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LIMITS IN THE CATEGORY OF GRAPHS

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Abstract

Limits in a category are defined by means of diagrams. We begin with the definitions of a diagram scheme and a diagram in an arbitrary category A over a scheme as in [1]. We have proved that the category of graphs G is finitely complete by using the fact that "A category with finite products and equalizers is finitely complete".

1. Introduction

A graph G consists of a pair $G = (V(G), E(G))$ (also written as $G = (V, E)$ whenever the context is clear) where $V(G)$ is a finite set whose elements are called vertices and $E(G)$ is a set of unordered pairs of distinct elements in $V(G)$ whose members are called edges. The graphs as we have defined above are called simple graphs. Throughout our discussions all graphs are considered to be simple graphs. Let G and G_1 be graphs. A homomorphisms $f : G \to G_1$ is a pair $f = (f^*, \widetilde{f})$ where $f^* : V(G) \to V(G_1)$ and $\widetilde{f} : E(G) \to E(G_1)$ are functions such that $\widetilde{f}((u, v)) = (f^*(u), f^*(v))$ for all edges $(u, v) \in E(G)$. For convenience if $(u, v) \in E(G)$ then $\widetilde{f}((u, v))$ is simply denoted as $\widetilde{f}((u, v))$ [3].

Then we have the category of graphs say $\mathcal{G}$, where objects are graphs and morphisms are as defined above, where equality, compositions and the

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identity morphisms are defined in the natural way. It is also proved that two homomorphisms $f = (f^*, \tilde{f})$ and $g = (g^*, \tilde{g})$ of graphs are equal if and only if $f^* = g^*$ Lemma 1.6 [3]

1. Limits

Definition 2.1. A diagram scheme is a triple (I, M, d) where I is a set whose elements are called vertices (not to be confused with the vertices of a Graph), M is a set whose elements are called arrows and $d : M \to I \times I$ is a function. If $m \in M$ and $d(m) = (i, j)$ then we call i the origin and j the extremity of m.

A diagram in a category A over the scheme Σ is a function D which assigns to each vertex $i \in I$ an object $D_i \in A$ and to each arrow $m \in M$ with origin i and extremity j a morphism $D(m)$ from D_i into D_j, i.e., $D(m) : D_i \to D_j$ is a morphism in A. If I and M are finite sets then we call a finite scheme and D a finite diagram over Σ.

Definition 2.2. Let D be a diagram in A over a scheme $\Sigma = (I, M, d)$. A family of morphisms $\{f_i : X \to D_i\}_{i \in I}$ is called a compatible family for D if for every arrow $m \in M$, with $d(m) = (i, j)$ the following diagram commutative.

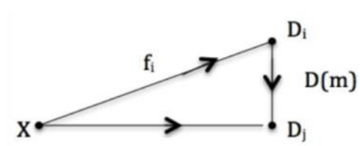

Figure 1

i.e. $D(m)f_i = f_j$ for all $m \in M$. The above family is said to be a limit for D if

(i) it is a compatible family for D, and

(ii) (Universal Mapping Property) for every compatible family $\{g_i : Y \to D_i\}$ $i \in I$, there is a unique morphism $\gamma : Y \to X$ such that $f_i\gamma = g$ for all $i \in I$.

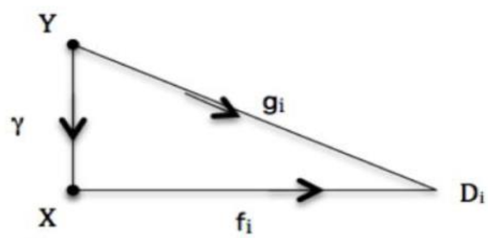

Figure 2.

Theorem 2.3. *Let $\mathcal{G}$ be the Category of graphs. Then every finite diagram in $\mathcal{G}$ (over a finite scheme) has a limit.*

Proof. We prove the theorem by actually constructing the limit of any finite diagram in $\mathcal{G}$. Let $\Sigma = (I, M, d)$ be any finite scheme and let D be a diagram in $\mathcal{G}$ over the scheme Σ. Let $I = \{1, 2, \ldots, n\}$.

