Advances and Applications in Mathematical Sciences
Volume 21, Issue 11, September 2022, Pages 6205-6212

COMMON FIXED POINT THEOREMS FOR MULTIVALUED CONTRACTIVE MAPPINGS IN FUZZY METRIC SPACES

S. MAHARASI[1], G. UTHAYA SANKAR[2] and B. REVATHY[3]

[1]Assistant Professor
Department of Mathematics
Sri Parasakthi College for Women
Courtallam - 627802, Tamilnadu, India
E-mail: maha.rasi120@gmail.com

[2]Assistant Professor
Department of Mathematics
Manonmaniam Sundaranar University
College, Naduvakurichi, Sankarankoil
Tamilnadu, India - 627 862
E-mail: uthayaganapathy@yahoo.com

[3]Assistant Professor
Department of Mathematics
Sri Sarada College for Women
Tirunelveli 627011, Tamilnadu, India
E-mail: thaymmalb1983@gmail.com

Abstract

Zadeh introduced the concept of fuzzy sets in 1965. Kramosil and Michalek introduced the concept of fuzzy metric spaces in terms of t-norm. George and Veeramani modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek and defined the Hausdorff topology of fuzzy metric spaces. Kubiaczyk and Sushil Sharma introduced the notion of multivalued mappings for fuzzy metric space in the sense of Kramosil and Michalek. In this paper, we prove some common fixed point theorems for multivalued mappings in complete fuzzy metric spaces.

2020 Mathematics Subject Classification: 54H25, 47H10.
Keywords: Fuzzy metric spaces, Common fixed point.
Received January 12, 2022; Accepted March 5, 2022

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [7] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. Kramosil and Michalek [4] introduced the concept of fuzzy metric space and modified by George and Veeramani [2]. Many authors proved fixed point theorems for contractive maps in fuzzy metric space. Kubiaczyk and Sharma [5] introduced the following concept of multivalued mappings in fuzzy metric space in the sense of Kramosil and Michalek [4].

Let $CB(X)$ denote the set of all non empty closed and bounded subsets of fuzzy metric space X. Then for every $A, B \in CB(X)$ and $t > 0$, $M^{\nabla}(A, y, t) = \max\{M(x, y, t) : x \in A\}$ and $M^{\nabla}(A, B, t) = \min\{\min_{x \in A} M^{\nabla}(x, B, t), \min_{y \in B} M^{\nabla}(A, y, t)\}$. In this paper, we prove some common fixed point theorems for multivalued mappings in complete fuzzy metric spaces.

2. Preliminaries

Definition 2.1. A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, $\forall\, x, y, z \in X$ and $s, t > 0$

(1) $M(x, y, t) > 0$

(2) $M(x, y, t) = 1$ for all $t > 0 \leftrightarrow x = y$,

(3) $M(x, y, t) = M(y, x, t)$,

(4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous,

(6) $\lim_{i \to \infty} M(x, y, t) = 1$

Example 2.2. Let (X, d) be a metric space. Define $a * b = ab$ (or $a * b = \min\{a, b\}$) and for all $x, y \in X, t > 0$

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M, *)$ is a fuzzy metric space. It is called fuzzy metric space induced by the metric d.

Definition 2.3. Let $(X, M, *)$ be a fuzzy metric space and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ is said to be convergent to a point $x \in X$ if limn $\lim_{n\to\infty} M(x_n, x, t) = 1 \ \forall\ t > 0$.

(ii) $\{x_n\}$ is called Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p} + x_n\ t)$ $= 1 \ \forall\ t > 0$ and $p > 0$.

(iii) A fuzzy metric space X is said to be complete if every Cauchy sequence in X is convergent to a point in it.

