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ON FUZZY Oz-SPACES

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Abstract

In this paper, a new class of fuzzy topological spaces, namely fuzzy Oz-spaces, is introduced and studied. It is obtained that fuzzy extremally disconnected spaces are fuzzy Oz-spaces and fuzzy Oz and fuzzy P-spaces are fuzzy extremally disconnected spaces and fuzzy Moscow spaces. It is obtained that fuzzy Oz-spaces are not fuzzy hyperconnected spaces and fuzzy super-connected spaces are not fuzzy Oz-spaces. Also the notion of weak fuzzy Oz-spaces is introduced and studied. It is obtained that fuzzy fraction dense and weak fuzzy Oz-spaces are fuzzy Oz-spaces.

1. Introduction

In order to deal with uncertainties, the idea of fuzzy sets, fuzzy set operations was introduced by L. A. Zadeh [24] in 1965. Any application of mathematical concepts depends firmly and closely how one introduces basic ideas that may yield various theories in various directions. If the basic idea is suitably introduced, then not only the existing theories stand but also the possibility of emerging new theories increases and on these lines, C. L. Chang [7] introduced the notion of fuzzy topological spaces by means of fuzzy sets in 1967 and his work paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts.

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R. L. Blair [6], V. E. Scepin [12] and T. Terada [13] independently introduced the concept of Oz-spaces, also known as perfectly κ-normal spaces which is analogous to that of normality, in classical topology. R. L. Blair studied the class of Oz topological spaces in which every open set is z-embedded. Weak Oz-spaces in which the closure of each co-zero set is a zero set, were discussed to varying degrees in [2] and [8]. A. Chigogidze [8] investigated weak Oz-spaces under the name of almost Scepin spaces. The concept of Moscow space in classical topology was introduced by A. V. Arhangel'skii [1] and the notion of Moscow spaces generalizes the notion of perfectly κ-normal spaces.

The concept of fuzzy extremally disconnected spaces was introduced and studied by B. Ghosh [10]. The notion of fuzzy P-spaces was introduced and studied by G. Thangaraj and G. Balasubramanian [14]. In the recent years, there has been a growing trend among many fuzzy topologists to introduce and study various types of fuzzy topological spaces. The purpose of this paper is to introduce the concept of fuzzy Oz-spaces and to study its properties and applications. In section 3, several characterizations of fuzzy Oz-spaces are established. It is obtained that closure of fuzzy open sets are fuzzy G_δ-sets and interior of fuzzy closed sets are fuzzy F_σ-sets in fuzzy Oz-spaces. It is established that fuzzy regular open sets are not fuzzy σ-nowhere dense sets and boundaries of fuzzy regular closed sets are fuzzy G_δ-sets in fuzzy Oz-spaces.

In Section 4, it is obtained by an example that fuzzy P-spaces need not be fuzzy Oz-spaces. It is established that fuzzy extremally disconnected spaces are fuzzy Oz-spaces and fuzzy Oz and fuzzy P-spaces, are fuzzy extremally disconnected spaces and fuzzy Moscow spaces. A condition for fuzzy Oz-spaces to become fuzzy Moscow spaces is also obtained by means of fuzzy regular closed sets. Fuzzy semi-open sets are found to be fuzzy pre-open sets in fuzzy Oz and fuzzy P-spaces. It is obtained that fuzzy Oz-spaces are not fuzzy hyperconnected spaces and fuzzy super-connected spaces are not fuzzy Oz-spaces. Also it is obtained that fuzzy perfectly disconnected and fuzzy Oz- spaces are not fuzzy hyperconnected spaces.

In Section 5, the notion of weak fuzzy Oz-space is defined and studied. Several characterizations of weak fuzzy Oz-spaces are established. It is obtained that interiors of fuzzy σ-boundary sets are fuzzy F_σ-sets and closures of fuzzy σ-boundary sets are fuzzy G_δ-sets and fuzzy σ-boundary sets are fuzzy somewhere dense sets in weak fuzzy Oz-spaces.

In Section 6, the conditions under which fuzzy Oz-spaces become fuzzy weak fuzzy Oz-spaces, are obtained. It is obtained that fuzzy fraction dense and weak fuzzy Oz-spaces are fuzzy Oz-spaces. It is found that the notions of fuzzy Oz-spaces and weak fuzzy Oz-spaces are identical in fraction dense spaces.

2. Preliminaries

In order to make the exposition self-contained, some basic notions and results used in the sequel, are given. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). Let X be a non-empty set and I the unit interval [0, 1]. A fuzzy set λ in X is a mapping from X into I. The fuzzy set 0_X is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set 1_X is defined as $1_X(x) = 1$, for all $x \in X$.

Definition 2.1[7]**.** Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). The interior, the closure and the complement of λ are defined respectively as follows:

(i) $\operatorname{int}(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$,

(ii) $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1-\mu \in T\}$.

(iii) $\lambda'(x) = 1 - \lambda(x)$, for all $x \in X$.

For a family $\{\lambda_i / i \in I\}$ of fuzzy sets in (X, T), the union $\psi = \vee_i(\lambda_i)$ and the intersection $\delta = \wedge_i(\lambda_i)$, are defined respectively as

(iv) $\psi(x) = \sup_i\{\lambda_i(x)/x \in X\}$.

(v) $\delta(x) = \inf_i\{\lambda_i(x)/x \in X\}$.

Lemma 2.1[3]**.** *For a fuzzy set* λ *of a fuzzy topological space* X,

(i) $1 - \operatorname{int}(\lambda) = cl(1-\lambda)$ *and* (ii) $1 - cl(\lambda) = \operatorname{int}(1-\lambda)$.

Definition 2.2. A fuzzy set λ in a fuzzy topological space (X, T) is called a

(1) fuzzy regular-open if $\lambda = \operatorname{int} cl(\lambda)$ and fuzzy regular-closed if $\lambda = cl \operatorname{int}(\lambda)$ [3].

(2) fuzzy G_δ-set if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$ for $i \in I$ [4].

(3) fuzzy F_σ-set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $1 - \lambda_i \in T$ for $i \in I$ [4].