Step 1: By Theorem 2.2 [7], $\mathcal{G}$ has categorical products so let $P = X, D_r$ be the $r = 1$ categorical product of $\{D_r\}_{r=1}^{n}$ [Remark 2.4 in [7]] with the canonical projections $p_i : P \to D_i$ from the product P into the i^{th} component D_i.

Then from Theorem 2.2 in [7], we know that $V(P) = V(D_1) \times \ldots \times V(D_n)$ (Cartesian product) which is the set $\{(x_1, \ldots, x_n)/x_i \in V(D_i)$ for all $i = 1$ to $n\}$. Moreover $(x_1, \ldots, x_n)$ is adjacent to $(y_1, \ldots, y_n)$ in P if and only if each x_i is adjacent with y_i in G_i for all $i \in I$.

Let $m \in M$ be an arrow with origin j and extremity k. Since $D(m) : D_j \to D_k$, $D(m)p_j$ is a morphism from $P = X, D_r \to D_k$, $r = 1$.

$$Km \xrightarrow[u_m]{} P = \underset{r=1}{\overset{n}{X}} Dr \; \overset{D(m)p_j}{\underset{p_k}{\longrightarrow}} \; Dk$$

Let $u_m : K_m \to P$ be the equalizer of $D(m)p_j$ and p_k. Since $\mathcal{G}$ has equalizers this is meaningful. Moreover since any two equalizers are isomorphic (proposition 2.4 in [6]). We may take, without loss of generality

$$V(K_m) = \{(x_1, x_2, \dots, x_n) \in V(P) / D(m) * p_j * (x_1, \dots, x_n) = p_k^*(x_1, \dots, x_n)\}$$

$$\text{i.e., } V(K_m) = \{(x_1, \dots, x_n) / D(m)^* x_j = x_k\}. \qquad (1).$$

Also $u_m : K_m \to P$ may be taken as the inclusion map. By Proposition 4.3 in [6], $\mathcal{G}$ has finite intersection.

So let

$$X = \bigcap_{m \in M} K_m \xrightarrow{v_m} K_m \xrightarrow{u_m} P = \mathop{X}_{r=1}^{n} D_r$$

with the inclusion map v_m as shown above. Let

$$u = u_m v_m : X \to P. \qquad (2)$$

We have following picture

$$X = \bigcap_{m \in M} K_m \xrightarrow{v_m} K_m \xrightarrow{u_m} P = \mathop{X}_{r=1}^{n} D_r \xrightarrow{p_i} D_i$$

Let $f_i : X \to D_i$ be defined by the composition of the above morphisms; i.e. $f_i = p_i u_m v_m$ for all $i \in I$, (3)

Step 2: We claim that $\{f_i : X \to D_i\}_i \in I$ is a compatible family.

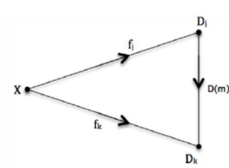

Figure 3

Now $D(m)^* f_j^*(x_1, \dots, x_n) = D(m)^* p_j^* u_m^* v_m^*(x_1, \dots, x_n)$ (by (3))

$$= D(m)^* p_j^*(x_1, \dots, x_n)$$

[since u_m, v_m are inclusion maps].

$$= D(m)^*(x_j) \ldots A$$

and

$$f_k^*(x_1, \ldots, x_n) = p_k^* u_m^* v_m^*(x_1, \ldots, x_n)$$

$$= p_k^*(x_1, \ldots, x_n)$$

$$= x_k \ldots (B).$$

From (1), A and $B\, D(m)^* f_j^* = f_k^*$ and so by Lemma 1.6 in [3]. $D(m)f_j = f_k$ providing that the above family is compatible for D.

Step 3. We now prove that $\{f_i : X \to D_i\}_{i \in I}$ is in fact the limit for the diagram D. Let $\{g_i : Y \to D_i\}_{i \in I}$ be any compatible family for D. Then by definition we have $D(m)g_j = g_k$ (4).