Lemma 2.4. *Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. If there exists a number $0 < k < 1$ such that $M(x_{n+2}, x_{n+1}, kt) \geq (x_{n+1}, x_n, t)$, $\forall\ t > 0$ and $n = 1, 2, 3, \ldots,$ then $\{x_n\}$ is a Cauchy sequence in X.*

Lemma 2.5. *Let $(X, M, *)$ be a fuzzy metric space and if there exists a number $0 < k < 1$ such that $M(x, y, kt) \geq M(x, y, t) \ \forall\ x, y, \in X$ and $t > 0$ then $x = y$.*

3. Main Results

Theorem 3.1. *Let $(X, M, *)$ be a complete fuzzy metric space and $S, T : X \to CB(X)$ be multi-valuedmaps satisfying the condition*

$$M^{\nabla}(Sx, Ty, t) \geq \min\{M(x, y, t/k), M^{\nabla}(x, Sx, t/k), M^{\nabla}(x, Ty, t/k),$$

$$\frac{M^{\nabla}(x, Sx, t/k) \cdot M^{\nabla}(y, Ty, t/k)}{M(x, y, t/k)}\}$$

for all $t > 0, x, y \in X$ and $0 < k < 1$. Then S and T have a unique common fixed point.

Proof of Theorem 3.1. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$ for $n = 0, 1, 2, \ldots$

For $n \geq 0$, we have

$$\geq \min\{M(x_{2n}, x_{2n+1}t/k), M^{\nabla}(x_{2n}, Sx_{2n}, t/k), M^{\nabla}(x_{2n+1}, Tx_{2n+1}, t/k),$$

$$\frac{M^{\nabla}(x_{2n}, Sx_{2n}, t/k) \cdot M^{\nabla}(x_{2n+1}, Tx_{2n+1}, t/k)}{M(x_{2n}, x_{2n+1}, t/k)}\}$$

$$\geq \min\{M(x_{2n}, x_{2n+1}, t/k), M(x_{2n}, x_{2n+1}, t/k), M(x_{2n+1}, x_{2n+2}, t/k),$$

$$\frac{M(x_{2n}, x_{2n+1}, t/k) \cdot M(x_{2n+1}, x_{2n+2}, t/k)}{M(x_{2n}, x_{2n+1}, t/k)}\}$$

$$\geq \min\{M(x_{2n}, x_{2n+1}, t/k), M(x_{2n-1}, x_{2n+2}, t/k), M(x_{2n+1}, x_{2n+2}, t/k)\}$$

$$= M(x_{2n}, x_{2n+1}, t/k)$$

Therefore, $M(x_{2n-1}, x_{2n+2}, t) \geq M(x_{2n}, x_{2n+1}, t/k)$ Similarly, $M(x_{2n-2}, x_{2n+3}, t) \geq M(x_{2n+1}, x_{2n+2}, t/k)$ and Hence $M(x_{2n-1}, x_{2n+2}, t) \geq M(x_n, x_{2n+1}, t/k)$, for all n. By Lemma 2.4, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, sequence $\{x_n\} \to x$. Hence $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are all converges to x. Now we have $M^{\nabla}(Sx, x_{2n+2}, t) \leq (Sx, Tx_{2n+1}, t)$

$$\geq \min\{M(x, x_{2n+1}, t/k), M^{\nabla}(x, Sx, t/k), M^{\nabla}(x_{2n+1}, Tx_{2n+1}, t/k),$$

$$\frac{M^{\nabla}(x, Sx, t/k) \cdot M^{\nabla}(x_{2n+1}, Tx_{2n+1}, t/k)}{M(x, x_{2n+1}, t/k)}\}$$

$$\geq \min\{M(x, x_{2n+1}, t/k), M^{\nabla}(x, Sx, t/k), M(x_{2n+1}, x_{2n+2}t/k),$$

$$\frac{M^{\nabla}(x, Sx, t/k) \cdot M(x_{2n+1}, x_{2n+2}, t/k)}{M(x, x_{2n+1}, t/k)}\}$$

Letting $n \to \infty$ we obtain

$$M^{\nabla}(Sx, x, t) \geq \min\{1, M^{\nabla}(x, Sx, t/k), 1, M^{\nabla}(x, Sx, t/k)\} = M^{\nabla}(x, Sx, t/k).$$

Hence $M^{\nabla}(Sx, x, t) \geq M^{\nabla}(Sx, x, t/k)$, for each $t > 0$. Therefore by Lemma 2.5, $x \in Sx$. Similarly, $x \in Tx$ Hence x is a common fixed point of S and T.

