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2-EDGE DOMINATING SETS AND 2-EDGE DOMINATION POLYNOMIALS OF STARS (S_n)

K. LAL GIPSON and ARUN WILLIAMS S

Assistant Professor
Research Department of Mathematics
Scott Christian College, (Autonomous)
Nagercoil, Kanyakumari District
Tamil Nadu, India

Research Scholar (Full time)
Scott Christian College, (Autonomous)
Nagercoil, Kanyakumari District
Tamil Nadu, India
Affiliated to Manonmaniam Sundaranar University
Abishekapatti, Tirunelveli-627012
Tamil Nadu, India

Abstract

Let S_n be the Star graph with n vertices and $n-1$ edges. Let $D_{2e}(G, i)$ be the family of 2-edge dominating sets in G with cardinality i. The polynomial $D_{2e}(G, x) = \sum_{i=\gamma_{2e}(G)}^{|E(G)|} d_{2e}(G, i)x^i$ is called the 2-edge domination polynomial of G. In this paper, we obtain a recursive formula for $d_{2e}(S_n, i)$. Using this recursive formula we construct 2-edge domination polynomial, $D_{2e}(G, x) = \sum_{i=1}^{n-1} d_{2e}(S_n, i)x^i$, where $d_{2e}(S_n, i)$ is the number of 2-edge dominating sets of S_n of cardinality i and obtain some properties of this polynomial.

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1. Introduction

Let G be a simple graph of order n. For any vertex, $v \in V$, the open neighbourhood of V is the set $N(v) = \{u \in V / uv \in E\}$ and the closed neighbourhood of V is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$ the open neighbourhood of S is $N(S) = N[S] \cup S$. A dominating set for a graph G is a subset D of V such that every vertex not in the D is adjacent to at least one member of D. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set of G.

An edge dominating set for a graph G is a set of $D \subseteq E$ such that every edge not in D is adjacent to at least one edge in D. An edge dominating set is also known as a line dominating set.

The star graph of order n, denoted S_n is a simple graph with n vertices with the following properties:

One distinguished vertex is of degree $n - 1$. The remaining vertices are all of degree 1 and are adjacent only to the distinguished vertex is of degree $n - 1$. The remaining vertices are all of degree 1 and are adjacent only to the distinguished vertex.

2. 2-Edge Dominating Sets of Star

In this section, we state the 2-edge domination number of Star and some of its properties.

Definition 2.1. Let G be a simple graph of order n. A set $D \subseteq E$ is a 2-edge dominating set of the graph G, if every edge $e \in E - D$ is adjacent to at least 2-edges in D. The 2-edge domination number $\gamma_{2e}(G)$ is the minimum cardinality among the 2-edge dominating sets of G.

Example for 2-edge dominating sets of Star

Let us consider S_6 as an example

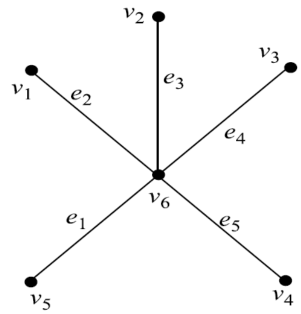

Figure 1.

Here $E = \{e_1, e_2, e_3, e_4, e_5\}$, we take $D = \{e_1, e_3, e_5\}$ then $E - D$ is $\{e_2\}$ and $\{e_4\}$ is adjacent to $\{e_1\}\{e_3\}\{e_4\}, \{e_5\}$ and $\{e_4\}$ is adjacent to $\{e_1\}\{e_2\}\{e_3\}$ and $\{e_5\}$. Therefore, the set $\{e_1, e_3, e_5\}$ is a 2-edge dominating set.

Theorem 2.2. *Let* S_n *be the star with n vertices and* $n - 1$ *edges then* $d_{2e}(S_n, i) = \binom{n-1}{i}$; *for all* $2 \leq i \leq n - 1$ *and* $n \geq 3$.

Proof. Let $S_n, n \geq 3$ be the star graph with n vertices and $n - 1$ edges. Let $v_1 \in V$ be the centre vertex and $v_2, v_3, \ldots, v_n$ be the remaining vertices. Let $e_1, e_2, \ldots, e_{n-1}$ be the edges of $S_n \cdot S_n$ contains $\binom{n-1}{i}$ number of subsets of edges with cardinality i. Since all the edges are incident with the centre vertex v_1 all the $\binom{n-1}{i}$ subsets of edges are 2-edge dominating sets. Therefore, $d_{2e}(S_n, i) = \binom{n-1}{i}$, for all $2 \leq i \leq n - 1$ and $n \geq 3$.

