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PRIME LABELING ON SOME CYCLE RELATED GRAPHS

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Abstract

A simple, finite, undirected and non-trivial graph G of size $\wp$ and order q with the vertex set $V(G)$ and the edge set $E(G)$ is said to admit prime labeling if $\hat{g} : V(G) \to \{1, 2, 3, \ldots, |V(G)|\}$ is an injective function such that the $\gcd(\hat{g}(u), \hat{g}(v)) = 1$. Then the graph G is a Prime graph. In this paper, few-cycle related graph such as $C_m(P_n)K_l, (C_n \odot K_1)l_m, C_3 \odot K_{1,n}, 2C_n(K_{1,m}), C_3(mP_n)$ are proved to be prime graphs.

1. Introduction

The concept of graph labeling was introduced in the nineties. Over 200 types of graph labeling have been studied over the last 60 years, with well almost 2500 articles published. Under certain conditions, graph labeling is the assignment of integers to vertices, edges, or both. A graph with n vertices admits a prime label if any two adjacent vertices can be labeled with the first n natural numbers in such a way that their labels are relatively prime.

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Gallian's 2016 paper [2] provides an in-depth examination of graph labeling. Rosa pioneered vertex labeling in graph theory in 1967 [6]. Prime graph labelling was proposed in 1982 by Tout, Dabboucy, and Howalla [8].

In this paper, we examine prime labeling on few-cycle related graphs.

2. Definitions

Definition 2.1. "Graph labeling" is the process of assigning values to the vertices or edges of a graph depending on certain conditions.

Definition 2.2. A prime labeling of a graph is an injective function $\hat{g} : V(G) \to \{1, 2, 3, \dots, |V(G)|\}$ such that the $\gcd(\hat{g}(u), \hat{g}(v)) = 1$ for each adjacent vertices u and v. A prime graph is a graph that permits prime labeling.

Definition 2.3. The graph $C_m(P_n)K_r$ is obtained by attaching a path P_n through a bridge to the end vertex of a cycle C_m and attaching l pendant edges to the end vertex of a path P_n.

Definition 2.4. The graph $(C_n \odot K_1)l_m$ is obtained by attaching m pendant edges to the first pendant edge of $C_n \odot K_1$.

Definition 2.5. The graph $C_3 \odot K_{1,n}$ is obtained by attaching n pendant edges to every vertices of the cycle C_3.

Definition 2.6. The graph $2C_n(K_{1,m})$ is obtained by joining two copies of cycle C_n through an edge to the apex vertex of the star graph $K_{1,m}$.

Definition 2.7. The graph $C_3(mP_n)$ is obtained by attaching m path of length n to the first vertex of the cycle C_3.

3. Main Results

Theorem 3.1. *The graph $C_m(P_n)K_l$ is a prime graph for $m \geq 3$.*

Proof. Consider $C_m(P_n)K_l$ $V(C_m(P_n)K_l) = \{c_i : 1 \leq i \leq m\} \cup \{p_j : 1 \leq j \leq n\} \cup \{k_r : 1 \leq r \leq l\}$ and $E(C_m(P_n)K_l) = \{c_i c_{i+1} : 1 \leq i \leq m-1\}$

$\cup \{c_m c_1\} \cup \{c_m p_1\} \cup \{p_j p_{j+1} : 1 \leq j \leq n-1\} \cup \{p_n k_r : 1 \leq r \leq l\}$.

$|V(C_m(P_n)K_l)| = |E(C_m(P_n)K_l)| = m + n + l.$

We define $\hat{g} : V(C_m(P_n)K_l) \to \{1, 2, 3, \ldots, |V(C_m(P_n)K_l)|\}$ as follows.

Case (i). If m is even.