Uniqueness:

Let us take y be another common fixed point of S and T other than x. Then

$$M(x, y, t) \geq M^{\nabla}(Sx, Ty, t)$$

$$\geq \min\{M(x, y, t/k), M^{\nabla}(x, Sx, t/k), M^{\nabla}(y, Ty, t/k),$$

$$\frac{M^{\nabla}(x\ Sx, t/k) \cdot M^{\nabla}(y, Ty, t/k)}{M(x, y, t/k)}\}$$

$$= \min\{M(x, y, t/k), 1, 1, [1/M(x, y, t/k) = M(y, y, t/k)$$

Hence $M(x, y, t) \geq M(x, y, t/k)$.

Therefore by Lemma 2.5, $x = y$. This completes the proof.

Theorem 3.2. *Let* $(X, M, *)$ *be a complete fuzzy metric space and* $S, T : X \to CB(X)$ *be multi-valuedmaps satisfying the condition*

$$4M_{\nabla}(Sx, Ty, t) \geq \{M(x, y, t/k) + M^{\nabla}(x, Sx, t/k) + M^{\nabla}(y, Ty, t/k)$$

$$+ \frac{M^{\nabla}(x\ Sx, t/k) \cdot M^{\nabla}(y, Ty, t/k)}{M(x, y, t/k)}\}$$

for all $t > 0, x, y \in X$ *and* $0 < k < 1$. *Then* S *and* T *have a unique common fixed point*.

Proof of Theorem 3.2. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$ for $n = 0, 1, 2, \ldots$. For $n \geq 0$, we have

$$4M(x_{2n+1}, x_{2n+2}, t) \geq 4M_{\nabla}(Sx_{2n}, Tx_{2n+1}, t)$$

$$\geq \min\{M(x_{2n}, x_{2n+1}, t/k) + M^{\nabla}(x_{2n}, Sx_{2n}, t/k) + M^{\nabla}(x_{2n+1}, Tx_{2n+1}t/k),$$

$$+ \frac{M(x_{2n}, x_{2n+1}, t/k) \cdot M(x_{2n+1}, x_{2n+2}, t/k)}{M(x_{2n}, x_{2n+1}, t/k)}\}$$

$$\geq \{M(x_{2n},\, x_{2n+1},\, t/k) + M(x_{2n},\, x_{2n+1},\, t/k) + M(x_{2n+1},\, x_{2n+2},\, t/k),$$

$$+ \frac{M(x_{2n},\, x_{2n+1},\, t/k) \cdot M(x_{2n+1},\, x_{2n+2},\, t/k)}{M(x_{2n},\, x_{2n+1},\, t/k)}\}$$

$$= \{2M(x_{2n+1},\, x_{n+1},\, t/k) + 2M(x_{2n+1},\, x_{2n+2},\, t/k)\}$$

Therefore, $2M(x_{2n+1},\, x_{2n+2},\, t) \geq 2M(x_{2n},\, x_{2n+1},\, t/k)$. Similarly, $M(x_{2n+2}, x_{2n+3}, t) \geq M(x_{2n+1}, x_{2n+2}, t/k)$ and $M(x_{n+1},\, x_{n+2},\, t) \geq M(x_{2n},\, x_{2n+1},\, t/k)$, for all n. By Lemma 2.4, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, sequence $\{x_n\} \to x$. Hence $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are all converges to x.

