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ON SOLVING THE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS OF THE NONLINEAR COVID-19 MODEL

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Abstract

This paper re-investigates a nonlinear COVID-19 model. The given two ordinary differential equations (ODEs), governing this model, are successfully combined to a single nonlinear differential equation. A direct series solution is established for the reduced model and hence, approximate analytical expressions are determined for the infected and recovered individuals. It is declared that the exact solution of the current model is also available at a specific restriction of the given initial conditions. The accuracy of our results are examined through several comparisons with another accurate numerical method. In addition, it is shown in this paper that our approach enjoys better accuracy in contrast to the homotopy perturbation method (HPM) in the literature. Moreover, the numerical results using the present Pade approximations revealed a complete coincidence with the Runge-Kutta numerical method if compared with the HPM in the literature.

1. Introduction

The Corona pandemic still occupies the attention of many researchers worldwide. Many mathematical models ([1]-[16]) have been published to describe this pandemic and try to understand its current and future behavior. In the current research, we want to shed some light on one of these models and present the accurate/exact solution, which may come as an alternative to the solution proposed in previous studies. The present nonlinear COVID-19 model was formulated in Ref. [5] and expressed by the following system of

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ODEs:

$$\frac{dR}{d\tau} = I(\tau), \tag{1}$$

$$\frac{dR}{d\tau} = \sigma[1 - R(\tau) - I(\tau)]I(\tau) - I(\tau), \tag{2}$$

where $\tau = t/T$, t is the time in days and T is the time of transmission of the virus. The symbols $I(t)$ and $R(t)$ stand for the infected individuals and the recovered individuals, respectively. Beside, $S(t)$ denotes the susceptible individuals: $S(t) = 1 - R(t) - I(t)$, where σ is the transmission rate (physical contact number between susceptible and infected individuals). The model is governed by the initial conditions (ICs) [6]:

$$R(0) = A,\ I(0) = B. \tag{3}$$

In the literature, a number of analytical approaches were discussed to solve linear and nonlinear ODEs. Samples of such approaches are known as the Adomian decomposition method (ADM) ([17]-[32]) and the HPM ([33]-[39]). The main notice on the ADM and the HPM is the requirement of finding/calculating the Adomian's polynomials to deal with the nonlinear terms contained in the governing system. Thus, a considerable effort is needed to calculate such polynomials. An alternate procedure is suggested in this paper in order to avoid these difficulties.

So, a simple analytical approach is proposed to directly solve the nonlinear system (1)-(3). The suggested approach is based on reducing the ODEs (1-2) to a single nonlinear ODE in only $R(\tau)$.

Then, the series solution of the reduced nonlinear ODE in $R(\tau)$ shall be determined by means the direct Maclaurin expansion (ME). The validity of the ME-approximations will be examined by performing several comparisons with the numerical solution using the Runge-Kutta method.

In addition, it will be shown that the present approach has many advantages over the HPM used in Ref. [6] to solve the nonlinear COVID-19 model (1)-(3). To improving the accuracy of the present results, several Pade-approximations are to be constructed. Moreover, it will be declared that the diagonal Pade-approximations are coincide with the Runge-Kutta method in the whole domain. Let us begin our analysis by reducing the model (1)-(3) as indicated in the next section.

2. The Equivalent Model

Differentiating Equation (1) once with respect to τ and then substituting into Equation (2), we obtain the 2nd-order nonlinear ODE:

$$\frac{d^2R}{d\tau^2} = \sigma\left[1 - R(\tau) - \frac{dR}{d\tau}\right]\frac{dR}{d\tau} - \frac{dR}{d\tau}, \tag{4}$$

which is subjected to the ICs:

$$R(0) = A,\ \frac{dR(0)}{d\tau} = B. \tag{5}$$

Equation (4) can be rewritten as

$$\frac{\frac{d}{d\tau}\left(R(\tau) + \frac{dR}{d\tau}\right)}{1 - \left(R(\tau) + \frac{dR}{d\tau}\right)} = \sigma\frac{dR}{d\tau}, \tag{6}$$

