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ON DISTINCT UNIT FRACTIONS WHOSE SUM EQUALS 1 WHEN $x_i \nmid x_j$ FOR $i \neq j$

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Abstract

In this paper we consider the title equation when $x_i = pq$ where p, q are distinct primes, and also when some values of x_i are of the form p^2q. Four solutions, old and new are exhibited. We also introduce the concept of the "basic set of primes" which enables us to achieve the sum of 1 without the use of a computer. In particular, this concept is applied in the analysis of the Johnson's solution–unpublished, in which the reciprocals and their sum of 1 are obtained only by the use of a computer.

1. Introduction

This article is concerned with the Diophantine equation

$$\sum_{i=1}^{k} \frac{1}{x_i} = 1, \quad x_1 < x_2 < \cdots < x_k, \quad x_i \nmid x_j \text{ for } i \neq j \tag{1}$$

which was considered by the late Paul Erdös, R. L. Graham, E. G. Barbeau, A. Wm. Johnson, the author and others.

It is easily verified that no integer x_t in (1) can be a power of a prime. Hence, all values x_i in (1) must be products of at least two prime factors.

If the integers x_i in (1) are of the form $x_i = pq$ where p, q are distinct primes, then these integers yield the direct consequence that

$$x_i \nmid x_j \quad \text{for} \quad i \neq j.$$

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Thus, we may view such a case as a particular case of the general equation (1).

The existence of a solution of (1) was independently raised by the author [2, 3] and R. L. Graham [8, 3]. The author [3] provided a solution of (1), for which he received a reward of \$10 offered by Erdös [2, 3]. In [3] $k = 79$, but not all x_i are of the form $x_i = pq$ where p, q are distinct primes. The author [4] provided another example of the same nature as that in [3] with $k = 68$. Barbeau applied a stronger condition, i.e. all x_i are products of exactly two distinct primes, and exhibited in [1] an example of (1) with $k = 101$ and $x_{101} = 1838171$. The author [5] improved Barbeau's solution with $k = 63$ and $x_{63} = 7909$ where all $x_i = pq$. The results of Barbeau [1] and of the author [3] are cited in [9]. In [6, 7], a solution in which $k = 52$ and $x_i = pq$ is demonstrated. All the solutions of the author which are mentioned here are obtained without a computer. The results [3, 5, 6] are also cited in [10]. Johnson [12] exhibited an example with $k = 48$ and $x_i = pq$. This result is unpublished and seems to be a solely computerized result. It is cited in [10, 11].

In Section 2 Johnson's example [12] is exhibited when $k = 48$ and $x_i = pq$. The author's Example 1 analyzes this result, and provides a structure of the numbers which yields the sum of 1. In Section 3, the author's result in [7] demonstrates in Example 2 another structure of the numbers when $k = 52$ and $x_i = pq$. Examples 1 and 2, represent two different structures in order to achieve the sum of 1 without a computer. Finally, in Section 4, when the restriction on the values $x_i = pq$ is slightly relaxed, new Examples 3 and 4 both with $k = 51$ are exhibited.

In each of the forthcoming Examples 1-4, the sum of 1 is attained without the aid of a computer.

2. The Case $k = 48$ (Johnson's Example)

It is cited in [11] that at least thirty-eight integers are required to obtain the sum of 1 when $x_i \nmid x_j$ for $i \neq j$. A. Wm. Johnson [12] manages it with forty-eight integers x_i where $x_i = pq$ and p, q are distinct primes. The forty-eight integers in ascending order cited in [10, 11] appear as follows:

6	21	34	46	58	77	87	115	155	215	287	391
10	22	35	51	62	82	91	119	187	221	299	689
14	26	38	55	65	85	93	123	203	247	319	731
15	33	39	57	69	86	95	133	209	265	323	901

Guy asks whether this is the smallest possible set, and further mentions that Richard Stong also solved this problem, but used a larger set. No reference as such is provided. It seems that the above result was obtained by a computer.

We shall now analyze Johnson's result by introducing the concept of the "basic set of primes" which will shed a new light on the structure of the above numbers, and will enable us in particular to obtain the sum of 1 without the use of a computer.

