Advances and Applications in Mathematical Sciences
Volume 21, Issue 4, February 2022, Pages 1773-1786

TWO PHASE RETRIAL QUEUE WITH STARTING FAILURES, CUSTOMER IMPATIENCE, FEEDBACK AND BERNOULLI VACATION

M. NILA and D. SUMITHA

Department of Mathematics
Avinashilingam Institute for Home Science
and Higher Education for Women
Coimbatore-641 043, Tamil Nadu, India

Department of Science and Humanities
Avinashilingam Institute for Home Science
and Higher Education for Women
Coimbatore-641 043, Tamil Nadu, India
E-mail: brightmoon02@gmail.com
sumitha4677@gmail.com

Abstract

This paper analyses batch arrival retrial queue with impatient customers, Bernoulli vacation, feedback and server subject to starting failure. The server renders two phases of heterogeneous service, first essential phase service (FEPS) and second optional phase service (SOPS). The customers balk and renege at particular times. On completion of SOPS, with certain probability the server may go for a vacation or remains idle. The customers are allowed to make a feedback after both the phases. We derive steady state solutions and performance measures for this model. We explore the various effects of system parameters numerically.

1. Introduction

Retrial queues are characterized by the fact that a customer arriving when all accessible servers are busy leaves the service area and repeats the demand after some time. This feature plays a vital role in telecommunication networks, cognitive networks, cloud computing systems, production,

2020 Mathematics Subject Classification: 60K25, 60K30, 68M20, 90B22, 60K20.
Keywords: Retrial queue, Impatient customers, Starting failures, Feedback, Bernoulli Vacation.
Received October 5, 2020; Accepted November 10, 2020

manufacturing systems etc. Single server retrial queues with starting failures are studied by Krishnakumar and Madheswari [7], Pavai Madheswari and Suganthi [10] and Ayyappan and Udaya Geetha [3]. Many retrial queues with non persistent (impatient) customers have attracted more attention. Such types of retrial models are considered by Ebenasar, et al. [4], Arivudainambi and Godhandaraman [2], and Bouchentouf and Guendouzi [1]. Some of the researchers like Jain and Upadhyaya [5], Li. et al. [8], Kalidass and Kasturi [6], Upadhyaya [11] and Pankaj Sharma [9] have developed two phase retrial queueing models with the concept of feedback and vacation. In this paper, we consider two phase retrial queueing model with customer impatience, feedback, starting failure and Bernoulli vacation. The article is organised as follows. In section 2, the prescribed model is formulated and analysed. Steady state equations are obtained in section 3. Various performance measures are discussed in section 4. Section 5, explores the numerical results for this model. Conclusion is given in section 6.

2. Model Description

Consider a single server batch arrival retrial queueing system. The customers arrive at the system according to Poisson stream of rate λ. Let Y be the batch size which is a random variable with $P\{Y = k\} = C_k$, $k = 1, 2, 3, \ldots$, $\sum_{k=1}^{\infty} C_k = 1$, and $C(z)$ is the probability generating function (PGF) with first two moments m_1 and m_2. We assume that there is no waiting line and therefore if the arriving customers find the server idle, and if the server is started successfully, one of the customer in the batch gets the service and all the other customers join the orbit. The probability of successful commencement of service is α. Otherwise the repair of the server commences and the arriving batch joins the orbit and make their retrials later. The retrial times follow a general distribution with distribution function $A(x)$, probability density function (pdf) $a(x)$ and corresponding Laplace Stieltjes transform (LST) $A^*(s)$. Repair times follow general distribution with distribution function $H(x)$. Let $h(x)$, $H^*(s)$ be the respective pdf and LST with first two moments γ_1 and γ_2. If the server is busy or unavailable, the new incoming batch may go to the orbit with probability p or leave the system with complementary probability. On account of arrival of primary customer, the retrial customer cancels his attempt for service and returns to the orbit with probability q or departs the system with complementary probability. The server provides FEPS to all the customers. It follows a general distribution with distribution function $B_0(x)$, pdf $b_0(x)$, LST $B_0^*(s)$ and first two moments μ_{01} and μ_{02}. As soon as the completion of FEPS with probability r_1 the customer move to the SOPS or move to the orbit as a feedback customer with probability β or may depart the system with probability $r_0 = (1 - \beta - r_1)$. After completing SOPS, the customer may enter to the orbit as a feedback customer for receiving the service again with probability δ or may exit the system with probability $1 - \delta$. The SOPS times follow a general distribution with distribution function $B_1(x)$, pdf $b_1(x)$ and LST $B_1^*(s)$ with first two moments μ_{11} and μ_{12}. At the completion of second optional phase service, the server may take a vacation with probability ω or waits for next service with probability $1 - \omega$. Vacation time follows a general distribution with distribution function $G(x)$, pdf $g(x)$, LST $G^*(s)$ with first two moments υ_1 and υ_2. Define the Markov process: $\{N(t);\ t \geq 0\} = \{X(t),\ C(t),\ \varepsilon_0(t),\ \varepsilon_1(t),\ \varepsilon_2(t),\ \varepsilon_3(t),\ \varepsilon_4(t);\ t \geq 0\}$, $X(t) =$ Orbit size at time t, $C(t) =$ Server state.