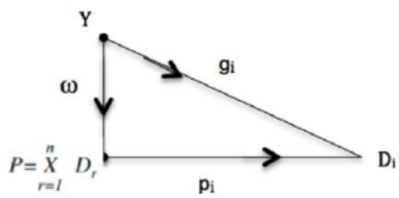

Figure 4

Then from the definition of products there exists a unique morphism say

$w : Y \to P = \overset{\omega}{\underset{r=1}{X}}, D_r$ such that $p_i = g_i$ for all $i \in I$. (5)

Next we have $D(m)p_j w$

$= D(m)g_j$ by (5).

$= g_k$ by (4)

$p_k w$ by (5).

Since k_m is the equalizer of $D(m)p_j$ and p_k, by definition of equalizer, there exists a unique morphism say $w_m : Y \to K_m$ such that

$$u_m w_m = w. \tag{6}$$

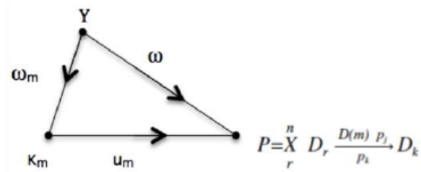

Figure 5

Since X is the intersection $\bigcap K_m$ and $w : Y \to P$ also factors through $u_m m \in M$ for each $m \in M$, by the definition of intersection there exists a unique morphism

$$\gamma : Y \to X \text{ such that } u\gamma = w. \quad (7).$$

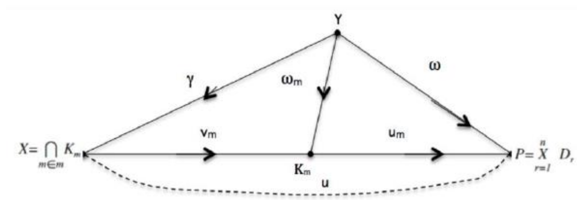

Figure 6

Finally we prove that $f_i\gamma = g_i$ for all $i \in I$.

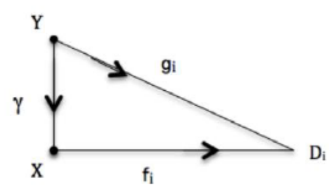

Figure 7.

Now $f_i\gamma = p_i u_m v_m \gamma$ (by definition of f_i)

$= p_i u\gamma$ (by definition of u)

$= p_i w$ by (7)

$= g_i$ by (5)

This completes the proof.

Definition 2.4. A category A is said to be Σ complete if every diagram in A over has a limit. If A is Σ-complete for all diagram schemes Σ, then A is called Complete. If A is Σ-complete for all finite diagram schemes Σ, then A is said to be finitely complete [1].

Note: The dual notions are that of colimits and finitely co-complete category.

Theorem 2.5. *The category of graphs* $\mathcal{G}$ *is finitely complete.*

Proof. The theorem is a direct consequence of the above Theorem.

Remarks 2.6. In [1, 4, 5] the authors have proved the above theorem by using the fact that "A category with finite products and equalizers is finitely complete". However in theorem 2.3, we have proved the same by actually constructing the limit of a finite diagram over a finite scheme Σ. This is found to be useful in application.

Remark 2.7. Again in [1, 4, 5] the authors have proved that the category of graphs is finitely co-complete by using the fact that "a category with coprodcuts and co-equalizers is finitely co-complete". However in our definition of the category $\mathcal{G}$ of simple graphs we have shown that $\mathcal{G}$ does not have coequalizers. Hence the above argument is not applicable in our case.

Definition 2.8. Let D and D' be diagrams in $\mathcal{G}$ over scheme $\Sigma = (I, M, d)$. Then D and D' are said to be isomorphic diagrams in $\mathcal{G}$ over Σ if

(i) there exists a family of isomorphisms $\{\theta_i : D_i \to D_i\}_i \in I$ and

(ii) to each arrow $m \in M$ with $d(m) = (i, j)$ the following diagram commutes [1].

