpha) \| x - y \| \\ &\leq \| x - Tx \| + \alpha (\| Tx - x \| + \| x - y \|) + \alpha \| x - Ty \| + (1 - 2\alpha) \| x - y \| \end{aligned}$$

It follows that

$$\| x - Ty \| \leq \left(\frac{1+\alpha}{1-\alpha}\right) \| x - Tx \| + \| x - y \|.$$

In the second case, by Lemma 2.6(a), we have

$$\begin{aligned} \| x - Ty \| &\leq \| x - Tx \| + \| Tx - T^2x \| + \| T^2x - Ty \| \\ &\leq 2 \| x - Tx \| + \alpha \| Tx - Ty \| + \alpha \| T^2x - y \| + (1 - 2\alpha) \| Tx - y \| \\ &\leq 2 \| x - Tx \| + \alpha (\| Tx - x \| + \| x - Ty \|) + \alpha (\| T^2x - Tx \| + \| Tx - y \|) \\ &\qquad + (1 - 2\alpha) \| Tx - y \| \\ &\leq (2 + \alpha) \| x - Tx \| + \alpha \| x - Ty \| + \alpha \| x - Tx \| + (1 - \alpha) \| Tx - y \| \\ &\leq (2 + \alpha) \| x - Tx \| + \alpha \| x - Ty \| + \alpha \| x - Tx \| + (1 - \alpha) \| Tx - x \| \\ &\qquad + (1 - \alpha) \| x - y \|. \end{aligned}$$

This implies

$$(1 + \alpha) \| x - Ty \| \leq (3 + \alpha) \| x - Tx \| + (1 - \alpha) \| x - y \|.$$

Since $(1 - \alpha) > 0$, we get

$$\| x - Ty \| \leq \left(\frac{3+\alpha}{1-\alpha}\right) \| Tx - x \| + \| x - y \|.$$

This completes the proof. □

Lemma 2.8. *Let T be a self-mapping on a nonempty subset C of a Banach space X satisfying Opial's condition. If T is generalized α-nonexpansive mapping, then the following holds*:

$$\{x_n\} \subseteq C,\ x_n z,\ \| x_n \rightharpoonup Tx_n \| \to 0 \Rightarrow Tz = z.$$

Proof. By Lemma 2.7, we have

$$\| x_n - Tz \| \leq \left(\frac{3+\alpha}{1-\alpha}\right) \| Tx_n - x_n \| + \| x_n - z \|.$$

It follows that

$$\liminf_{n \to \infty} \| x_n - Tz \| \leq \liminf_{n \to \infty} \| x_n - z \|.$$

From Opial's condition, we must have $Tz = z$. □

3. Main Results

In this section, we prove some strong and weak convergence theorems for generalized α-nonexpansive mappings in uniformly convex Banach space.

Lemma 3.1. *Let T be a self-mapping on a nonempty closed convex subset C of a uniformly convex Banach space X. If T is a generalized α-nonexpansive mappings with $F(T) \neq \emptyset$ and $\{x_n\}$ is the Picard-S hybrid iteration process defined by* (1.7), *then* $\lim_{n \to \infty} \| x_n - p \|$ *exists for all* $p \in F(T)$.

Proof. Let $p \in F(T)$. By Lemma 2.5, we have

$$\begin{aligned} \| z_n - p \| &= \| (1-\beta_n)x_n + \beta_n Tx_n - p \| \\ &\leq (1-\beta_n) \| x_n - p \| + \beta_n \| Tx_n - p \| \\ &\leq (1-\beta_n) \| x_n - p \| + \beta_n \| x_n - p \| \\ &= \| x_n - p \|, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \| y_n - p \| &= \| (1-\alpha_n)Tx_n + \alpha_n Tz_n - p \| \\ &\le (1-\alpha_n)\| Tx_n - p \| + \alpha_n \| Tz_n - p \| \\ &\le (1-\alpha_n)\| x_n - p \| + \alpha_n \| z_n - p \| \\ &\le (1-\alpha_n)\| x_n - p \| + \alpha_n \| x_n - p \| \\ &= \| x_n - p \|, \end{aligned}$$

which implies that

$$\begin{aligned} \| x_{n+1} - p \| &= \| Ty_n - p \| \\ &\le \| y_n - p \| \\ &\le \| x_n - p \|. \end{aligned}$$

Hence, the sequence $\{\| x_n - p \|\}$ is non-increasing and bounded, which implies that $\lim_{n\to\infty}\| x_n - p \|$ exists for all $p \in F(T)$. □

In the following theorem, we give the condition for the existence of a fixed point of generalized α-nonexpansive mappings on a closed convex subset of X.