Denote by S the "basic set of primes" which consists of the first eight smallest primes, i.e.

$$S = \{2, 3, 5, 7, 11, 13, 17, 19\},$$

and the product of the eight elements in S is

$$L = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 = 9699690$$

The above four rows of the forty-eight numbers may now be arranged in the following structure as shown in Example 1.

Example 1.

(1) $\frac{1}{2}\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}\right) = \frac{4633919}{L}$ **7** uf

(2) $\frac{1}{3}\left(\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}\right) = \frac{2011536}{L}$ **6** uf

(3) $\frac{1}{5}\left(\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}\right) = \frac{818934}{L}$ **5** uf

(4) $\frac{1}{7}\left(\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}\right) = \frac{387000}{L}$ **4** uf

(**5**) $\frac{1}{11}\left(\frac{1}{17}+\frac{1}{19}\right)$	$=\frac{98280}{L}$	**2** uf
(**6**) $\frac{1}{13}\left(\frac{1}{17}+\frac{1}{19}\right)$	$=\frac{83160}{L}$	**2** uf
(**7**) $\frac{1}{17}\left(\frac{1}{19}\right)$	$=\frac{30030}{L}$	**1** uf
(**8**) $\frac{1}{23}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{13}+\frac{1}{17}\right)$	$=\frac{493031}{L}$	**5** uf
(**9**) $\frac{1}{29}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{11}\right)$	$=\frac{356915}{L}$	**4** uf
(**10**) $\frac{1}{31}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}\right)$	$=\frac{323323}{L}$	**3** uf
(**11**) $\frac{1}{41}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{7}\right)$	$=\frac{230945}{L}$	**3** uf
(**12**) $\frac{1}{43}\left(\frac{1}{2}+\frac{1}{5}+\frac{1}{17}\right)$	$=\frac{171171}{L}$	**3** uf
(**13**) $\frac{1}{53}\left(\frac{1}{5}+\frac{1}{13}+\frac{1}{17}\right)$	$=\frac{61446}{L}$	**3** uf
	$1=\frac{\overline{9699690}}{L}$	**48** uf

where each of the above brackets contains members of S only. Furthermore, in rows (**8**)-(**13**), the denominator of the reciprocal outside the brackets is a prime which divides the numerator obtained by summing up the reciprocals inside the brackets.

Except for the product $11 \cdot 13$, the twenty-seven reciprocals of the form $\frac{1}{pq}$ which are obtained from rows (**1**)-(**7**) inclusive, consist of all the products of two different factors of L.

A hand calculator easily enables us now to obtain the thirteen partial sums appearing on the right-hand side of each row, and that all these sums add up to 1.

It is noted that the arrangement of the thirteen rows as in Example 1 is not the only such possibility. In Section 3 when $k = 52$, a different format of arrangement is employed, which yields the sum of 1 even in a simpler and easier way.

3. The Case $k = 52$

In this section, we present the author's result in [7] which contains fifty-two reciprocals satisfying (1), and each denominator is of the form pq.

As in Section 2, let T denote the "basic set of primes" which consists of the first six smallest primes, namely

$$T = \{2, 3, 5, 7, 11, 13\}.$$

The fifty-two numbers are arranged in fourteen rows as shown in Example 2, which is also unique.

Example 2.

(1')		6	10	14	15	21	35
(2')	$p = 13$	26	39	65	91		
(3')	$p = 19$	38	57	95	133		
(4')	$p = 31$	62	93	155			
(5')	$p = 41$	82	123	287			
(6')	$p = 71$	213	355	497			
(7')	$p = 11$	22	33	55	77		
(8')	$p = 17$	34	51	119	187		
(9')	$p = 29$	58	87	203	319		
(10')	$p = 53$	106	159	265	583		
(11')	$p = 61$	122	183	671			
(12')	$p = 23$	46	69	161	299		
(13')	$p = 101$	202	505	1313			
(14')	$p = 151$	453	1057	1963			

The least common multiple, in short R of the first row is $R = 2 \cdot 3 \cdot 5 \cdot 7 = 210.$ The numbers in row **(1')** are all the products of two different factors of R. The structure of each of the remaining thirteen rows is as follows. The members of each row are of the form Mp where $p \geq 11$ is a prime and all thirteen primes are distinct. Observe that except for rows **(1')** and **(7')**, in all other twelve rows the prime p indicated in front of the corresponding x_i divides the numerator of the sum of the reciprocals of that row.