$$C(t) = \begin{cases} 0 \text{ if the server is idle at time } t \\ 1 \text{ if the server is busy in FEPS at time } t \\ 2 \text{ if the server is busy in SOPS at time } t \\ 3 \text{ if the server is on vacation at time } t \\ 4 \text{ if the server is on repair at time } t. \end{cases}$$

3. Steady State Distribution

Define probability densities

$$R_n(x, t)dx = P\{C(t) = 0,\ X(t) = n,\ x \leq \varepsilon_0(t) < x + dx\};\ t \geq 0,\ x \geq 0,\ n \geq 1$$

$$S_n(x, t)dx = P\{C(t) = 1,\ X(t) = n,\ x < \varepsilon_1(t) < x + dx\};\ t \geq 0,\ x \geq 0,\ n \geq 0$$

$$W_n(x, t)dx = P\{C(t) = 2,\ X(t) = n,\ x < \varepsilon_2(t) \le x + dx\};\ t \ge 0,\ x \ge 0,\ n \ge 0$$

$$G_n(x, t)dx = P\{C(t) = 3,\ X(t) = n,\ x < \varepsilon_3(t) \le x + dx\};\ t \ge 0,\ x \ge 0,\ n \ge 0$$

$$H_n(x, t)dx = P\{C(t) = 4,\ X(t) = n,\ x < \varepsilon_4(t) \le x + dx\};\ t \ge 0,\ x \ge 0,\ n \ge 1.$$

The system of steady state equations that governs the model is given below

$$\lambda R_0 = r_0 \int_0^\infty S_0(x)\mu_0(x)dx + \overline{\omega}\overline{\delta}\int_0^\infty W_0(x)\mu_1(x)dx + \int_0^\infty G_0(x)v(x)dx \tag{1}$$

$$\frac{d}{dx} R_n(x) = -[\lambda + \eta(x)]R_n(x),\ n \ge 1 \tag{2}$$

$$\frac{d}{dx} S_0(x) = -[p\lambda + \mu_0(x)]S_0(x) \tag{3}$$

$$\frac{d}{dx} S_n(x) = -(p\lambda + \mu_0(x))S_n(x) + p\lambda \sum_{k=4}^{n} C_k S_{n-k}(x),\ n \ge 1 \tag{4}$$

$$\frac{d}{dx} W_0(x) = -[p\lambda + \mu_1(x)]W_0(x) \tag{5}$$

$$\frac{d}{dx} W_n(x) = -[p\lambda + \mu_1(x)]W_n(x) + p\lambda \sum_{k=1}^{n} C_k W_{n-k}(x),\ n \ge 1, \tag{6}$$

$$\frac{d}{dx} G_0(x) = -[p\lambda + v(x)]G_0(x) \tag{7}$$

$$\frac{d}{dx} G_n(x) = -[p\lambda + v(x)]G_n(x) + p\lambda \sum_{k=1}^{n} C_k G_{n-k}(x),\ n \ge 1 \tag{8}$$

$$\frac{d}{dx} H_1(x) = -[p\lambda + \gamma(x)]H_1(x) \tag{9}$$

$$\frac{d}{dx} H_n(x) = -[p\lambda + \gamma(x)]H_n(x) + p\lambda \sum_{k=1}^{n} C_k H_{n-k}(x),\ n \ge 2. \tag{10}$$

