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A STUDY ON SINGLE SERVER FEEDBACK PRIORITY QUEUE WHEN CATASTROPHES OCCUR

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Abstract

In this paper, a study has been made in a single server feedback queue with priority for feedback customers under steady state conditions when catastrophes occur. Priority in service is given to those customers who opt for a feedback and when catastrophes occur all the available customers are destroyed immediately. The server will be ready for service at the time of a new arrival. Under the above circumstances the asymptotic behaviour of the model and the average queue length are obtained. The stationary probability of 'n' and zero customers in the system are also obtained. Finally numerical examples are provided to check the correctness of the model.

1. Introduction

The queueing theory (or) waiting line theory owes its development to A. K. Erlang. He took up the problem on congestion at telephone traffic and published his work. A queue is a waiting line which demands service from a server. The queue does not include a customer being serviced. The Erlang work [3] on queueing stimulated many authors to develop a variety of queueing models. Takacs [17] has proposed to formulate queues with feedback mechanism which includes the possibility for a customer to return to the system for additional service. The customer may or may not opt for a feedback. A customer arriving for a feedback may be given priority in the

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waiting line and may be allowed to join the waiting line in front of the queue. Krishnamoorthy and Manjunath [8] have studied on queues with priority determined by feedback. In certain queueing models the server may have to do some preparatory work before starting a service. This sort of preparatory work for customers may occur in bank, hospitals, production process etc. Santhakumaran and Shanmugasundarm [15] have studied preparatory work on arriving customers with a single server feedback queue.

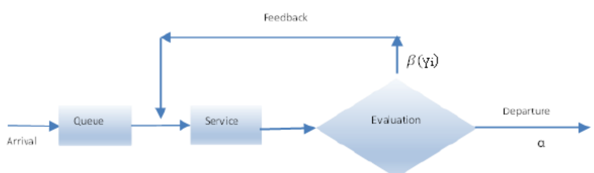
Time dependent analysis helps to understand the behaviour of a system when the parameters involved are perturbed and it can contribute to the costs and benefits of operating a system. Parthasarathy [12] made a study on transient state solution to a $M/M/1$ queue. Santhakumaran and Thangaraj [13] have proposed a single server queue with impatient and feedback customers. Santhakumaran, Ramasamy and Shanmugasundaram [14] have also studied a single queue with instantaneous Bernoulli feedback and setup time. Jaiswal [6] made a study on time dependent solution of the head of the line priority queue. Jewkes and Buzacott [7] have analysed flow time distributions in a k class $M/G/1$ priority feedback queue. Gautam Choudhury and Madhu Chandapaul [4] have proposed a two phase queueing system with Bernoulli feedback.

A catastrophe occurs in a queueing system if there happens a sudden calamity in the queue or service facility. When catastrophes occur, all the available customers are destroyed immediately and the server gets inactivated in the system. Analysis of catastrophes plays an important role in various areas of science and technology. Krishnakumar and Pavai Madheswari [10] have studied on transient analysis of an $M/M/1$ queue subject to catastrophes and server failures. Shanmugasundaram and Chitra [16] have made a study on time dependent solution of a single server feedback queue customer has a service with and without preparatory work when catastrophes occur. Chao [2] has proposed a queueing network model with catastrophes and product form solution. Jain and Kumar [5] have studied on $M/G/1$ queue with catastrophes. Krishnakumar and Arivudainambi [9] focussed on transient state solution to a $M/M/1$ queue with catastrophes. Thangaraj and Vanitha [18] have made a study on continued fraction approach to a $M/M/1$ queue with feedback.

Chandrasekaran and Saravanarajan [1] focussed their study on transient and reliability analysis of $M/M/1$ feedback queue subject to catastrophes, server failures and repairs. Krishnakumar, Krishnamoorthy, Pavai Madheswari and Sadiq Basha [11] studied a transient analysis of a single server queue with catastrophes failures and repairs.

2. Model Description

The figure given below illustrates the flow of customers through the queueing system with first come first served discipline. Customers arrive at the service station according to a poisson process with rate λ. The capacity of the queue is infinite. If the server is idle upon an arrival, the service begins instantaneously following general discipline with service rate μ.