$\hat{g}(c_i) = i + 1$ for $1 \leq i \leq m$

$\hat{g}(p_j) = m + 1 + j$ for $1 \leq j \leq n-1$

$\hat{g}(p_n) = 1$

$\hat{g}(k_r) = m + n + r$ for $1 \leq r \leq l$

Case (ii). If m is odd

$\hat{g}(c_i) = i + 1$ for $1 \leq i \leq m-1$

$\hat{g}(p_j) = m + 2 + j$ for $1 \leq j \leq n-1$

$\hat{g}(p_n) = 1$, $\hat{g}(c_m) = m + 2$

$\hat{g}(k_r) = m + n + 1 + r$ for $1 \leq r \leq l-1$

$\hat{g}(k_l) = m + 1.$

The following observations can be made based on the above labeling pattern.

The $\gcd(c_i, c_{i+1}) = 1$ for $1 \leq i \leq m-1$.

The $\gcd(c_m, c_1) = 1$

The $\gcd(c_m, p_1) = 1$

The $\gcd(p_j, p_{j+1}) = 1$ for $1 \leq j \leq n-1$.

The $\gcd(p_n, k_r) = 1$ for $1 \leq r \leq l$.

Hence by the definition 2.2, it is clear that the graph $C_m(P_n)K_l$ is a prime graph for $m \geq 3$.

Theorem 3.2. *The graph* $(C_n \odot K_1)l_m$ *is a prime graph for* $n \geq 3$.

Proof. Consider $(C_n \odot K_1)l_m$. $V((C_n \odot K_1)l_m) = \{c_i : 1 \leq i \leq n\} \cup \{c_j^1 : 1 \leq j \leq n\} \cup \{c_k^2 : 1 \leq k \leq m\}$ and $E((C_n \odot K_1)l_m) = \{c_i c_{i+1} : 1 \leq i \leq n-1\} \cup \{c_1 c_n\} \cup \{c_i c_j^1 : 1 \leq i = j \leq n\} \cup \{c_1^1 c_k^2 : 1 \leq k \leq m\}$.

$| V((C_n \odot K_1)l_m) | = | E((C_n \odot K_1)l_m) | = 2n + m$.

We define $\hat{g} : V((C_n \odot K_1)l_m) \to \{1, 2, 3, \ldots, | V((C_n \odot K_1)l_m) |\}$ as follows.

$\hat{g}(c_1) = 2$, $\hat{g}(c_1^1) = 1$

$\hat{g}(c_i) = 2i - 1$, for $2 \leq i \leq n$

$\hat{g}(c_j^1) = 2j$, for $2 \leq j \leq n$

$\hat{g}(c_k^2) = 2n + k$, for $1 \leq k \leq m$.

The following observations can be made based on the above labeling pattern.

The $\gcd(c_i, c_{i+1}) = 1$ for $1 \leq i \leq n-1$.

The $\gcd(c_n, c_1) = 1$

The $\gcd(c_i, c_j^1) = 1$ for $1 \leq i = j \leq n$

The $\gcd(c_1^1, c_k^2) = 1$, for $1 \leq k \leq m$.

Hence by the definition 2.2, It is clear that the graph $(C_n \odot K_1)l_m$ is a prime graph for $n \geq 3$.

Theorem 3.3. *The graph* $C_3 \odot K_{1,n}$ *is a prime graph.*

Proof. Consider $C_3 \odot K_{1,n}$. $V(C_3 \odot K_{1,n}) = \{c_1, c_2, c_3\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\}$ and $E(C_3 \odot K_{1,n}) = \{c_1c_2, c_2c_3, c_3c_1\} \cup \{c_1u_i : 1 \leq i \leq n\} \cup \{c_2v_i : 1 \leq i \leq n\} \cup \{c_3w_i : 1 \leq i \leq n\}$.

$| V(C_3 \odot K_{1,n}) | = | E(C_3 \odot K_{1,n}) | = 3n + 3.$

We define $\hat{g} : V(C_3 \odot K_{1,n}) \to \{1, 2, 3, \ldots, | V(C_3 \odot K_{1,n}) |\}$ as follows.