which can be easily integrated with respect to τ to give

$$R(\tau) + \frac{dR}{d\tau} = 1 + ce^{-\sigma R(\tau)}, \tag{7}$$

where c is a constant of integration. Applying the ICs (5) on Equation (7) yields

$$c = (B + A - 1)e^{\sigma A}. \tag{8}$$

Hence,

$$\frac{dR}{d\tau} = 1 - R(\tau) + ce^{-\sigma R(\tau)}. \tag{9}$$

This is a 1st-order nonlinear ODE in the single unknown $R(\tau)$. In the next section, a series solution of the nonlinear ODE (9) will be obtained via analytical approximations.

3. Direct Series Solution via Maclaurin Expansion

In this section, a direct series solution is to be obtained for Equation (9) by means of the Maclaurin expansion (ME):

$$R(\tau) = \sum_{n=0}^{\infty} R^{(n)}(0) \frac{\tau^n}{n!}, \tag{10}$$

where $R^{(n)}(0) = \frac{d^n R(0)}{d\tau^n}$. Firstly, we may rewrite Equation (9) in the form:

$$R^{(1)}(\tau) = 1 - R(\tau) + ce^{-\sigma R(\tau)}, \tag{11}$$

where $R^{(1)}(\tau) = \frac{dR}{d\tau}$. By this, we note that the first two terms of the expansion (10) are known, where $R(0) = A$ and $R^{(1)}(0) = B$. To find $R^{(2)}(0)$, we differentiate (11) once with respect to τ, so

$$R^{(2)}(\tau) = -R^{(1)}(\tau) - \sigma ce^{-\sigma R(\tau)} R^{(1)}(\tau), \tag{12}$$

and hence,

$$\begin{aligned} R^{(2)}(0) &= -R^{(1)}(0) - \sigma ce^{-\sigma R(0)} R^{(1)}(0), \\ &= -B - \sigma ce^{-\sigma A} B, \\ &= -B(1 + \sigma(B + A - 1)), \end{aligned}$$

where the value of the constant c is employed. Similarly, differentiating (12) once again with respect to τ we obtain

$$R^{(3)}(\tau) = -R^{(2)}(\tau) - \sigma ce^{-\sigma R(\tau)} (R^{(2)}(\tau) - \sigma(R^{(1)}(\tau))^2), \tag{14}$$

or

$$R^{(3)}(\tau) = -(1 + \sigma ce^{-\sigma R(\tau)}) R^{(2)}(\tau) + \sigma^2 ce^{-\sigma R(\tau)} (R^{(1)}(\tau))^2. \tag{15}$$

Thus

$$R^{(3)}(0) = -(1 + \sigma ce^{-\sigma A}) R^{(2)}(0) + \sigma^2 ce^{-\sigma A} B^2, \tag{16}$$

i.e.,

$$R^{(3)}(0) = (1 + \sigma(B + A - 1))^2 + \sigma^2 B^2 (B + A - 1). \tag{17}$$

Similarly, we can get $R^{(4)}(0)$ and $R^{(5)}(0)$ as

$$R^{(4)}(0) = -B(1 + \sigma(B + A - 1))^3 - 4\sigma^2 B^2 (1 + \sigma(B + A - 1))(B + A - 1)$$
$$- \sigma^3 B^3 (B + A - 1), \tag{18}$$

and

$$R^{(5)}(0) = B(1 + \sigma(B + A - 1))^4 + 11\sigma^2 B^2 (B + A - 1) + \sigma^3 B^3 (B + A - 1)$$
$$[7 + 11\sigma(B + A - 1)] + \sigma^4 B^4 (B + A - 1), \tag{19}$$

respectively. Inserting the above values into the series expansion (10), we obtain

$$R(\tau) = R(0) + R^{(1)}(0)\tau + \frac{R^{(2)}(0)}{2!}\tau^2 + \frac{R^{(3)}(0)}{3!}\tau^3 + \ldots, \tag{20}$$