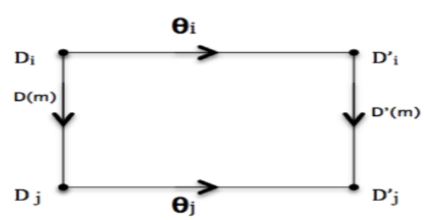

Figure 8.

Theorem 2.9. *Let D and D' be isomorphic diagrams in* $\mathcal{G}$ *over a scheme* $\Sigma = (I, M, d)$ *with* $\{\theta_i : D_i \to D'\}_i \in I$ *as the family of isomorphisms.*

If $\{f_i : X \to D_i\}_i \in I$ *is a limit for D over* Σ, *then* $\{\theta_i\ f_i : D_i \to D'\}_i \in I$ *is a limit for* D' *over* Σ.

Proof. By hypothesis we have the following commutative diagrams:

Thus $D(m)f_i = f_j$. (1) and $\theta_j D(m) = D'(m)\theta_i$. (2)

Consider the family of morphisms $\{\theta_i\ f_i : X \to D'_i\}_i \in I$.

Claim 1. The above family is compatible for the diagram D'.

For $D'(m)(\theta_i f_i) = (D'(m)\theta_i)f_i = \theta_j D(m) f_i$ by (1)

$= \theta_j f_j$ by (1) and hence the claim.

Claim 2: $\{\theta_i f_i : X \to D'_i\}_i \in I$ is a limit for D' over Σ. Let $\{g_i : Y \to D'_i\}_i \in I$ be any compatible family for D'_i. Then $D'(m)g_i = g_i$ (3). We claim that the family $\{\theta_i^{-1} g_i : X \to D_i\}_i \in I$ is a compatible family for the diagram D.

For $\theta_j^{-1} g_j = \theta_j^{-1} D'(m)\} g_i$ by (3).

$= \theta_j^{-1}(\theta_j D(m)\theta_i^{-1}) g_i$ by (1).

$= D(m)(\theta_i^{-1} g_i)$ and hence the claim. Since $\{f_i : X \to D_i\}_i \in I$ is a limit for the diagram D, there exists a unique morphism $\gamma : Y \to X$ such that $f_i \gamma = \theta_i^{-1} g_i$ for all $i \in I$ i.e. $(\theta_i, f_i)\gamma = g_i$ for all $i \in I$. Thus $\{\theta_i f_i : X \to D'_i\}_i \in I$ is a limit for the diagram D'.

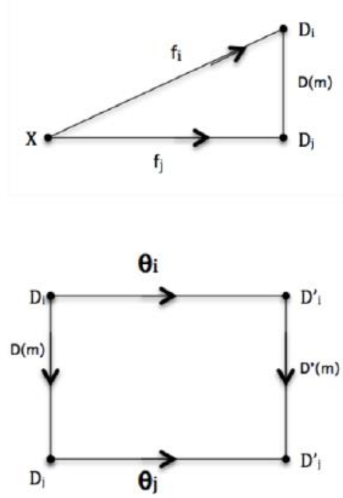

Figure 9.

We illustrate the above theorem by actually constructing the limit of a given diagram in $\mathcal{G}$ over a finite scheme.

Example 2.10. Let $\Sigma = (I, M, d)$ be a finite scheme where $I = \{1, 2, 3\}$, $M = \{m_1, m_2\}$, and $d(m_1) = (1, 2)$, $d(m_2) = (2, 3)$. Let D be the diagram in $\mathcal{G}$ over where D_1, D_2, D_3 and $D(m_1)$, $D(m_2)$ are as given below. $D(m_1) : D_1 \to D_2$ is given by the rule $x_1 \to y_1; x_2 \to y_2$. Similarly $D(m_2) : D_2 \to D_3$ $y_1 \to Z_1$; $y_2 \to Z_2$; $y_3 \to Z_3$. For convenience Let us denote the triple (x_i, y_i, z_k) simply as (i, j, k).