$\hat{g}(c_1) = 1$ $\hat{g}(c_2) = 2$, $\hat{g}(c_3) = 3$.

$\hat{g}(u_i) = i$ if; $i \not\equiv 1 (\text{mod } 3)$ and $i \equiv 0 (\text{mod } 2)$; $4 \leq i \leq 3n + 3$.

$\hat{g}(v_i) = i$ if; $i \not\equiv 1 (\text{mod } 3)$ and $i \equiv 1 (\text{mod } 2)$; $4 \leq i \leq 3n + 3$.

$\hat{g}(w_i) = i$ if; $i \equiv 1 (\text{mod } 3)$; $4 \leq i \leq 3n + 3$.

The following observations can be made based on the above labeling pattern.

The $\gcd(c_1, c_2) = 1$, $\gcd(c_2, c_3) = 1$ and $\gcd(c_3, c_1) = 1$.

The $\gcd(c_1, u_i) = 1$, for $1 \leq i \leq n$.

The $\gcd(c_1, v_i) = 1$, for $1 \leq i \leq n$.

The $\gcd(c_1, w_i) = 1$, for $1 \leq i \leq n$.

Hence by the definition 2.2, It is clear that the graph $C_3 \odot K_{1,n}$ is Prime graph.

Theorem 3.4. *The graph* $2C_n(K_{1,m})$ *is a prime graph for* $n \geq 3$.

Proof. Consider $2C_n(K_{1,m})$. $V(2C_n(K_{1,m})) = \{c_i^1 : 1 \leq i \leq n\} \cup \{c_i^2 : 1 \leq i \leq n\} \cup \{p\} \cup \{k_j ; 1 \leq j \leq m\}$ and $E(2C_n(K_{1,m})) = \{c_i^1 c_{i+1}^1 ; 1 \leq i \leq n-1\} \cup \{c_n^1 c_1^1\} \cup \{c_i^2 c_{i+1}^2 ; 1 \leq i \leq n-1\} \cup \{c_n^2 c_1^2\} \cup \{c_1^1 p\} \cup \{c_1^2 p\} \cup \{pk_j : 1 \leq j \leq m\}$.

$| V(2C_n(K_{1,m})) | = 2n + m + 1$ and $| E(2C_n(K_{1,m})) | = 2n + m + 2$.

We define $\hat{g} : V(2C_n(K_{1,m})) \to \{1, 2, 3, \ldots, | V(2C_n(K_{1,m})) |\}$ as follows.

