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LOCAL FINITENESS AND ITS PROPERTIES IN FUZZY TOPOLOGICAL SPACES

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Abstract

The concept point-finite, locally finite and discrete collections of sets in general topology play a very important role, particularly in the study of paracompact spaces, weaker and stronger forms of paracompact spaces etc. α-local finite, α-point finite and α-discrete finiteness of fuzzy topological space are introduced and some important results have been proved.

1. Introduction and Preliminaries

The study of local finiteness, paracompactness and related concepts in fuzzy topological space (fts) was initiated by S. R. Malghan and S. S. Benchalli [7] in and among other results, it was shown that a locally finite family of fuzzy sets is closer preserving. Further, paracompactness in fts was also introduced and studied.

Local finiteness in fts was also studied by J. G. Jiang [3] in, S. L. Pu [7] in Mao-Kang Luo [6] in, A. Bulbul and M. W. Wareen [6] in also local finiteness in fts which is based on the idea of quasi coincident introduced in [8]. All these concepts have some limitations.

T. E. Gantner, R. C. Steinlage and R. H. Warren [9] introduced the concept of α-compacteness and S. R. Malghan and S. S. Benchalli [8] introduced α-perfect map and proved that composition of two α-perfect maps is α-perfect map.

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In this paper the concept of α-local finiteness of a family of fuzzy sets has been introduced as a natural generalization of local finiteness in general topology. The concept of α-point finite and α-discrete families of fuzzy sets have also been introduced and studied. It is proved that every α-discrete family of fuzzy stets is α-locally finite and that every α-locally finite family is α-point finite. Further with corollary it is proved that every α-locally finite family of fuzzy sets is closure preserving and that arbitrary union of a α-locally finite family of closed fuzzy sets is a closed fuzzy set and α-local finiteness is invariant under α-perfect map.

Definition1.1 [9]**.** Let $\alpha \in [0, 1)$ (resp. $\alpha \in (0, 1]$). A fts (X, T) is said to be α-compact (resp. α^*- compact) if each open α-shading (resp. α^*- shading) of X has a finite α-subshading (resp. α^*- subshading).

Definition 1.2 [7]**.** Let $\mathcal{U} = \{A_\lambda : \lambda \in \Lambda\}$ and $\mathcal{V} = \{B_\gamma : \lambda \in \Gamma\}$ be two α-shading (resp. α^*- shading) of a fts (X, T). Then $\mathcal{U}$ is said to be a refinement of $\mathcal{V}$ written $\mathcal{U} < \mathcal{V}$ if for each $\lambda \in \Lambda$ there is some $\gamma \in \Gamma$ such that $A_\lambda \leq B_\gamma V_\gamma$.

Definition 1.3 [8]**.** Let $\alpha \in [0, 1)$ (resp. $\alpha \in (0, 1]$). An F-closed, F-continuous function $f : X \to Y$ from a fts X onto a fts Y is said to be α-Perfect (resp. α^*- perfect) if $f^{-1}(y)$ is α-compact (resp. α^*- compact) if each $y \in Y$.

2. Local Finiteness in Fuzzy Topological Space

Definition 2.1. Let $\alpha \in [0, 1)$ (resp. $\alpha \in (0, 1]$). A fuzzy set A in a fts (X, T) is said to be an empty fuzzy set of order α (resp. α^*) if $A(x) \leq \alpha$ resp. $A(x) < \alpha)$ for each $x \in X$. A fuzzy set A is said to be nonempty of order α (resp. α^*) if there exists $x_0 \in X$ such that $A(x_0) > \alpha$ (resp. $A(x_0) \geq \alpha$).

Definition 2.2. Let $\alpha \in [0, 1)$ (resp. $\alpha \in (0, 1]$). A family $\{A_\lambda : \lambda \in \Lambda\}$ of fuzzy sets in a fuzzy topological space (X, T) is said to be α-point finite (resp.

α^*-point finite) if for each $x \in X$, $A_\lambda(x) > \alpha$ (resp. $A_\lambda(x) \geq \alpha$) for at most finitely many $\lambda \in \Lambda$.

Definition 2.3. A family $\{A_\lambda : \lambda \in \Lambda\}$ of fuzzy sets in a fuzzy topological space (X, T) is said to be α-locally finite (resp. α^*-locally finite) in X if for each $x \in X$ there exists an open fuzzy set U in X such that $U(x) = 1$ and $U \wedge A_\lambda$ is non empty of order α (resp. non empty of order α^*) for at most finitely many $\lambda \in \Lambda$.