Then the product graph $P = \overset{3}{\underset{r=1}{X}} D_r$ is given by the following diagram.

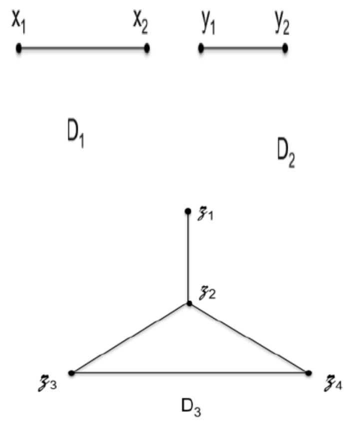

Figure 10

Let $p_1 : P = \overset{3}{\underset{r=1}{X}} D_r \to D_1, p_2 : P \to D_2$ and $p_3 : P \to D_3$ be the canonical projections. Then for all $1 \le i \le 2, 1 \le j \le 3, 1 \le k \le 4,$ we have

$$p_1^* : (xi,\ yj,\ z_k) \to xi;\ p_2^* : (xi,\ yj,\ z_k) \to yj\ p_3^* : (xi,\ yj,\ z_k) \to z_k.$$

Hence $K_{m1} = Eqn(D(m_1)p_1,\ p_2)$ has vertex set

$$V(K_{m1}) = \{(xi,\ y1,\ z_k), (xi,\ y2,\ z_k)\} 1 \le i \le 2;\ 1 \le k \le 4.$$

Similarly we can show that $K_{m2} = Eqn(D(m_2)p_2,\ p_3)$ has vertex set

$$V(K_{m2}) = \{(x_i,\ y_j,\ z) | 1 \le i \le 2$$

$$1 \le j \le 3$$

$$1 \le k \le 4\}$$

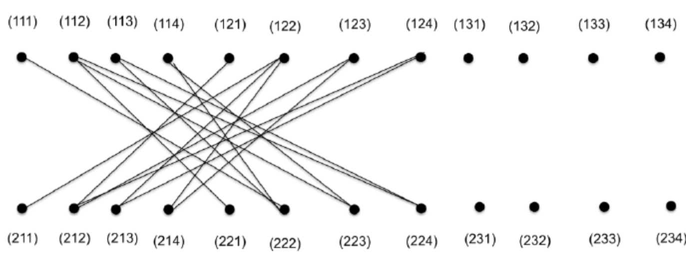

Figure 11.

Therefore $V(K_{m1}) \cap V(K_{m2}) = \{(x_i, y_j, z) | 1 \leq i \leq 2$

$$1 \leq j \leq 2$$

$$1 \leq k \leq 3\}.$$

Thus if X is the subgraph of $\overset{3}{\underset{r=1}{X,}} D_r$ whose vertex set is $V(K_{m1}) \cup V(K_{m2})$ then X is given by the following diagram.

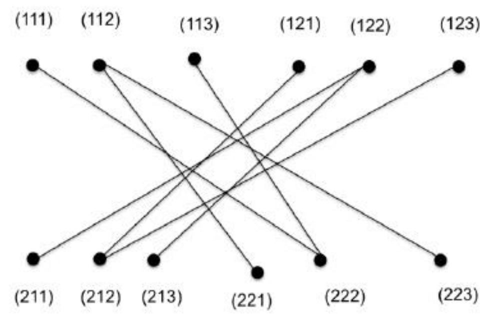

Figure 12.

For $i = 1$ to 3, let $f_i = p_i / X$ i.e. f_i is the restriction of p_i to X. Then by Theorem the family $\{f_i : X \to D_i\}_i \in I$ is the limit in $\mathcal{G}$ for the diagram D over the scheme Σ.

Conclusion 3.1. We have shown every finite diagram in $\mathcal{G}$. has a limit and have proved that the category of graphs $\mathcal{G}$. is finitely complete. Also we have shown that the forgetful functors is limit preserving.

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