Flow of customers

When the system is not empty catastrophes occur at the service facility according to a poisson process with rate δ. Whenever catastrophes occur all the customers in the system are flushed out or ejected instantly and the server is inactivated. The server will be ready for service at the time of a new arrival.

The customers after getting the service may opt for a feedback or to depart depending on the satisfaction of the customer. A customer opting for feedback joins the feedback stream with probability β and with priority at the rate of γ limited to three times and joins in front of the queue. If a customer does not feedback, he joins the departure process with probability α so that $\alpha + \beta = 1$. The motivation for this model comes from bank, hospital, production systems, restaurant etc.

Let $\{X(t) = n, n = 0, 1, 2, \ldots\}$ be the number of customers in the system at time t and let $P_n(t) = P[x(t) = n], n = 0, 1, 2, \ldots$ denote the probabilities that there are 'n' customers in the system at time t and let $P(x, t) = \sum_0^\infty P_n(t)x^n$ be its probability generating function. Without loss of generality assume that initially there are no customers in the system at time $t = 0$. i.e., $P_0(0) = 1$.

The system of differential difference equations for the probability P_n is

$$-\lambda P_0 + (\alpha + \gamma_i \beta)\mu P_1 + \delta(1 - P_0) = 0 \quad (1)$$

and for $n = 1, 2, 3, \ldots$

$$-\lambda P_{n-1} - [(\lambda + (\alpha + \gamma_i\beta)\mu + \delta)P_n + (\alpha + \gamma_i\beta)\mu P_{n+1} = 0 \quad (2)$$

Theorem 2.1. *The Stationary Probability distribution* $\{\pi_n, n \geq 0\}$ *for a single server queue when catastrophes occur is* $\pi_0 = 1 - \rho, \pi_n(1 - \rho)\rho^n$, $n = 1, 2, \ldots$ *Where*

$$\rho = \left[\frac{(\lambda + (\alpha + \gamma_i\beta)\mu + \delta) - \sqrt{\lambda^2 + [(\alpha + \gamma_i\beta)\mu]^2 + \delta^2 + 2\lambda\delta + 2(\alpha + \gamma_i\beta)\mu\delta - 2\lambda(\alpha + \gamma_i\beta)\mu}}{2(\alpha + \gamma_i\beta)\mu}\right]$$

Proof. The Laplace transform of the steady state probability for no customers in the system is

$$P_0'(x) = \frac{1 + \frac{\delta}{x}}{(x + \lambda + \delta) - \left(\frac{w - \sqrt{w^2 - 4\lambda(\alpha + \gamma_i\beta)\mu}}{2}\right)}$$

Where $w = (x + \lambda + (\alpha + \gamma_i\beta)\mu + \delta)$

$$\pi_0 = \lim_{x \to 0} P_0'(x) = \lim_{x \to 0} \frac{x + \delta}{(x + \lambda + \delta) - \left(\frac{w - \sqrt{w^2 - 4\lambda(\alpha + \gamma_i\beta)\mu}}{2}\right)}$$

$$\frac{2\delta}{[(\lambda + (\alpha + \gamma_i\beta)\mu + \delta)] - \sqrt{[(\lambda + (\alpha + \gamma_i\beta)\mu + \delta)]^2 - 4\lambda(\alpha + \gamma_i\beta)\mu}}$$

$\pi_0 = 1 -$

$$\left[\frac{(\lambda+(\alpha+\gamma_i\beta)\mu+\delta)-\sqrt{\lambda^2+[(\alpha+\gamma_i\beta)\mu]^2+\delta^2+2\lambda\delta+2(\alpha+\gamma_i\beta)\mu\delta-2\lambda(\alpha+\gamma_i\beta)\mu}}{2(\alpha+\gamma_i\beta)\mu}\right]$$

Also taking Laplace transform of the steady state probability for n customers in the system, we obtain

$$\lim_{x\to\infty} xP_n'(x) = \left(\frac{w-\sqrt{w^2-4\lambda(\alpha+\gamma_i\beta)\mu}}{2(\alpha+\gamma_i\beta)\mu}\right)^n \lim_{x\to\infty} P_0'(x)$$

$$\pi_n = \lim_{x\to\infty}\left(\frac{w-\sqrt{w^2-4\lambda(\alpha+\gamma_i\beta)\mu}}{2(\alpha+\gamma_i\beta)\mu}\right)^n \pi_0$$

$$\pi_n = \pi_0\left[\frac{(\lambda+(\alpha+\gamma_i\beta)\mu+\delta)-\sqrt{\lambda^2+[(\alpha+\gamma_i\beta)\mu]^2+\delta^2+2\lambda\delta+2\mu(\alpha+\gamma_i\beta)\delta-2\lambda(\alpha+\gamma_i\beta)\mu}}{2(\alpha+\gamma_i\beta)\mu}\right]$$