The following theorem follows from definitions.

Theorem 2.4. *Every α-locally finite (resp. α^*-locally finite) is a α-point finite (resp. α^*-point finite) in fts.*

Theorem 2.5. *Let $\alpha \in [0, 1)$ (resp. $\alpha \in (0, 1]$). If $\{A_\lambda : \lambda \in \Lambda\}$ and $\{B_\gamma : \gamma \in \Gamma\}$ are any two α-locally finite (resp. α^*-locally finite) families of fuzzy sets in a fts (X, T) then the family $\{A_\lambda \wedge B_\gamma : (\lambda, \gamma) \in \Lambda \times \Gamma\}$ is also α-locally finite (resp. α^*-locally finite) in X.*

Proof. Let $x \in X$, then there exists open fuzzy sets G and H in X such that $G(x) = 1$, $H(x) = 1$ and $G \wedge A_\lambda$ and $H \wedge B_\gamma$ are nonempty of order α for at most finitely many $\lambda \in \Lambda$. Suppose for each $z \in X$, $[(G \wedge H) \wedge (A_\lambda \wedge B_\gamma)](z) > \alpha$ is true for infinitely many $(\lambda, \gamma) \in \Lambda \times \Gamma$. Then it follows that, $[(G \wedge A_\lambda) \wedge (H \wedge B_\gamma)](z) > \alpha$ for infinitely many $(\lambda, \gamma) \in \Lambda \times \Gamma$. It contradicts that $G \wedge A_\lambda$ and $H \wedge B_\gamma$ are nonempty of order α for at most finitely many $\lambda \in \Lambda, \gamma \in \Gamma$. Therefore the family $\{A_\lambda \wedge B_\gamma : (\lambda, \gamma) \in \Lambda \times \Gamma\}$ is α-locally finite. The proof for α^*-case is similar.

Definition 2.6. Let $\alpha \in [0, 1)$ (resp. $\alpha \in (0, 1]$). A family $\{A_\lambda : \lambda \in \Lambda\}$ of fuzzy sets in a fuzzy topological space (X, T) is said to be α-discrete (resp. α^*-discrete) if for each $x \in X$ there exists an open fuzzy set U in X such that $U(x) = 1$ and $U \wedge A_\lambda$ is non empty of order α (resp. non empty of order α^*) for at most one member $\lambda \in \Lambda$.

It is obvious that every α-discrete (resp. α^*- discrete) family is α-locally finite (resp. α^*- locally finite).

The following is one of the important and interesting results in topology.

Theorem 2.7. *If a family* $\{A_\lambda : \lambda \in \Lambda\}$ *of fuzzy sets in a fts* (X, T) *is* α-*locally finite*, *then the following results hold good*.

1. $\{\overline{A}_\lambda : \lambda \in \Lambda\}$ *is also* α-*locally finite family*.

2. *For each subset* Λ' *of* Λ, $\vee \{\overline{A}_\lambda : \lambda \in \Lambda'\}$ *is a closed fuzzy set*.

Proof. (1) Let $x \in X$. There is an open fuzzy set U in X such that $U(x) = 1$ and $U \Lambda A_\lambda \Lambda$ is non empty of order α for at most finitely many $\lambda \in \Lambda$, there exists $x_0 \in X$ such that $(U \wedge A_\lambda)(x_0) > \alpha$ for at most finitely many $\lambda \in \Lambda$. Therefore $Min\, \{U(x_0) \wedge A_\lambda(x_0)\} > \alpha$, for at most finitely many $\lambda \in \Lambda$. Therefore $U \wedge A_\lambda$ is nonempty of order α for at most finitely many $\lambda \in \Lambda$. Thus $\{\overline{A}_\lambda : \lambda \in \Lambda\}$ is also is α-locally finite family in X.

(2) Let $\Lambda' \subset \Lambda$. Let $B = \vee\{\overline{A}_\lambda : \lambda \in \Lambda'\}$, we prove $1 - B$ is n open fuzzy set in X. Let $x \in X$ such that $(1 - B)(x) > 0$. Then $(1 - \bigvee_{\lambda \in \Lambda'}, \overline{A}_\lambda)(x) = \bigwedge_{\lambda \in \Lambda'} (1 - \overline{A}_\lambda)(x) = \inf_{\lambda \in \Lambda'} \in \Lambda' \{1 - \overline{A}_\lambda(x)\} > 0$. Therefore $1 - \overline{A}_\lambda(x) > 0$ for each $\lambda \in \Lambda'$. From (1), $U \Lambda \overline{A}_\lambda$ is nonempty of order α for at most finitely many $\lambda \in \Lambda'$, say $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \in \Lambda'$. Therefore, there exists $x_0 \in X$ such that $(U \Lambda \overline{A}_\lambda)(x_0) > \alpha$ for $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ and $(U \Lambda \overline{A}_\lambda)(x_0) \leq \alpha$ for $\lambda \neq \lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ (I).