Now we have

$$4M^{\nabla}(Sx,\, x_{2n+2},\, t) \geq 4M_{\nabla}(Sx,\, Tx_{2n+1},\, t/k)$$

$$\geq \{M(x,\, x_{2n+1},\, t/k) + M^{\nabla}(x,\, Sx,\, t/k) + M^{\nabla}(x_{2n+1},\, Tx_{2n+1},\, t/k)$$

$$+ \frac{M(x,\, Sx,\, t/k) \cdot M^{\nabla}(x_{2n+1},\, Tx_{2n+1},\, t/k)}{M(x,\, x_{2n+1},\, t/k)}\}$$

$$\geq \{M(x,\, x_{2n+1},\, t/k) + M^{\nabla}(x,\, Sx,\, t/k) + M(x_{2n+1},\, x_{2n+2},\, t/k)$$

$$+ \frac{M^{\nabla}(x,\, Sx,\, t/k) \cdot M(x_{2n+1},\, Tx_{2n+2},\, t/k)}{M(x,\, x_{2n+1},\, t/k)}\}$$

Letting $n \to \infty$ we obtain

$$4M^{\nabla}(Sx,\, x,\, t) \geq \{1 + M^{\nabla}(x,\, Sx,\, t/k) + 1 + M^{\nabla}(x,\, Sx,\, t/k)\}$$

$$= \{2 + 2M^{\nabla}(x,\, Sx,\, t/k)\} \geq \{2 + 2M^{\nabla}(x,\, Sx,\, t)\}$$

Therefore, $2M^{\nabla}(Sx,\, x,\, t) \geq 2$. That is, $M^{\nabla}(Sx,\, x,\, t) \geq 1$.

Hence $M^{\nabla}(Sx,\, x,\, t) = 1$, for all $t > 0$. Therefore, $x \in Sx$. Similarly, $x \in Tx$.

Hence x is a common fixed point of S and T.

Uniqueness:

Let us take y be another common fixed point of S and T other than x. Then

$$4M(x, y, t) \geq 4M_{\nabla}(Sx, Ty, t)$$

$$\geq \{M(x, y, t/k) + M^{\nabla}(x, Sx, t/k) + M^{\nabla}(x, Ty, t/k)$$

$$+ \frac{M^{\nabla}(x, Sx, t/k) \cdot M^{\nabla}(y, Ty, t/k)}{M(x, y, t/k)}\}$$

$$\geq \{M(x, y, t/k) + M(x, x, t/k) + M(y, y, t/k)$$

$$+ \frac{M(x, x, t/k) \cdot M(y, y, t/k)}{M(x, y, t/k)}\}$$

$$= \{M(x, y, t/k) + 1 + 1 + [1/M(x, y, t/k)]\}$$

$$= \{2 + M(x, y, t/k) + [1/M(x, y, t/k)]\}$$

$$\geq 2 + M(x, y, t/k) + 1 \geq 3 + M(x, y, t)$$

$$3M(x, y, t) \geq 3. \text{ That is, } M(x, y, t) \geq 1$$

Therefore, $M(x, y, t) = 1$ for all $t > 0$. Hence $x = y$.

This completes the proof.

Theorem 3.3. *Let* $(X, M *)$ *be a complete fuzzy metric space with continuous t-norm * is defined by* $a * b = \min\{a, b\}$ *and* $S, T : X \to CB(X)$ *be multi-valued maps satisfying the condition*

$$M_{\nabla}(Sx, Ty, t) \geq \min\{M(x, y, t/k) + M^{\nabla}(x, Sx, t/k) + M^{\nabla}(y, Ty, t/k),$$

$$\frac{M^{\nabla}(x, Tx, 2t/k), M^{\nabla}(y, Sx, 2t/k), M^{\nabla}(x, Sx, t/k) \cdot M^{\nabla}(y, Ty, t/k)}{M(x, y, t/k)}\}$$

for all $t > 0, x, y \in X$ *and* $0 < k < 1$. *Then S and T have a unique common fixed point*.

Proof of theorem 3.3. Proof is similar.

References

[1] B. C. Dhage, Generalised metric spaces and mappings with fixed point, Bull. Calcutta Math. Soc. 84(4) (1992), 329-336.

[2] A. George and P. Veeramani, On some results in fuzzy metric space, Fuzzy Sets and Systems 64 (1994), 395-399.

[3] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems 27 (1988), 385-389.

[4] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975), 326-334.

[5] I. Kubiaczyk and S. Sharma, Common coincidence point in fuzzy metric spaces, Journal of Fuzzy Mathematics 11(1) (2003), 1-5.

[6] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960), 314-334.

[7] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965), 338-353.