or

$$R(\tau) = A + B\tau - B(1 + \sigma(B + A - 1))\frac{\tau^2}{2!} + [B(1 + \sigma(B + A - 1))^2$$
$$+ \sigma^2 B^2 (B + A - 1)]\frac{\tau^3}{3!} + [- B(1 + \sigma(B + A - 1))^3 - 4\sigma^2 B^2 (1 + \sigma(B + A - 1))$$
$$(B + A - 1) - \sigma^3 B^3 (B + A - 1)]\frac{\tau^4}{4!} + [B(1 + \sigma(B + A - 1))^4$$
$$+ 11\sigma^2 B^2 (B + A - 1) + \sigma^3 B^3 (B + A - 1)[7 + 11\sigma(B + A - 1)]$$
$$+ \sigma^4 B^4 (B + A - 1)]\frac{\tau^5}{5!} + \ldots, \tag{21}$$

and hence, $I(\tau)$ can be obtained as

$$I(\tau) = B - B(1 + \sigma(B + A - 1))\tau + [B(1 + \sigma(B + A - 1))^2 + \sigma^2 B^2 (B + A - 1)\frac{\tau^2}{2!}$$

$$[-B(1+\sigma(B+A-1))^3 - 4\sigma^2 B^2(1+\sigma(B+A-1))(B+A-1)$$

$$-\sigma^3 B^3(B+A-1)]\frac{\tau^3}{3!} + [B(1+\sigma(B+A-1))^4 + 11\sigma^2 B^2(B+A-1)$$

$$+\sigma^3 B^3(B+A-1)[7+11\sigma(B+A-1)] + \sigma^4 B^4(B+A-1)]\frac{\tau^4}{4!} + \ldots, \tag{22}$$

and the number of terms in the above series solutions for $R(\tau)$ and $I(\tau)$ can be increased, using any software, to reach the desired accuracy. This point will be explained in a subsequent section.

4. Exact Solution at Special Cases

4.1 Case I. Zero initial susceptible individuals

The relation $S(0) = 1 - R(0) - I(0)$, i.e., $S(0) = 1 - A - B$ gives the initial susceptible individuals.

Accordingly, the case $A + B = 1$ corresponds to the zero initial susceptible individuals $S(0) = 0$.

In such a case, the exact solution is available and can be determined as follows. Substituting $A + B = 1$ into the series (21), we obtain

$$R(\tau) = A + B\tau - B\frac{\tau^2}{2!} + B\frac{\tau^3}{3!} - B\frac{\tau^4}{4!} + B\frac{\tau^5}{5!} + \ldots, \tag{23}$$

which can be summed up to infinity to give the following exact solution for $R(\tau)$:

$$R(\tau) = A + B(1 - e^{-\tau}). \tag{24}$$

Similarly, the exact solution for $I(\tau)$ can be evaluated from the series (22) as

$$I(\tau) = B - B\tau + B\frac{\tau^2}{2!} - B\frac{\tau^3}{3!} + B\frac{\tau^4}{4!} - B\frac{\tau^5}{5!} + \ldots, \tag{25}$$

which gives

$$I(\tau) = Be^{-\tau}. \tag{26}$$

It is to be noted from the above solutions that $I(\tau)$ is actually the first derivative of $R(\tau)$ and this satisfies the first differential equation of the present COVID-19 model. Also, one can easily check the validity of the above solutions by direct substitution into the governing system and the given ICS.