With boundary conditions

$$R_n(0) = r_0\int_0^\infty S_n(x)\mu_0(x) + \overline{\omega}\overline{\delta}\int_0^\infty W_n(x)\mu_1(x)dx + \beta\int_0^\infty S_{n-1}(x)\mu_0(x)dx$$

$$+ \overline{\omega}\delta\int_0^\infty W_{n-1}(x)\mu_1(x)dx + \int_0^\infty H_n(x)\gamma(x)dx + \int_0^\infty G_n(x)\upsilon(x)dx, \; n \geq 1 \quad (11)$$

$$S_0(0) = \alpha\lambda c_1 R_0 + \alpha\int_0^\infty R_1(x)\eta(x)dx + \alpha\lambda\overline{q}\int_0^\infty R_1(x)dx \quad (12)$$

$$S_n(0) = \alpha\lambda c_{n+1}R_0 + \alpha\int_0^\infty R_{n+1}(x)\eta(x)dx + \alpha\lambda\overline{q}\sum_{k=1}^{n+1} C_k\int_0^\infty R_{n-k+2}(x)dx$$

$$\alpha\lambda + q\sum_{k=1}^{n} C_k\int_0^\infty R_{n-k+1}(x)dx; \; n \geq 1 \quad (13)$$

$$W_n(0) = r_1\int_0^\infty S_n(x)\mu_0(x)dx; \; n \geq 0 \quad (14)$$

$$G_0(0) = \omega\overline{\delta}\int_0^\infty W_0(x)\mu_1(x)dx \quad (15)$$

$$G_n(0) = \omega\overline{\delta}\int_0^\infty W_n(x)\mu_1(x)dx + \omega\delta\int_0^\infty W_{n-1}(x)\mu_1(x)dx; \; n \geq 1 \quad (16)$$

$$H_1(0) = \overline{\alpha}\lambda R_0 + \overline{\alpha}\int_0^\infty R_1(x)\eta(x)dx \quad (17)$$

$$H_n(0) = \overline{\alpha}\lambda\sum_{k=1}^{n} C_k\int_0^\infty R_{n-k}(x)dx + \overline{\alpha}\int_0^\infty R_n(x)\eta(x)dx, \; n \geq 2. \quad (18)$$

The normalizing condition is

$$R_0 + \sum_{n=1}^\infty\int_0^\infty R_n(x)dx + \sum_{n=0}^\infty\int_0^\infty S_n(x)dx + \sum_{n=0}^\infty\int_0^\infty W_n(x)dx$$

$$+ \sum_{n=0}^\infty\int_0^\infty G_n(x)dx + \sum_{k=1}^\infty\int_0^\infty H_n(x)dx = 1. \quad (19)$$

Define the probability generating functions

$$R(x,\ z) = \sum_{n=1}^{n} R_n(x)z^n,\ S(x,\ z) = \sum_{n=0}^{\infty} S_n(x)z^n,\ H_n(x,\ z) = \sum_{n=0}^{\infty} H(x,\ z)$$

$$H_n\ (x)z^n,\ W(x,\ z) = \sum_{n=0}^{\infty} W_n(x)z^n,\ G(x,\ z) = \sum_{n=0}^{\infty} G_n(x)z^n.$$

Multiplying equations (1)-(18) by z^n and summing over n, $n = 0, 1, 2, 3, \ldots$ we obtain the following partial differential equations.

$$\left(\frac{d}{dx} + \lambda + \eta(x)\right)R(x,\ z) = 0 \tag{20}$$

$$\left(\frac{d}{dx} + p\lambda(1 - c(z)) + \mu_0(x)\right)S(x,\ z) = 0 \tag{21}$$

$$\left(\frac{d}{dx} + p\lambda(1 - c(z)) + \mu_1(x)\right)W(x,\ z) = 0 \tag{22}$$

$$\left(\frac{d}{dx} + p\lambda(1 - c(z)) + \upsilon(x)\right)G(x,\ z) = 0 \tag{23}$$

$$\left(\frac{d}{dx} + p\lambda(1 - c(z)) + \gamma(x)\right)H(x,\ z) = 0. \tag{24}$$