The values M consist of one prime factor of: R in rows **(2')**-**(6')**, $11R$ in rows **(7')**-**(11')**, $13R$ in rows **(12')**-**(14')**. Furthermore, the sum of the reciprocals of the 52 numbers above is as follows: $S_{(\mathbf{1'})\text{-}(\mathbf{6'})} = 147/R,$ $S_{(\mathbf{7'})\text{-}(\mathbf{11'})} = 561/11R = 51/R,$ $S_{(\mathbf{12'})\text{-}(\mathbf{14'})} = 156/13R = 12/R.$ The three partial sums yield $S_{(\mathbf{1'})\text{-}(\mathbf{14'})} = 210/R = 1,$ and the desired result is obtained without the use of a computer in a simpler way.

4. The Case $k = 51$

In this section, we slightly relax the restriction that $x_i = pq$ for all values of i. We demonstrate two such examples, namely Examples 3 and 4 with $k = 51.$ Each of the examples contains two unit fractions of the form $\frac{1}{p^2 q}$, and forty-nine unit fractions of the form $\frac{1}{pq}$. This is done by using Example 1.

Consider Example 1 with its thirteen rows and $k = 48.$ Delete the triplet in row **(12)**, i.e.

$$\textbf{(12)} \quad \frac{1}{43}\left(\frac{1}{2} + \frac{1}{5} + \frac{1}{17}\right) = \frac{3}{2 \cdot 5 \cdot 17} = \frac{171171}{L}.$$

Example 1 now contains forty-five unit fractions whose sum is less than 1. Add to these forty-five unit fractions the following two triplets:

$$(\mathbf{a})\ \frac{1}{43}\left(\frac{1}{4}+\frac{1}{7}+\frac{1}{17}\right)=\frac{5}{4\cdot 7\cdot 17}=\frac{101887.5}{L},$$

$$(\mathbf{b})\ \frac{1}{83}\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{7}\right)=\frac{1}{4\cdot 5\cdot 7}=\frac{69283.5}{L},$$

which satisfy

$$(\mathbf{a})+(\mathbf{b})=\frac{101887.5}{L}+\frac{69283.5}{L}=\frac{171171}{L}=(\mathbf{12}).$$

Example 3 is now established and consists of fourteen rows as follows.

Example 3 : rows (**1**)-(**11**), row (**a**), row (**13**), row (**b**), where

$$\sum_{i=1}^{51}\frac{1}{x_i}=1,\quad x_1<x_2<\cdots<x_{51},\quad x_i \nmid x_j \text{ for } i\neq j.$$

To obtain Example 4, we proceed in the same manner as above.

In Example 1 delete the triplet in row (**11**), namely:

$$(\mathbf{11})\ \frac{1}{41}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{7}\right)=\frac{1}{2\cdot 3\cdot 7}=\frac{230945}{L}.$$

To the remaining forty-five unit fractions add the following two triplets:

$$(\mathbf{c})\ \frac{1}{47}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)=\frac{1}{3\cdot 4\cdot 5}=\frac{161661.5}{L},$$

$$(\mathbf{d})\ \frac{1}{83}\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{7}\right)=\frac{1}{4\cdot 5\cdot 7}=\frac{69283.5}{L},$$

which satisfy

$$(\mathbf{c})+(\mathbf{d})=\frac{161661.5}{L}+\frac{69283.5}{L}=\frac{230945}{L}=(\mathbf{11}).$$

Example 4 is therefore comprised of fourteen rows as follows.

Example 4: rows (**1**)-(**10**), row (**12**), row (**c**), row (**13**), row (**d**), where

$$\sum_{i=1}^{51}\frac{1}{x_i}=1,\quad x_1<x_2<\cdots<x_{51},\quad x_i \nmid x_j \text{ for } i\neq j.$$

A direct consequence of Examples 1-4 is: when the number of primes in the "basic set of primes" increases, the value of k decreases. It seems therefore that the following conjecture may be raised.

Conjecture. Let N be a "basic set of primes" containing the first t smallest primes. If $t > 8,$ then (1) has a solution when $k \leq 48.$

It is noted: an example when $k = 48$ would imply that the Johnson's Example in Section 2 is not unique, and an example when $k < 48$ will provide the answer to Guy's question in [11] (see Section 2).

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