4.2 Case II. Zero contact number. In this case, the value of σ vanishes and it will be shown that the exact solutions for $R(\tau)$ and $I(\tau)$ are identical to the previous special case $A + B = 1$. However, our analysis for proving this point comes directly from the nonlinear transformed equation (9) along with the constant c defined in Equation (8). At $\sigma = 0$, we have $c = A + B - 1$ and consequently Equation (9) reduces to

$$R'(\tau) = A + B - R(\tau). \tag{27}$$

This is a 1st-order linear ordinary differential equation which can be easily solved by the separation of variables method:

$$\int_0^\tau \frac{dR(x)}{A + B - R(x)}\, dx = \int_0^\tau d\tau, \tag{28}$$

and hence,

$$\ln\left(\frac{A + B - R(\tau)}{A + B - R(0)}\right) = -\tau. \tag{29}$$

On using the IC $R(0) = A$ and performing some simplifications, we obtain the same expression given in Equation (24) for $R(\tau)$ and thus $I(\tau)$ also has the same expression (26).

5. Results and Validations

This section aims to validate the present approximate series solution given by the ME in section

3. The validation is based on extracting some numerical results and performing comparisons with another analytical approach in the literature in addition to the numerical solution.

5.1 Validation of the present ME-approximations.

Assume that $\Theta_m(\tau)$ and $\Phi_m(\tau)$ represent the m-term approximate solutions for $R(\tau)$ and $I(\tau)$, respectively. Then, the approximations $\Theta_m(\tau)$ and $\Phi_m(\tau)$ can be expressed as

$$\Theta_m(\tau) = \sum_{n=0}^{m-1} R^{(n)}(0)\frac{\tau^n}{n!}, \tag{30}$$

and

$$\Phi_m(\tau) = \sum_{n=0}^{m-1} R^{(n+1)}(0)\frac{\tau^n}{n!}, \tag{31}$$

respectively. Figures 1 and 2 show the comparisons between for the present approximations $\Theta_m(\tau)$ and $\Phi_m(\tau)$ using ten terms $(m = 10)$ and the numerical solution obtained using MATHEMATICA.

It can be seen in these figures that the approximations $\Theta_{10}(\tau)$ and $\Phi_{10}(\tau)$ are coincide with the numerical solution in a specific domain. Such a domain of coincidence can be enlarged via increasing the number of terms taken from the ME-approximations. Another way to achieve this task is to apply the Pade-approximations as indicated in the next section.

5.2 Improved results via Pade-approximations

This section is devoted to prove the effectiveness and efficiency of Pade-approximations over the standard ME-approximations. Also, it will be revealed that the present Pade-approximations.

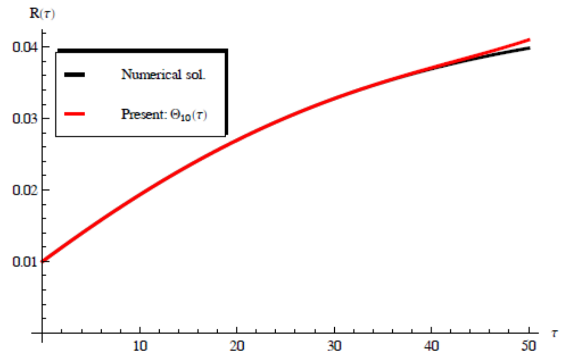

Figure 1. Comparison between the present $\Theta_m(\tau)$ and the numerical solution (Runge-Kutta) at initial recovered individuals $A = 0.01$ initial infected individuals $B = 0.001$ and transmission rate $\sigma = 1$ for the instantaneous recovered individuals $R(\tau)$.

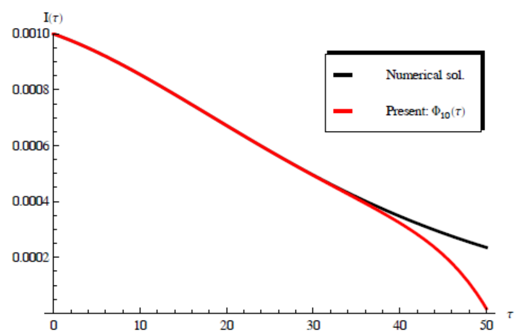

Figure 2. Comparison between the present $\Phi_{10}(\tau)$ and the numerical solution (Runge-Kutta) at initial recovered individuals $A = 0.01$, initial infected individuals $B = 0.001$, and transmission rate $\sigma = 1$ for the instantaneous infected individuals $I(\tau)$.