Solving the equations (20)-(24), we get

$$R(x,\ z) = R(0,\ z)e^{-\lambda x}(1 - A(x)) \tag{25}$$

$$S(x,\ z) = S(0,\ z)e^{-p\lambda[1-c(z)]x}(1 - B_0(x)) \tag{26}$$

$$W(x,\ z) = W(0,\ z)e^{-p\lambda[1-c(z)]x}(1 - B_1(x)) \tag{27}$$

$$G(x,\ z) = G(0,\ z)e^{-p\lambda[1-c(z)]x}(1 - G(x)) \tag{28}$$

$$H(x,\ z) = H(0,\ z)e^{-p\lambda[1-c(z)]x}(1 - H(x)) \tag{29}$$

$$R(0,\ z) = (r_0 + \beta z)\int_0^{\infty} S(x,\ z)\mu_0(x)dx + (\bar{\delta} + \delta z)\bar{\omega}\int_0^{\infty} W(x,\ z)\mu_1(x)dx$$

$$+ \int_0^{\infty} G(x,\ z)\upsilon(x)dx + \int_0^{\infty} H(x,\ z)\gamma(x)dx - \lambda R_0 \tag{30}$$

$$S(0, z) = \frac{\alpha c(z)\lambda R_0}{z} + \frac{\alpha}{z}\int_0^\infty R(x, z)\eta(x)dx + \frac{\lambda\alpha\bar{q}}{z^2}$$

$$\int_0^\infty R(x, z)c(z)dx + \frac{\lambda\alpha q}{z}\int_0^\infty R(x, z)c(z)dx \tag{31}$$

$$W(0, z) = r_1\int_0^\infty S(x, z)\mu_0(x)dx \tag{32}$$

$$G(0, z) = (\bar{\delta} + \delta z)\omega\int_0^\infty W(x, z)\mu_1(x)dx \tag{33}$$

$$H(0, z) = \bar{\alpha}\lambda\int_0^\infty R(x, z)c(z)dx + \bar{\alpha}\int_0^\infty R(x, z)\eta(x)dx + \bar{\alpha}\lambda R_0 c(z). \tag{34}$$

Solving equations (30)-(34) we obtain,

$$R(0, z) = S(0, z)K_1(z) - R_0\lambda\{(1 - \bar{\alpha}c(z)H^*(p\lambda - p\lambda c(z)))\}$$

$$/1 - Q_3(z)\bar{\alpha}H^*(p\lambda - p\lambda c(z)) \tag{35}$$

$$S(0, z) = \alpha\lambda R_0\{zc(z)(1 - \bar{\alpha}H^*(p\lambda - p\lambda c(z))Q_3(z)) - Q_1(z)$$

$$(1 - \bar{\alpha}c(z)H^*(p\lambda - p\lambda c(z)))\}/D(z) \tag{36}$$

$$W(0, z) = \alpha\lambda R_0 r_1 B_0^*(p\lambda - p\lambda c(z))\{zc(z)(1 - \bar{\alpha}H^*(p\lambda - p\lambda c(z))$$

$$Q_3(z)) - Q_1(z)(1 - \bar{\alpha}C(z)H^*(p\lambda - p\lambda c(z)))\}/D(Z) \tag{37}$$

$$G(0, z) = \alpha\lambda R_0 r_1\omega(\bar{\delta} + \delta z)B_0^*(p\lambda - p\lambda c(z))B_1^*(p\lambda - p\lambda c(z))$$

$$\{zc(z)(1 - \bar{\alpha}H^*(p\lambda - p\lambda c(z))$$

$$Q_3(z)) - Q_1(z)(1 - \bar{\alpha}c(z)H^*(p\lambda - p\lambda c(z)))\}/D(z) \tag{38}$$

$$H(0, z) = R(0, z)\bar{\alpha}Q_3(z) + \bar{\alpha}\lambda c(z)R_0 \tag{39}$$

where

$$D(z) = z^2\{1 - \bar{\alpha}Q_3(z)H^*(p\lambda - p\lambda c(z))\} - \alpha\{zA^*(\lambda) + c(z)(1 - A^*(\lambda)(\bar{q} + qz)\}K_1(z)$$