Theorem 3.2. *Let T be a self-mapping on a nonempty closed convex subset C of a uniformly convex Banach space X. If T is a generalized α-nonexpansive mappings and $\{x_n\}$ is the Picard-S hybrid iteration process defined by* (1.7), *then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and* $\lim_{n\to\infty}\| x_n - Tx_n \| = 0$.

Proof. Suppose that $F(T) \neq \emptyset$. From Lemma 3.1, $\lim_{n\to\infty}\| x_n - p \|$ exists for all $p \in F(T)$ and $\{\| x_n - p \|\}$ is bounded. We suppose $\lim_{n\to\infty}\| x_n - p \| = r$ for some $r \ge 0$.

From (3.1), we have

$$\liminf_{n\to\infty}\| z_n - p \| \le \liminf_{n\to\infty}\| x_n - p \| = r. \tag{3.2}$$

By Lemma 2.5, we have

$$\liminf_{n\to\infty}\| Tx_n - Tzp \| \le \liminf_{n\to\infty}\| x_n - p \| = r, \tag{3.3}$$

On the other hand

$$\begin{aligned}\| x_{n+1} - p \| &= \| Ty_n - p \| \\ &\leq \| y_n - p \| \\ &\leq (1-\alpha_n)\| Tx_n - p \| + \alpha_n \| Tz_n - p \| \\ &\leq (1-\alpha_n)\| x_n - p \| + \alpha_n \| z_n - p \|\end{aligned}$$

it follows that

$$\| x_n - p \| \leq \frac{\| x_n - p \| - \| x_{n+1} - p \|}{\alpha_n} + \| z_n - p \|$$

Taking the $\liminf$ on both sides, we obtain

$$r \leq \liminf_{n\to\infty} \| z_n - p \| \tag{3.4}$$

Combining (3.2) and (3.4), we get

$$r \leq \lim_{n\to\infty} \| z_n - p \| = \lim_{n\to\infty} \| (1-\beta_n)(x_n - p) + \beta_n (Tx_n - p) \|$$

Since $0 < \beta_n < 1$ for all $n \geq 1$, by Lemma 2.4, we have

$$\lim_{n\to\infty} \| x_n - Tx_n \| = 0.$$

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$. Let $p \in A(C, \{x_n\})$. By Lemma 2.7, we have

$$\begin{aligned}r(Tp, \{x_n\}) &= \limsup_{n\to\infty} \| x_n - Tp \| \\ &\leq \left(\frac{3+\alpha}{1-\alpha}\right) \limsup_{n\to\infty} \| x_n - Tx_n \| + \limsup_{n\to\infty} \| x_n - p \| \\ &= \limsup_{n\to\infty} \| x_n - p \| \\ &= r(p, \{x_n\}).\end{aligned}$$

It follows that $Tp \in A(C, \{x_n\})$. Since X is uniformly convex, set $A(C, \{x_n\})$ is a singleton. Hence, we have $Tp = p$ i.e., $F(T) \neq \emptyset$. □

Theorem 3.3. *Let T be a self-mapping on a nonempty compact convex subset C of a uniformly convex Banach space X. Let T be a generalized α-nonexpansive mappings with $F(T) \neq \emptyset$, then the Picard-S hybrid iteration process defined by* (1.7) *converges strongly to a fixed point of T.*

Proof. By Theorem 3.2, $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$. Since C is compact, we can find a strongly convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q$ for some $q \in C$. From Lemma 2.7, we have

$$\| x_{n_k} - Tq \| \leq \left(\frac{3+\alpha}{1-\alpha}\right) \| x_{n_k} - Tx_{n_k} \| + \| x_{n_k} - q \|$$

Taking limit $k \to \infty$, we get $Tq = q$. By using Lemma 3.1, $\lim_{n\to\infty} \| x_n - q \|$ exists for all $q \in F(T)$. Thus, $\{x_n\}$ converges strongly to a fixed point of T. □