(4) fuzzy semi-open if $\lambda \leq cl \operatorname{int}(\lambda)$ and fuzzy semi-closed if $\operatorname{int} cl(\lambda) \leq \lambda$ [3].

(5) fuzzy pre-open if $\lambda \leq \operatorname{int} cl(\lambda)$ and fuzzy pre-closed if $cl \operatorname{int}(\lambda) \leq \lambda$ [5].

Definition 2.3. A fuzzy set λ in a fuzzy topological space (X, T), is called a

(i) fuzzy dense set if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$. That is, $cl(\lambda) = 1$, in (X, T) [15].

(ii) fuzzy nowhere dense set if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $\operatorname{int} cl(\lambda) = 0$, in (X, T) [15].

(iii) fuzzy first category set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category [15].

(iv) fuzzy residual set if $1 - \lambda$ is a fuzzy first category set in (X, T) [16].

(v) fuzzy somewhere dense set if there exists a non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $\operatorname{int} cl(\lambda) \neq 0$, in (X, T) [19].

(vi) fuzzy σ-nowhere dense set if λ is a fuzzy F_σ-set with $\operatorname{int}(\lambda) = 0$, in (X, T) [17].

(vii). fuzzy σ-boundary set if $\lambda = \vee_{i=1}^{\infty}(\mu_i)$, where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and (λ_i)'s are fuzzy regular open sets in (X, T) [18].

Definition 2.4. A fuzzy topological space (X, T) is called a

(i) fuzzy P-space if each fuzzy G_δ-set in (X, T) is fuzzy open in (X, T) [14].

(ii) fuzzy hyperconnected space if every non-null fuzzy open subset of (X, T) is fuzzy dense in (X, T) [11].

(iii) fuzzy super-connected space if it has no proper fuzzy regular open set in (X, T) [9].

(iv) fuzzy extremally disconnected space if the closure of every fuzzy open set of (X, T) is fuzzy open in (X, T) [10].

(v) fuzzy Moscow space if for each fuzzy open set λ in (X, T), $cl(\lambda) = \vee_{i=1}^{\infty}(\delta_i)$, where (δ_i)'s are fuzzy G_δ-sets in (X, T) [22].

(vi) fuzzy F'-space if $\lambda \leq 1 - \mu$, where λ and μ are fuzzy F_σ-sets in (X, T), then $cl(\lambda) \leq 1 - cl(\mu)$, in (X, T) [21].

(vii) fuzzy perfectly disconnected space if for any two non-zero fuzzy sets λ and μ defined on X with $\lambda \leq 1 - \mu$, $cl(\lambda) \leq 1 - cl(\mu)$, in (X, T) [20].

(viii) fuzzy basically disconnected space if the closure of every fuzzy open F_σ-set of (X, T) is fuzzy open in (X, T) [14].

(ix) fuzzy fraction dense space if for each fuzzy open set λ in (X, T), $cl(\lambda) = cl(\mu)$, where μ is a fuzzy F_σ-set in (X, T) [23].

Theorem 2.1[3]**.** *In a fuzzy topological space*,

(a) *The closure of a fuzzy open set is a fuzzy regular closed set*.

(b) *The interior of a fuzzy closed set is a fuzzy regular open set*.

Theorem 2.2[19]**.** *If* λ *is a fuzzy somewhere dense set in a fuzzy topological space* (X, T), *then there exists a fuzzy regular closed set* λ *in* (X, T) *such that* $\eta \leq cl(\lambda)$.

Theorem 2.3[18]**.** *If* λ *is a fuzzy* σ*-boundary set in a fuzzy topological space* (X, T), *then* λ *is a fuzzy* F_σ*-set in* (X, T).

Theorem 2.4[10]. *For any fuzzy topological space* (X, T), *the following are equivalent*:

(a) *X is fuzzy extremally disconnected space*.

(b) *For each fuzzy closed set* λ, $\text{int}(\lambda)$ *is fuzzy closed*.

(c) *For each fuzzy open set* λ, $cl(\lambda) + cl[1 - cl(\lambda)] = 1$.

(d) *For every pair of fuzzy open sets* λ *and* μ *in* X with $cl(\lambda) + \mu = 1$, $cl(\lambda) + cl(\mu) = 1$.

Theorem 2.5[9]. *If X is a fuzzy topological space*, *then the following are equivalent*:

(a) *X is fuzzy super-connected space*.

(b) *Closure of every non-zero fuzzy open set in X is* 1.

(c) *Interior of every fuzzy closed set in X different from* 1, *is zero*.

(d) *X does not have non-zero fuzzy open sets* λ *and* μ *such that* $\lambda + \mu \leq 1$.

Theorem 2.6[22]. *If the fuzzy topological space* (X, T) *is a fuzzy extremally disconnected space*, *then* (X, T) *is a fuzzy Moscow space*.

Theorem 2.7[10]. *A fuzzy topological space* (X, T) *is fuzzy extremally disconnected if and only if* $FSO(X) \subset FPO(X)$.

Theorem 2.8[22]. *If each* $\lambda_i (i = 1 \text{ to } \infty)$ *is a fuzzy residual set in the fuzzy Moscow space* (X, T), *then there exists a fuzzy regular closed set* γ *in* (X, T) *such that* $\gamma \leq \vee_{i=1}^{\infty}(\lambda_i)$.

Theorem 2.9[23]. *If* μ *is a fuzzy regular closed set in a fuzzy fraction dense and fuzzy P-space* (X, T), *then* μ *is a fuzzy* F_σ*-set in* (X, T).

Theorem 2.10[21]. *If a fuzzy topological space* (X, T) *is a fuzzy perfectly disconnected space*, *then* (X, T) *is a fuzzy* F'*-space*.

Theorem 2.11[20]. *If* λ *is a fuzzy pre-closed set in a fuzzy perfectly disconnected space* (X, T), *then* $\text{int}(\lambda)$ *is a fuzzy regular closed set in* (X, T).

Theorem 2.12[16]. *If* λ *is a fuzzy first category set in a fuzzy topological space* (X, T), *then there is a fuzzy* F_σ*-set* η *in* (X, T) *such that* $\lambda \leq \eta$.