$$Q_1(z) = zA^*(\lambda) + c(z)(1 - A^*(\lambda))(\overline{q} + qz)$$

$$K_1(z) = B_0^*(p\lambda - p\lambda c(z))[r_0 + \beta z] + r_1 B_0^*(p\lambda - p\lambda c(z)) B_1^*(p\lambda - p\lambda c(z))$$

$$(\delta z + \overline{\delta})(\overline{\omega} + \omega G^*(p\lambda - p\lambda c(z)))$$

$$Q_3(z) = c(z) + (1 - c(z))A^*(z)$$

Inserting the expressions (35)-(39) in the equations (25)-(29) and integrating with respect to x from 0 to ∞ we obtain $R(z)$, $S(z)$, $W(z)$, $G(z)$, $H(z)$.

$$R(z) = R_0(1 - A^*(\lambda))z\{\alpha c(z)K_1(z) - z(1 - \overline{\alpha}c(z)H^*(p\lambda - p\lambda c(z)))\}/D(z)$$

$$S(z) = \alpha R_0\{1 - B_0^*(p\lambda - p\lambda c))\}\{zc(z)(1 - \overline{\alpha}H^*(p\lambda - p\lambda c(z))Q_3(z))$$

$$- Q_1(z)(1 - \overline{\alpha}C(z)H^*(p\lambda - p\lambda c(z)))\}/D(Z)(p - pC(z))$$

$$W(z) = \alpha R_0 r_1 B_0^*(p\lambda - p\lambda C(z))\{1 - B_0^*(p\lambda - p\lambda C(z))\}\{zC(z)$$

$$(1 - \overline{\alpha}H^*(p\lambda - p\lambda c(z))Q_3(z))$$

$$- Q_1(z)(1 - \overline{\alpha}C(z)H^*(p\lambda - p\lambda c(z)))\}/D(Z)(p - pC(z))$$

$$G(z) = \alpha R_0 r_1 \omega B_0^*(p\lambda - p\lambda c(z))\, B_1^*(p\lambda - p\lambda c(z))$$

$$(1 - G^*(p\lambda - p\lambda c(z)))(\delta z + \overline{\delta})$$

$$\{zc(z)(1 - \overline{\alpha}H^*(p\lambda - p\lambda c(z))Q_3(z)) - Q_1(z)(1 - \overline{\alpha}c(z)$$

$$H^*(p\lambda - p\lambda c(z)))\}/D(z)(p - pc(z))$$

$$H(z) = \overline{\alpha}R_0(1 - H^*(p\lambda - p\lambda c(z)))\{\alpha c(z)[zQ_3(z) - Q_1(z)]$$

$$K_1(z) - z^2(1 - c(z))A^*(\lambda)\}/D(z)(p - pc(z)).$$

4. Performance Measures

The probability that the server is idle during the retrial time is

$$R = R_0(1 - A^*(\lambda))\{\alpha[p\lambda m_1 N + \beta + r_1\beta + r_1\delta - 1] + \alpha p\lambda m_1\gamma_1 + m_1\}/D_1.$$

The probability that the server is busy in FEPS is given by

$$S = \alpha\lambda R_0\mu_{01}[m_1 A^*(\lambda) + \alpha\bar{q}(1 - A^*(\lambda))]/D_1.$$

The probability that the server is busy in SOPS is given by

$$W = \lambda R_0\alpha r_1\mu_{11}[m_1 A^*(\lambda) + \alpha\bar{q}(1 - A^*(\lambda))]/D_1.$$

The probability that the server is on vacation is given by

$$G = \alpha\lambda R_0 r_1\omega v_1[m_1 A^*(\lambda) + \alpha\bar{q}(1 - A^*(\lambda))]/D_1.$$

The probability that the server is under repair is given by

$$H = (1 - \alpha)\lambda R_0\gamma_1[m_1 A^*(\lambda) + \alpha\bar{q}(1 - A^*(\lambda))]/D_1.$$