Case 1. if n is even

Subcase 1.1. $n \equiv 0, 2 (\text{mod } 6)$

$\hat{g}(p) = 1$

$\hat{g}(c_i^1) = i + 1$ for $1 \le i \le n$

$\hat{g}(c_i^2) = n + i + 1$ for $1 \le i \le n$

$\hat{g}(k_j) = 2n + 1 + j$ for $1 \le j \le m$

Subcase 1.2. $n \equiv 4 (\bmod 6)$

$\hat{g}(p) = 1$

$\hat{g}(c_i^1) = i + 1$ for $1 \le i \le n$

$\hat{g}(c_i^2) = n + 2 + i$ for $1 \le i \le n$

$\hat{g}(k_j) = 2n + 2 + j$ for $1 \le j \le m - 1$

$\hat{g}(k_m) = n + 2$

Case 2. if n is odd

Subcase 2.1. $n \equiv 1 (\bmod 6)$

$\hat{g}(p) = 1$

$\hat{g}(c_i^1) = i + 1$ for $1 \le i \le n - 1$

$\hat{g}(c_n^1) = n + 2$

$\hat{g}(c_i^2) = n + 2 + i$ for $1 \le i \le n - 1$

$\hat{g}(c_n^2) = 2n + 3$

$\hat{g}(k_j) = 2n + 3 + j$ for $1 \le j \le m - 2$

$\hat{g}(k_{m-1}) = n + 1$

$\hat{g}(k_m) = 2(n + 1)$

Subcase 2.2. $n \equiv 3 (\bmod 6)$

$\hat{g}(p) = 1$

$\hat{g}(c_i^1) = i + 1$ for $1 \leq i \leq n - 1$

$\hat{g}(c_n^1) = n + 2$

$\hat{g}(c_i^2) = n + 3 + i$ for $1 \leq i \leq n$

$\hat{g}(k_j) = 2n + 3 + j$ for $1 \leq j \leq m - 2$

$\hat{g}(k_{m-1}) = n + 1$

$\hat{g}(k_m) = n + 3$

Subcase 2.3. $n \equiv 5 (\bmod 6)$

$\hat{g}(p) = 1$

$\hat{g}(c_i^1) = i + 1$ for $1 \leq i \leq n - 1$

$\hat{g}(c_n^1) = n + 2,$

$\hat{g}(c_i^2) = n + 2 + i$ for $1 \leq i \leq n - 1$

$\hat{g}(c_n^2) = 2n + 3$

$\hat{g}(k_j) = 2n + 3 + j$ for $1 \leq j \leq m - 2$

$\hat{g}(k_{m-1}) = n + 1$

$\hat{g}(k_m) = 2(n + 1)$

The following observations can be made based on the above labeling pattern.

The $\gcd(c_i^1, c_{i+1}^1) = 1$, for $1 \leq i \leq n - 1$.

The $\gcd(c_n^1 c_1^1) = 1$.

The $\gcd(c_i^2 c_{i+1}^2) = 1$ for $1 \leq i \leq n - 1$.

The $\gcd(c_n^2 c_1^2) = 1$

The $\gcd(c_1^1 p) = 1$

The $\gcd(c_1^2 p) = 1$

The $\gcd(p, k_j) = 1$ for $1 \leq j \leq m$.

Hence by the definition 2.2, It is clear that the graph $2C_n(K_{1,m})$ is a prime graph, for $n \geq 3$.

Theorem 3.5. *The graph $C_3(mP_n)$ is a prime graph.*

Proof. Consider $C_3(mP_n)$. $V(C_3(mP_n)) = \{v_1, v_2, v_3\} \cup \{u_i^j; 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(C_3(mP_n)) = \{v_1, v_2\} \cup \{v_2, v_3\} \cup \{v_3, v_1\} \{v_1 u_i^1; 1 \leq i \leq n\} \cup \{u_i^j u_i^{j+1} : 1 \leq i \leq m; 1 \leq j \leq n-1\}$.

$|V(C_3(mP_n))| = |E(C_3(mP_n))| = mn + 3.$

We define $\hat{g} : V(C_3(mP_n)) \to \{1, 2, 3, \ldots, |V(C_3(mP_n))|\}$ as follows.

$\hat{g}(v_1) = 1 \quad \hat{g}(v_2) = 2 \quad \hat{g}(v_3) = 3$

$\hat{g}(u_i^j) = (i-1)n + 3 + j$ for $1 \leq i \leq m;\ 1 \leq j \leq n$.

The following observations can be made based on the above labeling pattern.

The $\gcd(v_1, v_2) = 1$

The $\gcd(v_2, v_3) = 1$

The $\gcd(v_3, v_1) = 1$

The $\gcd(v_1 u_i^1) = 1$ for $1 \leq i \leq n$

The $\gcd(u_i^j u_i^{j+1}) = 1$ for $1 \leq i \leq n; 1 \leq j \leq m$

Hence by the definition 2.2, It is clear that the graph $C_3(mP_n)$ is a prime graph.

Conclusion

We studied the existence of prime labeling in some cycle related graphs in this paper such as $C_m(P_n)K_l$, $(C_n \odot K_1)l_m$, $C_3 \odot K_{1,n}$, $2C_n(K_{1,m})$, $C_3(mP_n)$ and proved to be prime graphs. Investigating the presence of prime labeling on some other types of graphs are our future work.

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