Theorem 3.4. *Let T be a self-mapping on a nonempty closed convex subset C of a uniformly convex Banach space X. Let T be a generalized α-nonexpansive mappings with $F(T) \neq \emptyset$, then the Picard-S hybrid iteration process defined by* (1.7) *converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x, F(T)) = \inf \{d(x, p) : p \in F(T)\}$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. From Lemma 3.1, we have $\lim_{n\to\infty} \| x_n - p \|$ exists for all $p \in F(T)$, so $\liminf_{n\to\infty} d(x_n, F(T))$ exists for all $p \in F(T)$. By hypothesis

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Now we show that $\{x_n\}$ is a Cauchy sequence in C. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ for any $\varepsilon > 0$ there exists a positive integer $n_0 = n_0(\varepsilon)$ such that

$$d(x_n, F(T)) < \frac{\varepsilon}{2}, \forall\, n \geq n_0.$$

Therefore, there exists $q \in F(T)$ such that

$$\| x_{n_0} - q \| < \frac{\varepsilon}{2}.$$

Thus, for all $m, n \geq n_0$, we get

$$\begin{aligned} \| x_m - x_n \| &\leq \| x_m - q \| + \| x_n - q \| \\ &\leq \| x_{n_0} - q \| + \| x_{n_0} - q \| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence. Since C is a closed subset of Banach space X, the sequence $\{x_n\}$ converges strongly to some $p \in C$. Also $F(T)$ is a closed subset of C and $\lim_{n\to\infty} d(x_n, F(T)) = 0$ we have $p \in F(T)$. Thus, the sequence $\{x_n\}$ converges strongly to a fixed point of T. This completes the proof. □

Senter and Dotson [16] introduced the condition $(\mathcal{I})$ as follows:

Definition 3.5[16]**.** A self-mapping T on a subset C of a Banach space X is said to satisfy condition $(\mathcal{I})$, if there exists a non-decreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$, for all $t \in (0, \infty)$ such that

$$\| x - Tx \| \geq \varphi(d(x, F(T)))$$

for all $x \in C$.

Theorem 3.6. *Let T be a self-mapping on a nonempty closed convex subset C of a uniformly convex Banach space X. Let T be a generalized α-nonexpansive mappings with $F(T) \neq \emptyset$. If T satisfies condition* $(\mathcal{I})$, *then the Picard-S hybrid iteration process defined by* (1.7) *converges strongly to a fixed point of T.*

Proof. From Theorem 3.2, it follows that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T satisfies condition $(\mathcal{I})$, we have

$$0 \leq \lim_{n\to\infty} \varphi(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

i.e.,

$$\lim_{n\to\infty} \varphi(d(x_n, F(T))) = 0.$$

Since the function $\varphi : [0, \infty) \to [0, \infty)$ is a non-decreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0,$ for all $t > 0,$ we get

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Consequently, $\{x_n\}$ converges strongly to a fixed point of T. □

Theorem 3.7. *Let T be a self-mapping on a nonempty closed convex subset C of a uniformly convex Banach space X satisfying the Opial condition. If T is generalized α-nonexpansive mappings with $F(T) \neq \emptyset$, then the Picard-S hybrid iteration process defined by* (1.7) *converges weakly to a fixed point of T.*

Proof. From Theorem 3.2, $\{x_n\}$ is bounded and $\lim_{n\to\infty} \| x_n - Tx_n \| = 0.$ Since every uniformly convex Banach space X is reflexive, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q$ for some $q \in C$. It follows by Lemma 2.8 that $q \in F(T)$. We suppose that q is not weak limit of $\{x_n\}$. Then, there exists another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \rightharpoonup q'$ and $q \neq q'$. Obviously, $q \in F(T)$. Now, using the Opial's condition, we have

$$\lim_{n\to\infty} \| x_n - q \| = \lim_{k\to\infty} \| x_{n_k} - q \| < \lim_{k\to\infty} \| x_{n_k} - q' \| = \lim_{n\to\infty} \| x_n - q' \|$$

but

$$\lim_{n\to\infty} \| x_n - q' \| = \lim_{l\to\infty} \| x_{n_l} - q' \| < \lim_{l\to\infty} \| x_{n_l} - q \| = \lim_{n\to\infty} \| x_n - q \|$$

which is a contradiction. Hence, $\{x_n\}$ converges weakly to q. □

4. Conclusion

We have proved some fixed point convergence results for generalized α-nonexpansive mappings via Picard-S hybrid iteration in the setting of uniformly convex Banach space. In future research, the readers can prove some fixed point convergence results for generalized α-nonexpansive mappings in other settings. Moreover, the readers can suggest new iterative methods and consider convergence analysis of these methods under certain suitable conditions.

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