Where

$$D_1 = \alpha\{2 - (p\lambda m_1 N + \beta + r_1\delta) - [A^*(\lambda) + (1 - A^*(\lambda))(m_1 + q)]\} - p\lambda m_1\gamma_1\bar{\alpha}$$

$$- \bar{\alpha}m_1(1 - A^*(\lambda)).$$

$$N = \mu_{01} + r_1\mu_{11} + \omega r_1 v_1.$$

Substituting the expressions of R, S, W, G, H in the normalizing condition (19) and solving we obtain as

$$R_0 = D_1/\alpha\{1 - q(1 - q(1 - A^*(\lambda)) - A^*(\lambda)(p\lambda m_1 N + \beta + r_1\delta)\} - A^*(\lambda)\bar{\alpha}p\lambda m_1\gamma_1.$$

$$+ (\alpha\lambda N + \bar{\alpha}\lambda\gamma_1)[m_1 A^* + \alpha\bar{q}(1 - A^*(\lambda))].$$

The PGF of the number of customers in the orbit is

$$P_q(z) = R_0 + R(z) + S(z) + W(z) + G(z) + H(z)$$

$$P_q(z) = R_0\{zA^*(\lambda)(1 - c(z))[p(z - \alpha K_1(z)c(z)) - \bar{\alpha}z(1 - (1 - p)H^*(p\lambda - p\lambda c(z))]$$

$$+ \alpha(zc(z) - Q_1(z))[1 - K_1(z)(1 - p(1 - c(z))) - (1 - z)B_0^*(p\lambda - p\lambda c(z))$$

$$(\beta + r_1\delta B_1^*(p\lambda - p\lambda c(z))] + \alpha\bar{\alpha}c(z)(zQ_3(z) - Q_1(z))[K_1(z)$$

$$- H^*(p\lambda - p\lambda c(z)) + (1 - z)$$

$$B_0^*(p\lambda - p\lambda c(z))H^*(p\lambda - p\lambda c(z))(\beta + r_1\delta B_1^*(p\lambda - p\lambda c(z))]\}/D(z)(p - pc(z)).$$

The PGF of the number of customers in the system is

$$P_s(z) = R_0 + R(z) + zS(z) + zW(z) + G(z) + H(z).$$

The mean number of customer in the orbit is

$$L_q = \lim_{z\to 1} \frac{d}{dz} P_q(z) = \frac{N_4N_3 - N_5N_2}{3N_4^2}$$

$$N_2 = 2R_0\{m_1 A^*(\lambda)[\alpha p(p\lambda m_1 N + \beta + r_1\delta) - 1) - \overline{\alpha}(1-p)p\lambda m_1\gamma_1 + \alpha pm_1\}$$

$$\alpha\{(p\lambda m_1 N + pm_1)[A^*(\lambda) + (1 - A^*(\lambda))(m_1 + q) - (m_1 + 1)\}$$

$$+ \alpha\overline{\alpha}\overline{q}(1 - A^*(\lambda))p\lambda m_1(N - \gamma_1)\}$$

$$N_3 = 3R_0\{\alpha pm_1A^*(\lambda)[m_2 + 2m_1M + k_2] - \overline{\alpha}A^*(\lambda)(1-p)[m_1(p^2\lambda^2m_1^2\gamma_2$$

$$+ p\lambda m_2\gamma_1 + 2p\lambda m_1\gamma_1) + (m_2 + 2m_1)p\lambda m_1\gamma_1] + \alpha pA^*(\lambda)(m_2 + 2m_1)$$

$$[M + m_1 - 1] + \alpha(p\lambda m_1N + pm_1)[(1 - A^*(\lambda))(m_2 + 2m_1q) - (m_2 + 2m_1)]$$

$$+ \alpha[(A^*(\lambda) + (1 - A^*(\lambda))(m_1 + q) - (m_1 + 1)][2pm_1M + (p\lambda m_2N + pm_2)$$

$$+ p^2\lambda^2m_1^2\{\mu_{02} + r_1\mu_{12} + r_1\omega_2v_2\} + 2r_1p^2\lambda^2m_1^2(\mu_{01}\mu_{11} + \omega\mu_{01}v_1 + \omega\,\mu_{11}v_1)$$

$$+ 2r_1\delta\omega p\lambda m_1v_1] + \alpha\overline{\alpha}\overline{q}(1 - A^*(\lambda))[p^2\lambda^2m_1^2(\mu_{02} + r_1\mu_{12} + r_1\omega v_2)$$

$$+ 2r_1p^2\lambda^2m_1^2\{\mu_{01}\mu_{11} + \omega\mu_{01}v_1 + \omega\mu_{11}v_1) + p\lambda m_2N + 2r_1\delta\omega p\lambda m_1v_1$$