$\pi_0 = (1-\rho)\rho^2$, $n = 1, 2, 3\ldots$ and the stationary probability distribution exists if and only if $\rho < 1$.

Theorem 2.2. *The asymptotic behaviour of average queue length* $H(t)$ *when* $\delta > 0$ *is*

$$H(t) = \left(\frac{\lambda-(\alpha+\gamma_i\beta)\mu}{\delta}\right)$$

$$+\frac{2(\alpha+\gamma_i\beta)\mu}{2(\lambda+\delta)-[(\lambda+(\alpha+\gamma_i\beta)\mu+\delta)-\sqrt{(\lambda+(\alpha+\gamma_i\beta)\mu+\delta)^2-4\lambda(\alpha+\gamma_i\beta)\mu}]}$$

as $t\to\infty$.

Proof. Consider the probability generating function $P(x, t) = \sum_{n=0}^{\infty} P_n(t)x^n$ together with initial conditions and using the equations (1) and (2), the probability generating function $P(x, t)$ becomes

$$\frac{\partial P(x,\, t)}{\partial t} = \left[\lambda + \frac{(\alpha + \gamma_i\beta)\mu}{x} - (\lambda + (\alpha + \gamma_i\beta)\mu + \delta)\right] P(x,\, t)$$

$$+ (\alpha + \gamma_i\beta)\mu\left(1 - \frac{1}{x}\right)P_0 + \delta$$

The average queue length is

$$H(t) = \sum_{n=1}^{\infty} nP_n(t) = \frac{\partial P(x,\, t)}{\partial t} \text{ at } x = 1$$

$$\frac{dH(t)}{dt} + \delta H(t) = \lambda - (\alpha + \gamma_i\beta)\mu(1 - P_0)$$

This differential equation is linear in $H(t)$ and solving for $H(t)$ we get

$$H(t)e^{\int \delta dt} = \int_0^t [\lambda - (\alpha + \gamma_i\beta)\mu(1 - P_0)]e^{\int \delta dt} dt + c$$

$$H(t) = \frac{\lambda}{\delta}(1 - e^{\delta t}) - \frac{(\alpha + \gamma_i\beta)\mu}{\delta}(1 - e^{\delta t}) + (\alpha + \gamma_i\beta)\mu\int_0^{\tau} P_0(u)e^{-\delta(t-u)}du$$

Taking Laplace transform for the above expression, we get

$$H^*(x) = \frac{\lambda}{x(x + \delta)} - \frac{(\alpha + \gamma_i\beta)\mu}{x(x + \delta)} + \frac{(\alpha + \gamma_i\beta)\mu}{(x + \delta)} P_0^*(x)$$

$$\lim_{t\to\infty} H(t) = \lim_{x\to 0} xH^*(x) = \lim_{x\to 0} \frac{\lambda - (\alpha + \gamma_i\beta)\mu(t)}{x + \delta}$$

$$+ \frac{(\alpha + \gamma_i\beta)\mu(t)}{(x + \lambda + \delta) - [(x + \lambda + (\alpha + \gamma_i\beta)\mu(t) + \delta) \\ \frac{- \sqrt{(x + \lambda + (\alpha + \gamma_i\beta)\mu(t) + \delta)^2 - 4\lambda(\alpha + \gamma_i\beta)\mu(t)}}{2}]}$$

$$= \frac{\lambda - (\alpha + \gamma_i\beta)\mu}{\delta}$$

$$+ \frac{2(\alpha + \gamma_i\beta)\mu}{2(\lambda + \delta) - [(\lambda + (\alpha + \gamma_i\beta)\mu + \delta) - \sqrt{(\lambda + (\alpha + \gamma_i\beta)\mu(t) + \delta)^2 - 4\lambda(\alpha + \gamma_i\beta)\mu(t)}]}$$