Theorem 2.13[23]. *A fuzzy topological space* (X, T) *is a fuzzy fraction dense space if and only if for each fuzzy regular closed set* μ *in* (X, T), $\mu = cl(\eta)$, *where* η *is a fuzzy* F_σ*-set in* (X, T).

Theorem 2.14[21]. *If* $\lambda \leq 1 - \mu$, *where* λ *and* μ *are fuzzy* σ*-boundary sets in a fuzzy* F'*-space* (X, T), *then* $cl(\lambda) \leq 1 - cl(\mu)$ *in* (X, T).

3. Fuzzy Oz-Spaces

Motivated by the works of R. L. Blair [6], V. E. Scepin [12] and T. Terada [13], the notion of Oz-space in fuzzy setting is defined as follows:

Definition 3.1. A fuzzy topological space (X, T) is called a fuzzy Oz-space if each fuzzy regular closed set is a fuzzy G_δ-set in (X, T).

Example 3.1. Let $X = \{a, b, c\}$. Let $I = [0, 1]$. The fuzzy sets α, β and γ are defined on X as follows:

$\alpha : X \to I$ is defined by $\alpha(a) = 0.5, \alpha(b) = 0.5, \alpha(c) = 0.6$,

$\beta : X \to I$ is defined by $\beta(a) = 0.5, \beta(b) = 0.6, \beta(c) = 0.5$,

$\gamma : X \to I$ is defined by $\gamma(a) = 0.6, \gamma(b) = 0.4, \gamma(c) = 0.5$.

Then, $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \gamma \vee [\alpha \wedge \beta], \alpha \vee \beta \vee \gamma, 1\}$ is a fuzzy topology on X. By computation, one can find that

$cl(\alpha) = 1; \mathrm{int}(1 - \alpha) = 0;$

$cl(\beta) = 1 - (\alpha \wedge \gamma) = \beta; \mathrm{int}(1 - \beta) = \alpha \wedge \gamma;$

$cl(\gamma) = 1; \mathrm{int}(1 - \gamma) = 0;$

$cl(\alpha \vee \beta) = 1; \mathrm{int}(1 - [\alpha \vee \beta]) = 0;$

$cl(\alpha \vee \gamma) = 1; \mathrm{int}(1 - [\alpha \vee \gamma]) = 0;$

$cl(\beta \vee \gamma) = 1; \mathrm{int}(1 - [\beta \vee \gamma]) = 0;$

$cl(\alpha \wedge \beta) = 1 - (\alpha \wedge \beta) = \alpha \wedge \beta; \text{int}(1 - [\alpha \wedge \beta]) = \alpha \wedge \beta;$

$cl(\alpha \wedge \gamma) = 1 - \beta = \alpha \wedge \gamma; \text{int}(1 - [\alpha \wedge \gamma]) = \beta;$

$cl(\gamma \vee [\alpha \wedge \beta]) = 1; \text{int}[1 - (\gamma \vee [\alpha \wedge \beta)] = 0$

$cl(\alpha \vee \beta \vee \gamma) = 1; \text{int}(1 - [\alpha \vee \beta \vee \gamma]) = 0.$

The fuzzy regular closed sets in (X, T) are $1 - \beta, 1 - (\alpha \wedge \beta)$ and $1 - (\alpha \wedge \gamma)$.

Now $1 - \beta = \beta \wedge \gamma \wedge (\alpha \wedge \beta) = \alpha \wedge \gamma;$

$1 - (\alpha \wedge \beta) = \alpha \wedge (\alpha \vee \gamma) \wedge [\gamma \vee (\alpha \wedge \beta)] = \alpha \wedge \beta;$

$1 - (\alpha \wedge \gamma) = (\alpha \vee \beta) \wedge (\beta \vee \gamma) \wedge (\alpha \vee \beta \vee \gamma) = \beta.$

Then, $1 - \beta, 1 - (\alpha \wedge \beta)$ and $1 - (\alpha \wedge \gamma)$ are fuzzy G_δ-sets in (X, T). Thus the fuzzy regular closed sets $1 - \beta, 1 - (\alpha \wedge \beta)$ and $1 - (\alpha \wedge \gamma)$ are fuzzy G_δ-sets in (X, T). Hence (X, T) is a fuzzy Oz-space.

Example 3.2. Let $X = \{a, b, c\}$. Let $I = [0, 1]$. The fuzzy sets α, β and γ are defined on X as follows.

$\alpha : X \to I$ is defined by $\alpha(a) = 0.4, \alpha(b) = 0.6, \alpha(c) = 0.4,$

$\beta : X \to I$ is defined by $\beta(a) = 0.5, \beta(b) = 0.4, \beta(c) = 0.5,$

$\gamma : X \to I$ is defined by $\gamma(a) = 0.6, \gamma(b) = 0.5, \gamma(c) = 0.4.$

Then, $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \beta \vee [\alpha \wedge \gamma], \gamma \wedge (\alpha \vee \beta), \alpha \vee \beta \vee \gamma, 1\}$ is a fuzzy topology on X. By computation, one can find that

$cl(\alpha) = 1 - \beta; \text{int}(1 - \alpha) = \beta;$

$cl(\beta) = 1 - (\alpha \vee \beta); \text{int}(1 - \beta) = \alpha \vee \beta;$

$cl(\gamma) = 1 - (\alpha \wedge \gamma); \text{int}(1 - \gamma) = \alpha \wedge \gamma;$

$cl(\alpha \vee \beta) = 1 - \beta; \text{int}(1 - [\alpha \vee \beta]) = \beta;$

$cl(\alpha \vee \gamma) = 1 - (\alpha \wedge \beta); \text{int}(1 - [\alpha \vee \gamma]) = \alpha \wedge \beta;$

$cl(\beta \vee \gamma) = 1 - (\alpha \wedge \gamma); \operatorname{int}(1 - [\beta \vee \gamma]) = \alpha \wedge \gamma;$

$cl(\alpha \wedge \beta) = 1 - (\alpha \wedge \beta \vee \gamma); \operatorname{int}(1 - [\alpha \wedge \beta]) = \alpha \wedge \beta \vee \gamma;$

$cl(\alpha \wedge \gamma) = 1 - (\beta \vee \gamma); \operatorname{int}(1 - [\alpha \vee \gamma]) = \beta \vee \gamma;$