$$- 2p\lambda m_1\gamma_1(\beta + r_1\delta) + 4m_1p\lambda m_1(N - \gamma_1) - (p^2\lambda^2m_1^2\gamma_2 + p\lambda m_2\gamma_1)]\}$$

$$N_4 = -2pm_1D_1$$

$$N_5 = -3p[D_1m_2 + D_2m_1]$$

where

$$D_2 = 2\alpha[1 - m_1q(1 - A^*(\lambda)) - (A^*(\lambda) + (1 - A^*(\lambda))(m_1 + q))(p\lambda m_1N + \beta + r_1\delta)]$$

$$- \overline{\alpha}p\lambda\gamma_1[m_2 + 4m_1 + 2m_1^2(1 - A^*(\lambda))]$$

$$- (1 - A^*(\lambda))(m_2 + 4\overline{\alpha}m_1) - \overline{\alpha}p^2\lambda^2m_1^2\gamma_2 - \alpha k_2$$

$$k_2 = [p^2\lambda^2 m_1^2(\mu_{02} + r_1\mu_{12} + r_1\omega v_2) + 2r_1 p^2\lambda^2 m_1^2(\mu_{01}\mu_{11} + \omega\mu_{01}v_1 + \omega\mu_{11}v_1)$$
$$+2p\lambda m_1(\beta\mu_{01} + r_1\delta(\mu_{01} + \mu_{11} + \omega v_1)) + p\lambda m_2 N]$$

$$M = p\lambda m_1 N + \beta + r_1\delta.$$

The Mean number of customer in the system is given by

$$L_s = \lim_{z\to 1}\frac{d}{dz}p_s(z)$$
$$= L_q + S + W.$$

5. Numerical Results

Numerical evaluation of the effect of various system parameters on the performance measures are done and the results are shown in tables. Here we assume that retrial time, first essential phase service time, second optional phase service time and vacation time follow exponential distribution with rates η, μ_0, μ_1 and v. We choose the following parameters.

$$\lambda = 1.5;\ p = 0.4;\ q = 0.6;\ t = 0.5;\ a = 0.4;\ c = 0.4;\ b = 0.2;\ c_1 = 0.5,$$

$$c_2 = 0.5;\ g = 0.4;\ h = 0.6;\ f = 0.8;\ \mu_0 = 20;\ \mu_1 = 10;\ v = 10;\ r = 10;\ \eta = 20.$$

Performance measures such as R_0 the probability that the system is empty, R-the probability that the server is idle in nonempty system, S-the probability that the server is busy in essential service, W-the probability that the server is busy in optional service, G-the probability that the server is on vacation and L_s-the mean system size are computed for different values of λ, μ_0, η, p and q and are given respectively in tables (1-5).

The values predict that

1. Increase in λ decrease R_0 and increases all other performance measures.

2. Increase in μ_0 decreases S and L_s and increases in R_0, W, G

3. Increase in η increases R_0, W, G and G and decreases in R and L_s

4. Increase in p and q decreases R_0 and increases all other performance measures.

Table 1. Performance measures versus λ.

λ	R_0	R	S	W	G	L_s
1.1	0.6265	0.0912	0.1046	0.0836	0.0418	0.6115
1.3	0.5722	0.1053	0.1194	0.0956	0.0478	0.7431
1.5	0.5212	0.1187	0.1334	0.1067	0.0533	0.8859
1.7	0.4732	0.1316	0.1464	0.1171	0.0586	1.0429
1.9	0.4278	0.1439	0.1586	0.1269	0.0634	1.2179

Table 2. Performance measures versus μ_0.

μ_0	R_0	R	S	W	G	L_s
20	0.5212	0.1187	0.1334	0.1067	0.0533	0.8859
25	0.5413	0.1198	0.1084	0.1084	0.0542	0.8313
30	0.5550	0.1206	0.0914	0.1096	0.0548	0.7955
35	0.5650	0.1212	0.0789	0.1105	0.0553	0.7703
40	0.5726	0.1216	0.0695	0.1112	0.0556	0.7514

Table 3. Performance measures versus η.