Define $N = U \Lambda (\bigwedge_{i=1}^{n} (1 - \overline{A}_{\lambda_i} \overline{A}_{\lambda_i}))$. Clearly N is an open fuzzy set and $N(x) > (1 - B)(x)$ (II)

Suppose $N \leq 1 - B$ is not true. Then there exists $z \in X$ such that $N(z) > (1 - B)(z)$ that is $[U \Lambda (\bigwedge_{i=1}^{n} (1 - \overline{A}_{\lambda_i}))](z) > \bigwedge_{\lambda \in \Lambda'} (1 - \overline{A}_\lambda)(z)$.

Suppose $\bigwedge_{\lambda \in \Lambda'} (1 - \overline{A}_\lambda)(z) = \inf_{\lambda \in \Lambda'} \in \Lambda' (1 - \overline{A}_\lambda(z)) = (1 - \overline{A}_{\lambda_0})(z)$ for some

$\lambda_0 \in \Lambda'$. Then $U(z) \wedge (\bigwedge_{i=1}^{n}(1 - \overline{A}_{\lambda_i}\overline{A}_{\lambda i}))(z) > (1 - \overline{A}_{\lambda_0})(z)$ and $\bigwedge_{i=1}^{n}(1 - \overline{A}_{\lambda_i}\overline{A}_{\lambda i})(z) > (1 - \overline{A}_{\lambda_0})(z)$ ... (III).

If $\lambda_0 = \lambda_i$ for some $i = 1, 2, 3, \ldots, n$, then from (III), $(1 - \overline{A}_{\lambda i})(z) > (1 - \overline{A}_{\lambda_0})(z)$, for each $i = 1, 2, 3, \ldots, n$, which implies $(1 - \overline{A}_{\lambda_0})(z) > (1 - \overline{A}_{\lambda_0})(z)$, (z) which is impossible, and there is a contradiction.

If $\lambda_0 \neq \lambda_i$ for $i = 1, 2, 3, \ldots, n$, then we have $\bigwedge_{i=1}^{n}(1 - \overline{A}_{\lambda_i i})(z) \geq \bigwedge_{\lambda \in \Lambda''}^{n}(1 - \overline{A}_{\lambda})(z)$ and $\bigwedge_{\lambda \in \Lambda''}^{n}(1 - \overline{A}_{\lambda})(z) = (1 - \overline{A}_{\lambda_0})(z)$. Therefore $\bigwedge_{i=1}^{n}(1 - \overline{A}_{\lambda_i i})(z) \geq (1 - \overline{A}_{\lambda_0})(z)$ for $\lambda_0 \neq \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\lambda_n$ ... (IV).

From (I), $Min\,\{U(x), \overline{A}_{\lambda}(x)\} \leq \alpha$, for each $x \in X$ and for $\lambda_0 \neq \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ that is $\overline{A}_{\lambda^0}(z) \leq \alpha$ for $z \in X$ and for $\lambda_0 \neq \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$.

Now from (IV), $\wedge_{\lambda=1}^{n}(1 - \overline{A}_{\lambda})(z) \geq 1 - \alpha$ and $(1 - \overline{A}_{\lambda_i})(z) \geq 1 - \alpha$ for $i = 1, 2, 3, \ldots, n$. That is $1 - \overline{A}_{\lambda}(z) \geq 1 - \alpha$ for $\lambda = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$, which implies $\overline{A}_{\lambda}(z) \leq \alpha$ for $\lambda = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\lambda_n$ ... (V).

Again from (I), we have $(U \wedge \overline{A}_{\lambda})(z) > \alpha$ for $\lambda = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$. That is $[U(z) \wedge \overline{A}_{\lambda}] > \alpha$, for $\lambda = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$, which implies $\overline{A}_{\lambda}(z) > \alpha$ for $\lambda \neq \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ which is a contraction to (V). Therefore $N > 1 - B$ is not true. Hence $N \leq 1 - B$. (IV).