Theorem 2.3. *Let* S_n *be a star with order n and size* $n - 1$ *then*

1. $d_{2e}(S_n, i) = d_{2e}(S_{n-1}, i) + d_{2e}(S_{n-1}, i - 1)$ *for* $3 \leq i \leq n$

2. $d_{2e}(S_n, i) = d_{2e}(S_{n-1}, i) + (n - 2)$ *if* $i = 2$.

Proof. By Theorem 2.2, we have,

$$d_{2e}(S_n, i) = \binom{n-1}{i}$$

We have,

$d_{2e}(S_n, i) = \binom{n-1}{i}$

$d_{2e}(S_{n-1}, i) = \binom{n-2}{i}$ and

$d_{2e}(S_{n-1}, i-1) = \binom{n-2}{i-1}$

We know that,

$$\binom{n-1}{i} = \binom{n-2}{i} + \binom{n-2}{i-1}$$

Therefore,

$$d_{2e}(S_n, i) = d_{2e}(S_{n-1}, i) + d_{2e}(S_{n-1}, i-1) \text{ for } 3 \leq i \leq n$$

2. We have,

$$d_{2e}(S_n, i) = \binom{n-1}{i}$$

When $i = 2$

$$d_{2e}(S_n, 2) = \binom{n-1}{2}$$

Consider, $d_{2e}(S_{n-1}, 2) + (n-2) = \binom{n-2}{2} + n - 2$

$$= \binom{n-1}{i} + n - 2$$

$$= \frac{(n-2)(n-3)}{2} + n - 2$$

$$= \frac{n^2 - 3n + 2}{2}$$

$$= \frac{(n-1)(n-2)}{2}$$

$$= \binom{n-1}{2}$$

$$d_{2e}(S_{n-1}, 2) + (n-2) = \binom{n-1}{2}$$

$$d_{2e}(S_{n-1}, i) + (n-2) = \binom{n-1}{i} \text{ where } i = 2$$

3. 2-Edge Domination Polynomials of Star

Definition 3.1. Let $D_{2e}(S_n, i)$ be the family of 2-edge dominating sets of star S_n with cardinality i and let $d_{2e}(S_n, i) = |D_{2e}(S_n, i)|$. Then the 2-edge domination polynomial $D_{2e}(S_n, x)$ of S_n is defined as $D_{2e}(S_n, x) = \sum_{i=\gamma_{2e}(S_n)}^{n-1} d_{2e}(S_n, i)x^i$, where $\gamma_{2e}(S_n)$ is the 2-edge domination number of S_n.

Lemma 3.2. $\gamma_{2e}(S_n) = 2$.

Proof. Let S_n be a star graph with n vertices and $n \geq 3$. Since all the edges are incident with the centre vertex and edges 2-edge dominating set must contain at least 2 edges.

Therefore, $\gamma_{2e}(S_n) = 2$.

Theorem 3.3.

1. $D_{2e}(S_n, x) = (1+x)D_{2e}(S_{n-1}, x) + (n-2)x^2$ *with* $D_{2e}(S_4, x) = 3x^2 + x^3$.

2. $D_{2e}(S_n, x) = \sum_{i=2}^{n-1} \binom{n}{i} x^i + \binom{n-1}{i-1} x^i$

Proof. We have $D_{2e}(S_n, x) = \sum_{i=2}^{n-1} d_{2e}(S_n, i)\, x^i$

$$= d_{2e}(S_n, 2)x^2 + \sum_{i=3}^{n-1} d_{2e}(S_n, i)x^i$$

$$= \binom{n-1}{2}x^2 + \sum_{i=3}^{n-1}[d_{2e}(S_{n-1}, i) + d_{2e}(S_{n-1}, i-1)]x^i$$

$$= \binom{n-1}{2}x^2 + \sum_{i=3}^{n-1} d_{2e}(S_{n-1}, i)x^i + \sum_{i=3}^{n-1} d_{2e}(S_{n-1}, i-1)x^i$$