η	R_0	R	S	W	G	L_s
20	0.5212	0.1187	0.1334	0.1067	0.0533	0.8859
22	0.5308	0.1083	0.1337	0.1069	0.0535	0.8545
24	0.5388	0.0996	0.1339	0.1071	0.0536	0.8290
26	0.5457	0.0922	0.1341	0.1073	0.0537	0.8079
28	0.5516	0.0858	0.1343	0.1075	0.0537	0.7900

Table 4. Performance measures versus p.

p	R_0	R	S	W	G	L_s
0.2	0.5579	0.1059	0.1245	0.0996	0.0498	0.7060
0.4	0.5212	0.1187	0.1334	0.1067	0.0533	0.8859
0.6	0.4790	0.1335	0.1435	0.1148	0.0574	1.1414
0.8	0.4298	0.1507	0.1554	0.1243	0.0622	1.5235
1.0	0.3717	0.1710	0.1694	0.1355	0.0677	2.1383

Table 5. Performance measures versus q.

q	R_0	R	S	W	G	L_s
0.2	0.5345	0.1141	0.1302	0.1041	0.0521	0.8188
0.4	0.5280	0.1163	0.1317	0.1054	0.0527	0.8511
0.6	0.5212	0.1187	0.1334	0.1067	0.0533	0.8859
0.8	0.5142	0.1212	0.1351	0.1080	0.0540	0.9235
1.0	0.5069	0.1237	0.1368	0.1095	0.0547	0.9641

6. Conclusion

In this paper, we analysed batch arrival retrial queue with second optional service, starting failure, Bernoulli vacation, feedback and impatient customers. The PGF of system size when it is idle, busy on FEPS or SOPS, repair or on vacation is found by supplementary variable technique. Performance measures like the probability of the server is idle, busy on both phases, under repair or on vacation, mean orbit size, mean system size are derived. The numerical results are analysed for this model.

References

[1] Amina Angelika Bouchentouf and Abdelhak Guendouzi, The $M^x/M/C$ Bernoulli Feedback queue with variant multiple working vacations and impatient customers, Performance and economic analysis, Arabian Journal of Mathematics, http://doi.org/10.1007/s40065-019-0260-x, (2019).

[2] D. Arivudainambi and P. Godhandaraman, Retrial queueing system with balking, optional service and vacation, Annals of Operations Research 229(1) (2015), 67-84.

[3] G. Ayyappan and J. Udaya Geetha, Analysis of pre-emotive priority retrial queueing system with starting failures, modified Bernoulli vacation with vacation interruption, repair, immediate feedback and impatient customers, International Journal of Mathematics and Applications 6(1-E) (2018), 1013-1023.

[4] Ebenasar Anna Bagyam, K. Udaya Chandrika and K. Prakash Rani, Bulk arrival two phase retrial queueing system with impatient customers, orbital search, active breakdown and delayed repair, International Journal of Computer Applications 73(11) (2013), 13-17.

[5] M. Jain and S. Upadhyaya, Optimal repairable $M^x/G/1$ queue with multioptional services and Bernoulli vacation, International Journal of Operational Research 7(1) (2010), 109-132.

[6] K. Kalidass and R. Kasturi, A two phase service $M/G/1$ queue with a finite number of immediate Bernoulli feedbacks, OPSEARCH 51(2) (2014), 201-218.

[7] K. B. Krishna Kumar and S. V. Madheswari, $M^x/G/1$ retrial queue with multiple vacations and starting failures, OPSEARCH 40(2) (2013), 115-137.

[8] T. Li, Z. Wang and Z. Liu, Geo/Geo/1 retrial queue with working vacations and vacation interruption, Journal of Applied Mathematics computing 39 (2012), 131-143.

[9] Pankaj Sharma, $M/G/1$ Retrial queueing system with Bernoulli feedback and modified vacation, International Journal of Mathematics Trends and Technology 61(1) (2018), 10-21.

[10] S. Pavai Madheswari and P. Sugathi, An $M/G/1$ retrial queue with second optional service and starting failure under modified Bernoulli vacation, Transylvanian Review 24(10) (2016), 1602-1621.

[11] S. Upadhyaya, Performance prediction of a discrete time batch arrival retrial queue with Bernoulli feedback, Application of Mathematical Modeling 283 (2016), 108-119.