$cl(\beta \wedge \gamma) = 1 - (\alpha \vee \beta); \operatorname{int}(1 - [\alpha \vee \gamma]) = \alpha \vee \beta;$

$cl(\alpha \vee [\beta \wedge \gamma]) = 1 - \beta; \operatorname{int}(1 - [\alpha \vee (\beta \wedge \gamma)]) = \beta;$

$cl(\beta \vee [\alpha \wedge \gamma]) = 1 - (\beta \vee [\alpha \wedge \gamma]); \operatorname{int}(1 - [\beta \vee (\alpha \wedge \gamma)]) = \beta \vee [\alpha \wedge \gamma];$

$cl(\gamma \wedge [\alpha \vee \beta]) = 1 - (\beta \vee [\alpha \wedge \gamma]); \operatorname{int}(1 - [\gamma \wedge (\alpha \vee \beta)]) = \beta \vee [\alpha \wedge \gamma];$

$cl(\alpha \vee \beta \vee \gamma) = 1 - (\alpha \wedge \beta); \operatorname{int}(1 - [\alpha \vee \beta \vee \gamma]) = \alpha \wedge \beta.$

The fuzzy regular closed sets in (X, T) are $1 - \beta, 1 - (\alpha \vee \beta), 1 - (\beta \vee \gamma), 1 - (\alpha \wedge \beta), 1 - (\alpha \wedge \gamma), 1 - [\beta \vee (\alpha \wedge \gamma)]$ and $1 - [\alpha \vee \beta \vee \gamma]$ and the fuzzy regular closed set $1 - (\alpha \wedge \gamma)$ is not a fuzzy G_δ-set in (X, T). Hence (X, T) is not a fuzzy Oz-space.

Proposition 3.1. *If* λ *is a fuzzy regular open set in a fuzzy Oz-space*, *then* λ *is a fuzzy* F_σ*-set in* (X, T).

Proof. Let λ be a fuzzy regular open set in (X, T). Then, $1 - \lambda$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy Oz-space, $1 - \lambda$ is a fuzzy G_δ-set in (X, T). Then, λ is a fuzzy F_σ-set in (X, T).

Proposition 3.2. *If* λ *is a fuzzy regular open set in a fuzzy Oz-space*, *then* λ *is not a fuzzy* σ*-nowhere dense set in* (X, T).

Proof. Let λ be a fuzzy regular open set in (X, T). Then, $\operatorname{int} cl(\lambda) = \lambda$, in (X, T) and int $[\operatorname{int} cl(\lambda)] = \operatorname{int}(\lambda)$. Then, $\operatorname{int} cl(\lambda) = \operatorname{int}(\lambda)$ and $\operatorname{int}(\lambda) \neq 0$, in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.1, λ is a fuzzy F_σ-set in (X, T). Thus, λ is a fuzzy F_σ-set with $\operatorname{int}(\lambda) \neq 0$ in (X, T). Hence λ is not a fuzzy σ-nowhere dense set in (X, T).

Remark. In view of the above proposition, one will have the following result: "Fuzzy regular open sets are not fuzzy σ-nowhere dense sets in fuzzy Oz-spaces."

Proposition 3.3. *If* λ *is a fuzzy open set in a fuzzy Oz-space* (X, T) *such that* $cl(\lambda) \neq 1$, *then* $cl(\lambda)$ *is a fuzzy* G_δ*-set in* (X, T).

Proof. Let λ be a fuzzy open set in (X, T) such that $cl(\lambda) \neq 1$. Then, by Theorem 2.1, $cl(\lambda)$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy Oz-space, $cl(\lambda)$ is a fuzzy G_δ-set in (X, T).

Proposition 3.4. *If* μ *is a fuzzy closed set in a fuzzy Oz-space* (X, T) *such that* $\operatorname{int}(\mu) \neq 0$, *then* $\operatorname{int}(\mu)$ *is a fuzzy* F_σ*-set in* (X, T).

Proof. Let μ be a fuzzy closed set in (X, T) such that $\operatorname{int}(\mu) \neq 0$. Then, $1 - \mu$ is a fuzzy open set in (X, T) such that $cl(1 - \mu) \neq 1$. Since (X, T) is a fuzzy Oz-space, $cl(1 - \mu)$ is a fuzzy G_δ-set in (X, T) and thus $1 - \operatorname{int}(\mu)$ [1] is a fuzzy G_δ-set in (X, T). Hence $\operatorname{int}(\mu)$ is a fuzzy F_σ-set in (X, T).

Proposition 3.5. *If* λ *is a fuzzy open set in a fuzzy Oz-space* (X, T), *such that* $cl(\lambda) \neq 1$, *then* $bd(\lambda) = cl(\lambda) \wedge cl(1 - \lambda)$, *where* δ *is a fuzzy* G_δ*-set in* (X, T).

Proof. Let λ be a fuzzy open set in (X, T) such that $cl(\lambda) \neq 1$. Now $bd(\lambda) = cl(\lambda) \wedge cl(1 - \lambda)$, in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.3, $cl(\lambda)$ is a fuzzy G_δ-set in (X, T). Let $\delta = cl(\lambda)$. Thus, $bd(\lambda) = \delta \wedge cl(1 - \lambda)$, where δ is a fuzzy G_δ-set in (X, T).

Corollary 3.1. *If* λ *is a fuzzy open set in a fuzzy Oz-space* (X, T) *such that* $cl(\lambda) \neq 1$, *then there exists a fuzzy* G_δ*-set* δ *in* (X, T) *such that* $bd(\lambda) \leq \delta$.

Proposition 3.6. *If* μ *is a fuzzy closed set in a fuzzy Oz-space* (X, T) *such that* $\operatorname{int}(\mu) \neq 0$, *then* $bd(\mu) = \mu \wedge \gamma$, *where* γ *is a fuzzy* G_δ*-set in* (X, T).