Consider $\sum_{i=3}^{n-1} d_{2e}(S_{n-1}, i)x^i = \sum_{i=2}^{n-1} d_{2e}(S_{n-1}, i)x^i - d_{2e}(S_{n-1}, 2)x^2$

$$= D_{2e}(S_{n-1}, x) - \binom{n-2}{2}x^2$$

Consider

$$\sum_{i=3}^{n-1} d_{2e}(S_{n-1}, i-1)x^i = x\sum_{i=3}^{n-1} d_{2e}(S_{n-1}, i-1)x^{i-1}$$

$$= x\sum_{i=2}^{n-1} d_{2e}(S_{n-1}, 2)x^i$$

$$= xD_{2e}(S_{n-1}, i)x^i$$

$$D_{2e}(S_n, x) = \binom{n-1}{2}x^2 + D_{2e}(S_{n-1}, x) - \binom{n-2}{2}x^2 + xD_{2e}(S_{n-1}, x)$$

$$= (1+x) + D_{2e}(S_{n-1}, x) + \left[\binom{n-1}{2} - \binom{n-2}{2}\right]x^2$$

$$= (1+x)D_{2e}(S_{n-1}, x) + \binom{n-2}{1}x^2$$

$$= (1+x)D_{2e}(S_{n-1}, x) + (n-2)x^2.$$

Example. We consider S_6 as shown in Figure 1

By Theorem 3.3 (1) we have,

$$D_{2e}(S_n, x) = (1+x)D_{2e}(S_5, x) + 4x^2$$

$$= (1+x)(6x^2 + 4x^3 + x^4) + 4x^2$$

$$= 6x^2 + 4x^3 + x^4 + 6x^3 + 4x^4 + x^5 + 4x^2$$

$$= 10x^2 + 10x^3 + 5x^4 + x^5$$

$$D_{2e}(S_n, x) = 10x^2 + 10x^3 + 5x^4 + x^5$$

2. We have $D_{2e}(S_n, x) = \sum_{i=2}^{n-1} \binom{n-1}{i} x^i$

$$= \sum_{i=2}^{n-1} \left[\binom{n}{i} + \binom{n-1}{i-1}\right] x^i$$

$$= \sum_{i=2}^{n-1} \binom{n}{i} x^i + \sum_{i=2}^{n-1} \binom{n-1}{i-1} x^i$$

Table 1. $d_{2e}(S_n, i)$ the number of 2-edge dominating sets of S_n with cardinality i for all $4 \le n \le 15$ and $2 \le i \le 14$ as shown in following table.

n \ i	2	3	4	5	6	7	8	9	10	11	12	13	14
S_4	3	1											
S_5	6	4	1										
S_6	10	10	5	1									
S_7	15	20	15	6	1								
S_8	21	35	35	21	7	1							
S_9	28	56	70	56	28	8	1						
S_{10}	36	84	126	126	84	36	9	1					

S_{11}	45	120	210	252	210	120	45	10	1				
S_{12}	55	165	330	462	462	330	165	55	11	1			
S_{13}	66	220	495	792	924	792	495	220	66	12	1		
S_{14}	78	286	715	1287	1716	1716	1287	715	286	78	13	1	
S_{15}	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1

Theorem 3.4. *The following properties hold for the coefficients of* $D_{2e}(S_n, x)$.

1. $d_{2e}(S_n, 1) = 0$
2. $d_{2e}(S_n, n-1) = 1$
3. $d_{2e}(S_n, n-2) = n-1$
4. $d_{2e}(S_n, n-3) = \frac{1}{2}[n^2 - 3n + 2]$; *for all* $n \geq 5$
5. $d_{2e}(S_n, n-4) = \frac{1}{6}[n^3 - 6n^2 + 11n - 6]$; *for all* $n \geq 6$
6. $d_{2e}(S_n, i) = d_{2e}(S_n, n-i-1)$; *for every* $2 \leq i \leq n-2$.

Proof. Proof of the theorem is obvious by using the above Table 1.

Conclusion

In this paper 2-edge domination sets of Stars and 2-edge domination polynomials of Stars are studied and obtained some properties. We can generalize this study to any S_n.

References

[1] Y. Caro and Y. Roditty, A note on the k-domination number of a graph, J. Math and Math. Sci. 13(1) (1990), 205-206.

[2] Saeid Alikhani and Yee-Hock Peng, Dominating sets and domination polynomials of certain graphs, II, Opuscula Mathematica 30 (2010).

[3] Saeid Alikhani and Yee-Hock Peng, Dominating sets and domination polynomials of paths, International Journal of Mathematics and Mathematical Sciences (2009), Article

ID; 542040.

[4] A. Vijayan and K. Lal Gipson, Dominating sets and domination polynomials of square of paths, Open Journal of Discrete Mathematics (2013), 60-69.

[5] K. Lal Gipson and Arun Williams S, 2-Edge dominating sets and 2-Edge domination polynomials of paths, Malaya Journal of mathematik 9(1) (2021), 844-849.