enjoy better accuracy than those approximations in Ref. [6] using the HPM and given by

$$R(t) = A - Be^{-\tau} + B + \sigma B\left[-B(-\tau e^{-\tau} - e^{-\tau}) - \frac{1}{2}Be^{-2\tau} + Be^{-\tau} - \tau e^{-\tau} - e^{-\tau}\right]$$

$$\sigma B\left(\frac{3}{2}B - 1\right), \tag{32}$$

$$I(t)=Be^{-\tau}+e^{-\tau}[-\sigma B(B\tau-Be^{-\tau}-\tau)-\sigma B^2]-\frac{1}{2}\sigma Be^{-\tau}(4\sigma B^2-2\sigma B+2B)$$

$$-\frac{1}{2}\sigma Be^{-\sigma}[4\sigma B^2\tau e^{-\sigma}+6\sigma B^2e^{-\tau}-2\sigma B^2e^{-2\tau}-\sigma B^2\tau^2-2\sigma B^2\tau^{-}$$

$$-2\sigma Be^{-\tau}+2\sigma B\tau+2\sigma B\tau^2-4\sigma B\tau e^{-\tau}+2B\tau-2Be^{-\tau}-\sigma\tau^2].$$

$$I(t)=Be^{-\tau}+e^{-\tau}[-\sigma B(B\tau-Be^{-\tau} \tag{33}$$

The diagonal Pade-approximations $[r/r](\tau)$ is an effective tool to enlarge the domain of applicability of a series solution. As examples, the diagonal Pade-approximations $[1/1](\tau)$ and $[2/2](\tau)$ are constructed as

$$[1/1](\tau)=\frac{P_1(\tau)}{Q_1(\tau)}, \tag{34}$$

$$[2/2](\tau)=\frac{P_2(\tau)}{Q_2(\tau)}, \tag{35}$$

where $P_1(\tau)$ and $Q_1(\tau)$ are polynomials of first degree in τ while $P_2(\tau)$ and $Q_2(\tau)$ are polynomials of second degree. These polynomials can be obtained as

$$P_1(\tau)=2R(0)R^{(1)}(0)+[2(R^{(1)}(0))^2-R(0)R^{(2)}(0)]\tau, \tag{36}$$

$$Q_1(\tau)=2R^{(1)}(0)-R^{(2)}(0)\tau, \tag{37}$$

$$P_2(\tau)=[36R(1)(R^{(2)}(0))^2-24(R^{(1)}(0))R^{(3)}(0)-12R(0)R^{(2)}(0)R^{(3)}(0)$$

$$+6R(0)R^{(1)}(0)R^{(4)}(0)]\tau+[18(R^{(2)}(0))^3-24R^{(1)}(0)R^{(2)}(0)R^{(3)}(0)$$

$$+4R(0)(R^{(3)}(0))^2+6(R^{(1)}(0))^2R^{(4)}(0)-3R(0)-3R(0)R^{(2)}(0)R^{(4)}(0)]\tau^2$$

$$36R(0)(R^{(2)}(0))^2 24R(0)R^{(1)}(0)R^{(3)}(0), \tag{38}$$

$$Q_2(\tau)=36(R^{(2)}(0))-24R^{(1)}(0)R^{(3)}(0)+[6R^{(1)}(0)R^{(4)}(0)-12R^{(2)}(0)R^{(3)}(0)]$$

$$\tau+[(R^{(3)}(0))^2-3R^{(2)}(0)R^{(4)}(0)]\tau^2. \tag{39}$$

Higher-order diagonal Pade-approximations $[r/r](\tau)(r > 2)$ are also available but ignored here for lengthy results. Figures 3 and 4 depict the diagonal Pade-approximations $[r/r](\tau)$ for $r = 2, 4, 6$ and compared with the numerical solution using the Runge-Kutta method. It is observed from