Proof. Let μ be a fuzzy closed set in (X, T) such that $\operatorname{int}(\mu) \neq 0$. Now $bd(\mu) = cl(\mu) \wedge cl(1 - \mu)$, in (X, T). Then, $bd(\mu) = cl(\mu) \wedge (1 - \operatorname{int}(\mu))$ [1]. Since (X, T) is a fuzzy Oz-space, by Proposition 3.4, $\operatorname{int}(\mu)$ is a fuzzy F_σ-set in (X, T) and then $1 - \operatorname{int}(\mu)$ is a fuzzy G_δ-set in (X, T). Let $\gamma = 1 - \operatorname{int}(\mu)$.

Since μ is a fuzzy closed set in (X, T), $cl(\mu) = \mu$. Hence $bd(\mu) = \mu \wedge \gamma$, where γ is a fuzzy G_δ-set in (X, T).

Corollary 3.2. *If* μ *is a fuzzy closed set in a fuzzy Oz-space* (X, T) *such that* $\text{int}(\mu) \neq 0$, *then there exists a fuzzy* G_δ*-set* γ *in* (X, T) *such that* $bd(\mu) \leq \gamma$.

Proposition 3.7. *If* λ *is a fuzzy open set in a fuzzy Oz-space* (X, T), *such that* $cl(\lambda) \neq 1$, *then there exists a fuzzy* F_σ*-set* λ *in* (X, T) *such that* $\delta \leq \text{int}\, cl(\lambda)$.

Proof. Let λ be a fuzzy open set in (X, T) such that $cl(\lambda) \neq 1$. Now $\lambda \vee [1 - cl(\lambda)]$ is a fuzzy open set in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.3, $cl(\lambda \vee [1 - cl(\lambda)])$ is a fuzzy G_δ-set in (X, T). Now $cl(\lambda \vee [1 - cl(\lambda)]) = cl(\lambda) \vee cl(1 - cl(\lambda)) = cl(\lambda) \vee (1 - \text{int}\, cl(\lambda))$. Then, $1 - \text{int}\, cl(\lambda) \leq cl(\lambda \vee [1 - cl(\lambda)])$ and $1 - cl(\lambda \vee [1 - cl(\lambda)]) \leq \text{int}\, cl(\lambda)$. Let $\delta = 1 - cl(\lambda \vee [1 - cl(\lambda)])$ and thus δ is a fuzzy F_σ-set in (X, T) and $\delta \leq \text{int}\, cl(\lambda)$.

Corollary 3.3. *If* λ *is a fuzzy open set in a fuzzy Oz-space* (X, T) *such that* $cl(\lambda) \neq 1$, *then there exists a fuzzy* F_σ*-set* δ *and a fuzzy* G_δ*-set* θ *in* (X, T) *such that* $\delta \leq \text{int}\, cl(\lambda) \leq \theta$.

Proof. Let λ be a fuzzy open set in (X, T) such that $cl(\lambda) \neq 1$. Then, by Proposition 3.7, there exists a fuzzy F_σ-set δ in (X, T) such that $\delta \leq \text{int}\, cl(\lambda)$. Now $\text{int}\, cl(\lambda) \leq cl(\lambda)$. By proposition 3.3, $cl(\lambda)$ is a fuzzy G_δ-set in (X, T). Let $\theta = cl(\lambda)$. Hence there exists a fuzzy F_σ-set δ and a fuzzy G_δ-set θ in (X, T) such that $\delta \leq cl(\lambda) \leq \theta$.

Corollary 3.4. *If* λ *is a fuzzy open set in a fuzzy Oz-space* (X, T) *such that* $cl(\lambda) \neq 1$, *then there exists a fuzzy* F_σ*-set* δ *and a fuzzy* G_δ*-set* θ *and a fuzzy open set* $\gamma \geq \lambda$ *in* (X, T) *such that* $\delta \leq \gamma \leq \theta$.

Proof. Let λ be a fuzzy open set in the fuzzy Oz-space (X, T). Then, $\lambda = \text{int}(\lambda) \leq \text{int}\, cl(\lambda)$. By Corollary 3.3, there exists a fuzzy F_σ-set δ and a fuzzy G_δ-set θ in (X, T) such that $\delta \leq \operatorname{int} cl(\lambda) \leq \theta$. Let $\gamma = \operatorname{int} cl(\lambda)$ and then γ is a fuzzy open set in (X, T). Thus there exists a fuzzy F_σ-set δ and a fuzzy G_δ-set θ in (X, T) such that $\delta \leq \gamma \leq \theta$.

Proposition 3.8. *If* μ *is a fuzzy closed set in a fuzzy Oz-space* (X, T) *such that* $\operatorname{int}(\mu) \neq 0$, *then there exists a fuzzy* G_δ*-set* θ *in* (X, T) *such that* $cl \operatorname{int}(\mu) \leq \theta$.

Proof. Let μ be a fuzzy closed set in (X, T). Then, $1 - \mu$ is a fuzzy open set in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.7, there exists a fuzzy F_σ-set δ in (X, T) such that $\delta \leq \operatorname{int} cl(1 - \mu)$. Then, $\delta \leq 1 - cl \operatorname{int}(\mu)$ [1]. This implies that $cl \operatorname{int}(\mu) \leq 1 - \delta$, in (X, T). Let $\theta = 1 - \delta$. Then, θ is a fuzzy G_δ-set in (X, T). Hence for the fuzzy closed set μ, there exists a fuzzy G_δ-set θ in (X, T) such that $cl \operatorname{int}(\mu) \leq \theta$.

Corollary 3.5. *If* μ *is a fuzzy closed set in a fuzzy Oz-space* (X, T), *such that* $\operatorname{int}(\mu) \neq 0$, *then there exists a fuzzy* F_σ*-set* δ *and a fuzzy* G_δ*-set* θ *in* (X, T) *such that* $\delta \leq cl \operatorname{int}(\mu) \leq \theta$.

Proof. Let μ be a fuzzy closed set in (X, T) such that $\operatorname{int}(\mu) \neq 0$. Then, by Proposition 3.8, there exists a fuzzy G_δ-set θ in (X, T) such that $cl \operatorname{int}(\mu) \leq \theta$. Now $\operatorname{int}(\mu) \leq cl \operatorname{int}(\mu)$ in (X, T). By Proposition 3.4, $\operatorname{int}(\mu)$ is a fuzzy F_σ-set in (X, T). Let $\delta = \operatorname{int}(\mu)$. Hence there exists a fuzzy F_σ-set δ and a fuzzy G_δ-set θ in (X, T) such that $\delta \leq cl \operatorname{int}(\mu) \leq \theta$.