Therefore from (II) and (IV), $N(x) = (1 - B)(x)$. Thus it follows that $N(x) = (1 - B)(x)$ and $N \leq 1 - B$. According to [3] B is a closed fuzzy set. Hence the theorem.

Corollary 2.8. *If* $\{A_{\lambda} : \lambda \in \Lambda\}$ *be a* α*-locally finite family of fuzzy sets in a fts* (X, T) *then it is closure preserving.*

Proof. If $\{A_\lambda : \lambda \in \Lambda\}$ be a α-locally finite fuzzy sets in a fts (X, T) then for each $A_\lambda \leq \vee_{\lambda \in \Lambda} A_\lambda A_\lambda$. Therefore $\overline{A_{\bar{\lambda}}} \leq \overline{\bigvee_{\lambda \in \Lambda} A_\lambda}$ for each $\lambda \in \Lambda$ so $\bigvee_{\lambda \in \Lambda} \overline{A}_\lambda \leq \overline{\bigvee_{\lambda \in \Lambda} A_\lambda}$.

On the other hand, it is known that $A_\lambda \leq \overline{A}_\lambda$, for each $\lambda \in \Lambda$, which implies that $\bigvee_{\lambda \in \Lambda} A_\lambda \leq \bigvee_{\lambda \in \Lambda} \overline{A}_\lambda$. From above theorem we have $\vee_{\lambda \in \Lambda} \overline{A}_\lambda \overline{A}_\lambda$ is a closed fuzzy set which contains $\bigvee_{\lambda \in \Lambda} A_\lambda$. But $\overline{\bigvee_{\lambda \in \Lambda} A_\lambda}$ is a smallest closed fuzzy set containing $\bigvee_{\lambda \in \Lambda} A_\lambda$. Therefore $\overline{\bigvee_{\lambda \in \Lambda} A_\lambda} \leq \bigvee_{\lambda \in \Lambda} \overline{A}_\lambda$. Thus $\bigvee_{\lambda \in \Lambda} \overline{A}_\lambda = \overline{\bigvee_{\lambda \in \Lambda} A_\lambda}$. Hence $\{A_\lambda : \lambda \in \Lambda\}$ closer preserving.

Corollary 2.9. *If $\{A_\lambda : \lambda \in \Lambda\}$ be an α-locally finite family of closed fuzzy sets in X then $\bigvee A_\lambda$ is a closed fuzzy set in X.*

Theorem 2.10. *If $f : X \to Y$ is a α-Perfect map and $\{A_\lambda : \lambda \in \Lambda\}$ is a α-locally finite family of fuzzy subsets of X, then $\{f(A) : \lambda \in \Lambda\}$ is a α-locally finite family of fuzzy subsets of Y.*

Proof. Let $y \in Y$. Then $f^{-1}(y)$ is α-compact in X. For each $x \in f^{-1}(y)$ there is an open fuzzy set U_x with $U_x = 1$ and $U_x \Lambda A$ is nonempty of order α for at most finitely many $\lambda \in \Lambda$. That is $\Lambda_x = \{\lambda \in \Lambda : U_x \Lambda A\}$ is nonempty of order $\alpha\}$ is finite. Therefore $\{U_x : x \in f^{-1}(y)\}$ is an open α-shading of $f^{-1}(y)$ and hence has a finite α-subshading say $\{U_x : x \in B\}$, where B is a finite subset of $f^{-1}(y)$. Thus $f^{-1}(y) \leq \vee\{\vee \{U_x : x \in B\}$ which is an open fuzzy set in X. There exists an open fuzzy set V_y in Y such that $V_y(y) = 1$. Now $\{\lambda \in \Lambda : V_y \Lambda f(A_\lambda)$ is nonempty of order $\alpha\}$ is finite; If $V_y \Lambda f(A_\lambda)$ is nonempty of order α then $f^{-1}(V_y) \Lambda A_\lambda$ is nonempty of order α which implies $U_{x0} \Lambda A_\lambda$ is nonempty of order α for some $x_{.0} \in B$. Therefore $\lambda \in \Lambda x_0$ and $\lambda \in M = \bigcup\{A_k : x \in B\}$ is a finite subset of Λ. Therefore $\{\lambda \in \Lambda : V_y \Lambda f(A_\lambda)$ is nonempty of order $\alpha\}$ is finite. Thus $\{f(A_\lambda) : \lambda \in \Lambda\}$ is a α-locally finite in Y. Hence the theorem.

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