As $t \to \infty$

$$H(t) = \left(\frac{\lambda - (\alpha + \gamma_i\beta)\mu}{\delta}\right)$$

$$+ \frac{2(\alpha + \gamma_i\beta)\mu}{2(\lambda + \delta) - [(\lambda + (\alpha + \gamma_i\beta)\mu + \delta) - \sqrt{(\lambda + (\alpha + \gamma_i\beta)\mu + \delta)^2 - 4\lambda(\alpha + \gamma_i\beta)\mu}]}$$

If the service is carried out with and without preparatory work and If the customers without preparatory work are only allowed to feedback with probability β and not given any priority for the feedback customers but the customers with preparatory work are not allowed to feedback and depart from the system with probability α such that $\alpha + \beta = 1$ with service rates μ_1 and μ_2 respectively then the asymptotic behaviour of the average queue length coincides with the result of [16].

3. Numerical Study

In this section a numerical study has been made based on the average queue length of the model. For this purpose an illustration is done by taking $\alpha = 0.4$ and $\beta = 0.6$ with $\gamma_i = 0.3$ by varying the values of λ and δ by keeping μ fixed.

Table 3.1. The average length of the system $H(t)$ is computed for catastrophic effect of $\delta = 0.3, 0.5, 0.7$ with $\mu = 15$.

λ	$\delta(0.3)$	$\delta(0.5)$	$\delta(0.7)$
1	0.124419492	0.121057625	0.117889459
2	0.282298886	0.272616677	0.263692743
3	0.488088482	0.466333325	0.446905163
4	0.764896043	0.719459435	0.68068005
5	1.150692933	1.057375771	0.982722594
6	1.708203932	1.515930449	1.375293416
7	2.535605621	2.140506883	1.882765232
8	3.759606865	2.976122604	2.525417908

9	5.477225575	4.047352116	3.311336661
10	7.675918792	5.343126677	4.232436273

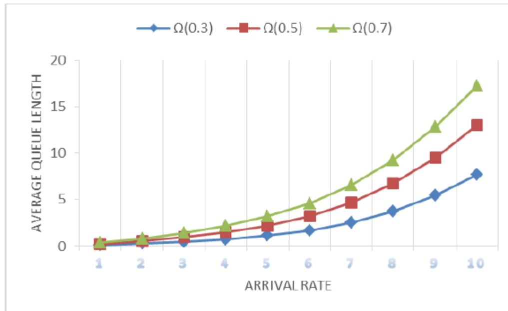


Figure 1.

Table 3.2. The stationary probability distribution for fixed value of $\mu = 10$ and $\delta = 0.3, 0.5, 0.7$ for various values of λ are computed as follows.

λ	$\delta(0.3)$	$\delta(0.5)$	$\delta(0.7)$
1	0.889347799	0.892014806	0.89454283
2	0.779849387	0.785782568	0.7913316
3	0.672003051	0.68197318	0.6911303
4	0.566605611	0.581578128	0.594997245
5	0.464966423	0.486056079	0.50435699
6	0.369248411	0.397467267	0.42100062
7	0.282836975	0.318419936	0.346889157
8	0.210101386	0.251501299	0.283654314
9	0.154387089	0.198123685	0.231946628
10	0.115261568	0.157650958	0.191115562

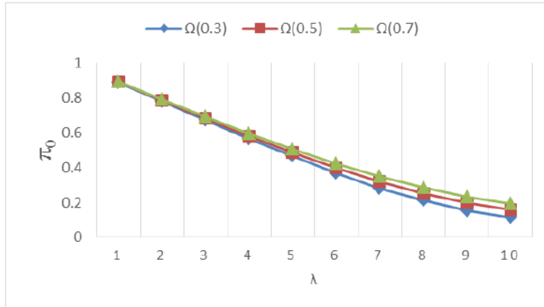


Figure 2.

Conclusion

We have derived the stationary probability of 'n' number of customers in the system and no customer in the system. The asymptotic behaviour of average queue length has been studied. In the numerical study, the graph shows when the arrival rate increases the average queue length increases (as in Figure 1). As the arrival rate increases the stationary probability of no customers in the system decreases (as in Figure 2). It shows the correctness of the result.

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