Proposition 3.9. *If* λ *is a fuzzy somewhere dense set in a fuzzy Oz-space* (X, T), *then there exists a fuzzy* G_δ*-set* η *in* (X, T) *such that* $\eta \leq cl(\lambda)$.

Proof. Let λ be a fuzzy somewhere dense set in (X, T). Then, by Theorem 2.2, there exists a fuzzy regular closed set η in (X, T) such that $\eta \leq cl(\lambda)$. Since (X, T) is a fuzzy Oz-space, η is a fuzzy G_δ-set in (X, T). Hence there exists a fuzzy G_δ-set η in (X, T) such that $\eta \leq cl(\lambda)$.

Proposition 3.10. *If* λ *is a fuzzy set defined on X in a fuzzy Oz-space* (X, T) *such that* $cl \operatorname{int}(\lambda) \neq 1$, *then* $cl \operatorname{int}(\lambda) \leq \mu$, *where* μ *is a fuzzy open set in* (X, T).

Proof. Let λ be a fuzzy set defined on X such that $cl\,\text{int}(\lambda) \neq 1$. Then, $\text{int}(\lambda)$ is a fuzzy open set in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.3, $cl[\text{int}(\lambda)]$ is a fuzzy G_δ-set in (X, T). Then, $cl[\text{int}(\lambda)]$ $= \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$. This implies that $cl[\text{int}(\lambda)] \leq \lambda_i$ for each $i \in I$. Let $\mu = \lambda_i$. Hence $cl\,\text{int}(\lambda) \leq \mu$, where μ is a fuzzy open set in (X, T).

Proposition 3.11. *If λ is a fuzzy semi-open set in a fuzzy Oz-space (X, T), then there exists a fuzzy G_δ-set μ in (X, T) such that $\lambda \leq \mu$.*

Proof. Let λ be a fuzzy semi-open set in (X, T). Then, $\lambda \leq cl\,\text{int}(\lambda)$ in (X, T). Now $\text{int}(\lambda)$ is a fuzzy open set in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.3, $cl[\text{int}(\lambda)]$ is a fuzzy G_δ-set in (X, T). Let $cl[\text{int}(\lambda)] = \mu$. Then, μ is a fuzzy G_δ-set in (X, T). Thus, for a fuzzy semiopen set λ, there exists a fuzzy G_δ-set μ in (X, T) such that $\lambda \leq \mu$.

Proposition 3.12. *If λ is a fuzzy regular closed set in a fuzzy Oz-space (X, T), then $bd(\lambda)$ is a fuzzy G_δ-set in (X, T).*

Proof. Let λ be a fuzzy regular closed set in (X, T). Then, $cl\,\text{int}(\lambda) = \lambda$, in (X, T). Now $cl[cl\,\text{int}(\lambda)] = cl(\lambda)$ and then $cl\,\text{int}(\lambda) = cl(\lambda)$ (A). Since $\text{int}(\lambda)$ is a fuzzy open set in (X, T). Then, by Theorem 2.1, $cl\,\text{int}(\lambda)$ is a fuzzy regular closed set in (X, T). From (A), $cl(\lambda)$ is a fuzzy regular closed set in (X, T). Since a fuzzy regular closed set is a fuzzy closed set in a fuzzy topological space, λ is a fuzzy closed set in (X, T) and then $1 - \lambda$ is a fuzzy open set in (X, T). By Theorem 2.1, $cl(1 - \lambda)$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy Oz-space, the fuzzy regular closed sets $cl(\lambda)$ and $cl(1 - \lambda)$ are fuzzy G_δ-sets in (X, T). Since $bd(\lambda) = cl(\lambda) \wedge cl(1 - \lambda)$, $bd(\lambda)$ is a fuzzy G_δ-set in (X, T). Hence $bd(\lambda)$ is a fuzzy G_δ-set, for the fuzzy regular closed set λ in (X, T).

Proposition 3.13. *If λ and μ are fuzzy open sets in a fuzzy Oz-space (X, T) such that $\lambda \leq 1 - \mu$, then there exist fuzzy F_σ-sets δ and λ in (X, T) such that $\lambda \leq \delta$, $\mu \leq \eta$ and $\delta \leq 1 - \eta$.*

Proof. Let λ and μ be any two fuzzy open sets in (X, T) such that $\lambda \leq 1-\mu$. Now $\lambda \leq 1-\mu$, implies that $cl(\lambda) \leq cl(1-\lambda)$ and then $cl(\lambda) \leq 1-\text{int}(\mu)$ and $\text{int}(\mu) \leq 1-cl(\lambda)$. Since $\mu = \text{int}(\mu), \mu \leq 1-cl(\lambda)$, in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.3, $cl(\lambda)$ is a fuzzy G_δ-set in (X, T). Then, $1-cl(\lambda)$, is a fuzzy F_σ-set in (X, T). Let $\eta = 1-cl(\lambda)$ and then η is a fuzzy F_σ-set in (X, T) such that $\mu \leq \eta$. Also $\lambda \leq 1-\mu$, implies that $\mu \leq 1-\lambda$ and $cl(\mu) \leq cl(1-\mu) = 1-\text{int}(\lambda)$. Then, $\text{int}(\lambda) \leq 1-cl(\mu)$. This implies that $\lambda = \text{int}(\lambda) \leq 1-cl(\mu)$. By Proposition 3.3, $cl(\mu)$ is a fuzzy G_δ-set in (X, T). Then, $1-cl(\mu)$, is a fuzzy F_σ-set in (X, T). Let $\delta = 1-cl(\mu)$ and then δ is a fuzzy F_σ-set in (X, T) such that $\lambda \leq \delta$.

Now $\lambda \leq 1-cl(\mu) \leq 1-\mu$, implies that $\lambda \leq \delta \leq 1-\mu$ (1).