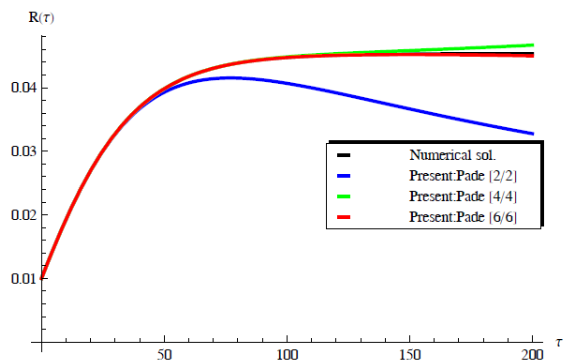

Figure 3. Comparison between the present diagonal Pade-approximations $[r/r](\tau)(r = 2, 4, 6)$ and the numerical solution (Runge-Kutta) at initial recovered individuals $A = 0.01$, initial infected individuals $B = 0.001$, and transmission rate $\sigma = 1$ for the instantaneous infected individuals $R(\tau)$.

these figures that the diagonal Pade-approximations $[6/6](\tau)$ agree with the numerical solution in the whole domain. This is one of the advantages of the present analysis.

In order to confirm the accuracy of the diagonal Pade-approximations over those in the literature, the numerical results are listed in table 1 for the purpose of comparison between the present $[6/6](\tau)$, the HPM-approximations [6], and the obtained numerical ones using MATHEMATICA.

It is obvious that our analysis is much accurate than the HPM [6].

6. Conclusions

In this paper, the approximate series solution was obtained for a nonlinear COVID-19 model based on the ME. It was shown that the obtained ME-series solution transforms to an exact solution at a specific condition for the sum of the initial values of the infected and recovered individuals. In addition, several comparisons were accomplished to stand on the accuracy of the current results. Regarding, it was proved that our analysis enjoys better accuracy in contrast to another analytical solution in the literature via the HPM [6]. Moreover, the tabulated values for the recovered individuals using the Pade-approximations revealed that our numerical results are much accurate than those of the HPM [6]. This conclusion was based on implementing the

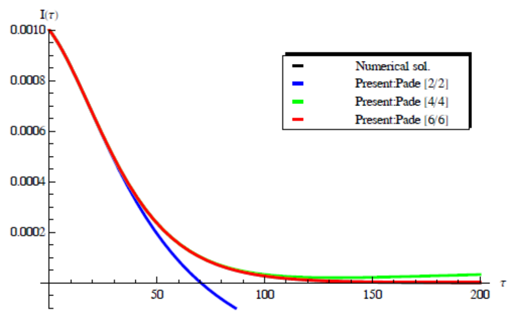

Figure 4. Comparison between the present diagonal Pade-approximations $[\tau/\tau](\tau)(r = 2, 4, 6)$ and the numerical solution (Runge-Kutta) at initial recovered individuals $A = 0.01$, initial infected individuals $B = 0.001$, and transmission rate $\sigma = 1$ for the instantaneous infected individuals $I(\tau)$.

Table 1. Comparisons between the approximate values of $R(\tau)$ using the HPM-approximations [6], the present Pade $[6/6](\tau)$, and the numerical solution using Runge-Kutta (MATHEMATICA) at $\sigma = 1, A = 0.01$ and $B = 0.001$.

τ	HPM [6]	$[6/6](\tau)$ (Present)	Runge-Kutta
5	0.011951	0.014846	0.014846
10	0.011998	0.019325	0.019325
15	0.011999	0.023376	0.023376
20	0.011999	0.026968	0.026968
25	0.011999	0.030097	0.030097
30	0.011999	0.032782	0.032782

35	0.011999	0.035056	0.035056
40	0.011999	0.036962	0.036962
45	0.011999	0.038543	0.038543
50	0.011999	0.039846	0.039846

Runge-Kutta method as a reference numerical method. Therefore, the current numerical results may give better predictions for the progress of the outbreak than those in the relevant literature.

Competing interests

The author declares that there is no competing interests.

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