Also $\mu \leq 1-cl(\lambda) \leq 1-\lambda$, implies that $\mu \leq \eta \leq 1-\lambda$ and $1 \leq \mu \leq 1-\eta \geq 1-(1-\lambda)$. That is, $\lambda \leq 1-\eta \leq 1-\mu$ (2). From (1) and (2), $\lambda \leq \delta \leq 1-\eta \leq 1-\mu$. Hence, for the fuzzy open sets λ and μ in (X, T) such that $\lambda \leq 1-\mu$, there exist fuzzy F_σ-sets δ and η in (X, T) such that $\lambda \leq \delta$, $\mu \leq \eta$ and $\delta \leq 1-\eta$.

4. Fuzzy Oz-Spaces and Other Fuzzy Topological Spaces

When analyzing the properties of fuzzy Oz-spaces, it is obtained that fuzzy Oz-spaces need not be fuzzy P-spaces. For, consider the following example:

Example 4.1. Let $X = \{a, b, c\}$. Let $I = [0, 1]$. The fuzzy sets α, β, γ and δ are defined on X as follows:

$\alpha : X \to I$ is defined by $\alpha(a) = 0.6; \alpha(b) = 0.4; \alpha(c) = 0.5$,

$\beta : X \to I$ is defined by $\beta(a) = 0.4; \beta(b) = 0.5; \beta(c) = 0.6$,

$\gamma : X \to I$ is defined by $\gamma(a) = 0.5; \gamma(b) = 0.7; \gamma(c) = 0.4$,

$\delta : X \to I$ is defined by $\delta(a) = 0.5; \delta(b) = 0.5; \delta(c) = 0.5$.

Then, $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma],$

$\beta \vee [\alpha \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \alpha \wedge [\beta \vee \gamma], \beta \wedge [\alpha \vee \gamma], \gamma \wedge [\alpha \vee \beta], \alpha \wedge \beta \wedge \gamma,$ $\alpha \vee \beta \vee \gamma, 1\}$ is a fuzzy topology on X. Also by computation, one can find that

$$cl(\alpha) = 1 - [\beta \wedge [\alpha \vee \gamma]]; \operatorname{int}(1 - \alpha) = \beta \wedge [\alpha \vee \gamma];$$

$$cl(\beta) = 1 - [\gamma \wedge [\alpha \vee \beta]]; \operatorname{int}(1 - \beta) = \gamma \wedge [\alpha \vee \beta];$$

$$cl(\alpha) = 1; \operatorname{int}(1 - \gamma) = 0;$$

$$cl(\alpha \vee \beta) = 1 - (\beta \wedge \gamma); \operatorname{int}(1 - [\alpha \vee \beta]) = \beta \wedge \gamma;$$

$$cl(\alpha \vee \gamma) = 1; \operatorname{int}(1 - [\alpha \vee \gamma]) = 0;$$

$$cl(\beta \vee \gamma) = 1; \operatorname{int}(1 - [\beta \vee \gamma]) = 0;$$

$$cl(\alpha \wedge \beta) = 1 - [\alpha \vee (\beta \wedge \gamma)]; \operatorname{int}(1 - [\alpha \wedge \beta]) = \alpha \vee (\beta \wedge \gamma);$$

$$cl(\alpha \wedge \gamma) = 1 - [\beta \vee (\alpha \wedge \gamma)]; \operatorname{int}(1 - [\alpha \wedge \gamma]) = \beta \vee (\alpha \wedge \gamma);$$

$$cl(\beta \wedge \gamma) = 1 - (\alpha \vee \beta); \operatorname{int}(1 - [\beta \wedge \gamma]) = \alpha \vee \beta;$$

$$cl(\alpha \vee [\beta \wedge \gamma]) = 1 - [\beta \wedge (\alpha \vee \gamma)]; \operatorname{int}(1 - [\alpha \vee (\beta \wedge \gamma)]) = \beta \wedge (\alpha \vee \gamma);$$

$$cl(\beta \vee [\alpha \wedge \gamma]) = 1 - [\gamma \wedge (\alpha \vee \beta)]; \operatorname{int}((1 - [\beta \vee [\alpha \wedge \gamma]])) = \gamma \wedge (\alpha \vee \beta);$$

$$cl(\gamma \vee [\alpha \wedge \beta]) = 1; \operatorname{int}(1 - [\gamma \vee (\alpha \wedge \beta)]) = 0;$$

$$cl(\alpha \wedge [\beta \vee \gamma]) = 1 - [\gamma \wedge (\alpha \vee \beta)]; \operatorname{int}(1 - [\alpha \vee (\beta \wedge \gamma)]) = \gamma \wedge (\alpha \vee \beta);$$

$$cl(\beta \wedge [\alpha \vee \gamma]) = 1 - [\alpha \vee (\alpha \wedge \beta)]; \operatorname{int}(1 - [\beta \wedge (\alpha \vee \gamma)]) = \alpha \vee (\beta \wedge \gamma);$$

$$cl(\gamma \wedge [\alpha \vee \beta]) = 1 - [\beta \vee (\alpha \wedge \gamma)]; \operatorname{int}(1 - [\gamma \wedge (\alpha \vee \beta)]) = \beta \vee (\alpha \wedge \gamma);$$

$$cl(\alpha \wedge \beta \vee \gamma) = 1; \operatorname{int}(1 - [\alpha \vee \beta \vee \gamma]) = 0;$$

$$cl(\alpha \wedge \beta \wedge \gamma) = 1 - (\alpha \vee \beta); \operatorname{int}(1 - [\alpha \vee \beta \vee \gamma]) = \alpha \vee \beta.$$

The fuzzy regular closed sets in (X, T) are $1 - (\alpha \vee \beta), 1 - (\beta \wedge \gamma),$ $1 - (\alpha \vee [\beta \wedge \gamma]), 1 - (\beta \vee [\alpha \wedge \gamma]), 1 - [\beta \wedge (\alpha \vee \gamma)]$ and $1 - (\gamma \wedge [\alpha \vee \beta])$. Also by computation, one can find that the fuzzy regular closed sets are fuzzy G_δ-sets. Hence (X, T) is a fuzzy Oz-space. By computation, $\delta = (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$ $\wedge (\beta \vee \gamma)$ and thus δ is a fuzzy G_δ-set in (X, T). But δ is not a fuzzy open set in (X, T) and hence (X, T) is not a fuzzy P-space.

Proposition 4.1. *If a fuzzy topological space (X, T) is a fuzzy extremally disconnected space, then (X, T) is a fuzzy Oz-space.*

Proof. Let λ be a fuzzy regular closed set in (X, T). Then, $cl\,\mathrm{int}(\lambda) = \lambda$ (A). Now $\mathrm{int}(\lambda)$ is a fuzzy open set in (X, T). Since (X, T) is a fuzzy extremally disconnected space, $cl(\mathrm{int}(\lambda))$ is a fuzzy open set in (X, T) and then from (A), λ is a fuzzy open set in (X, T). Also for the fuzzy open set λ in the fuzzy extremally disconnected space (X, T), $cl(\lambda)$ is a fuzzy open set in (X, T). Now $\lambda \leq cl(\lambda)$, implies that $\lambda = \lambda \wedge cl(\lambda)$ and $\lambda, cl(\lambda) \in T$, implies that λ is a fuzzy G_δ-set in (X, T). Hence the fuzzy regular closed set λ is a fuzzy G_δ-set in (X, T), implies that (X, T) is a fuzzy Oz-space.

Proposition 4.2. *If a fuzzy topological space (X, T) is a fuzzy Oz and fuzzy P-space, then (X, T) is a fuzzy extremally disconnected space.*

Proof. Let λ be a fuzzy open set in (X, T). If $cl(\lambda) = 1$, then clearly $cl(\lambda) \in T$. In this case (X, T) is a fuzzy extremally disconnected space. Suppose that λ is a fuzzy open set in (X, T) such that $cl(\lambda) \neq 1$. Since (X, T) is a fuzzy Oz-space, by Proposition 3.3, $cl(\lambda)$ is a fuzzy G_δ-set in (X, T). Also since (X, T) is a fuzzy P-space, the fuzzy G_δ-set $cl(\lambda)$ is a fuzzy open set in (X, T). Hence $cl(\lambda)$ is a fuzzy open set in (X, T), for the fuzzy open set λ in (X, T), implies that (X, T) is a fuzzy extremally disconnected space.

Remark 4.1. The converse of the above proposition need not be true. That is, a fuzzy extremally disconnected space need not be a fuzzy Oz and fuzzy P-space. For, consider the following example:

Example 4.1. Let $X = \{a, b, c\}$. Let $I = [0, 1]$. The fuzzy sets α, β and γ are defined on X as follows:

$\alpha : X \to I$ is defined by $\alpha(a) = 0.5; \alpha(b) = 0.6; \alpha(c) = 0.4,$

$\beta : X \to I$ is defined by $\beta(a) = 0.4; \beta(b) = 0.5; \beta(c) = 0.6,$

$\gamma : X \to I$ is defined by $\gamma(a) = 0.6; \gamma(b) = 0.4; \gamma(c) = 0.5$,

$\delta : X \to I$ is defined by $\delta(a) = 0.5; \delta(b) = 0.5; \delta(c) = 0.5$.

Then, $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \beta \vee [\alpha \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \alpha \wedge [\beta \vee \gamma], \beta \wedge [\alpha \vee \gamma], \gamma \wedge [\alpha \vee \beta], \alpha \wedge \beta \wedge \gamma, \alpha \vee \beta \vee \gamma, 1\}$ is a fuzzy topology on X. By computation, one can find that the closure of each fuzzy open set is fuzzy open in (X, T) and thus (X, T) is a fuzzy extremally disconnected space. By computation, one can find that $\delta = (\alpha \vee \beta) \wedge (\alpha \vee \gamma) \wedge (\beta \vee \gamma) \wedge (\alpha \vee \beta \vee \gamma) \wedge (\alpha \vee [\beta \wedge \gamma]) \wedge (\beta \vee [\alpha \wedge \gamma])$ and thus δ is a fuzzy G_δ-set in (X, T). But δ is not fuzzy open in (X, T), implies that (X, T) is not a fuzzy P-space. Also $1 - (\alpha \wedge \beta \wedge \gamma)$ is a fuzzy regular closed set in (X, T). Now $1 - (\alpha \wedge \beta \wedge \gamma) \neq \wedge_{i=1}^{\infty}(\theta_i)$, where (θ_i)'s are fuzzy open sets in (X, T), implies that (X, T) is not a fuzzy Oz-space.

Proposition 4.3. *If a fuzzy topological space (X, T) is a fuzzy Oz and fuzzy P-space, then (X, T) is a fuzzy Moscow space.*

Proof. Let (X, T) be a fuzzy Oz and fuzzy P-space. Then, by Proposition 4.2, (X, T) is a fuzzy extremally disconnected space. By Theorem 2.6, (X, T) is a fuzzy Moscow space.

Remark 4.2. It is to be noted that a fuzzy Moscow space need not be a fuzzy Oz-space. For, consider the following example:

Example 4.2. Let $X = \{a, b, c\}$. Let $I = [0, 1]$. The fuzzy sets α, β and γ are defined on X as follows:

$\alpha : X \to I$ is defined by $\alpha(a) = 0.6; \alpha(b) = 0.4; \alpha(c) = 0.6$,

$\beta : X \to I$ is defined by $\beta(a) = 0.5; \beta(b) = 0.5; \beta(c) = 0.5$,

$\gamma : X \to I$ is defined by $\gamma(a) = 0.4; \gamma(b) = 0.6; \gamma(c) = 0.4$.

Then, $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, 1\}$ is a fuzzy topology on X. By computation, one can find that $\alpha, \gamma, \alpha \wedge \beta, \alpha \wedge \gamma$ and $\beta \wedge \gamma$ are fuzzy G_δ-sets in (X, T) and for each fuzzy open set

$\theta = (\alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma)$ in (X, T), $cl(\theta) = \vee_{i=1}^{\infty}(\delta_i)$, where (δ_i)'s $= (\alpha, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma)$ are fuzzy G_δ-sets in (X, T), implies that (X, T) is a fuzzy Moscow space. Also by computation $cl\,\mathrm{int}[1 - (\alpha \wedge \gamma)] = 1 - (\alpha \wedge \gamma)$ implies that $1 - (\alpha \wedge \gamma)$ is a fuzzy regular closed set in (X, T). Now $1 - (\alpha \wedge \beta \wedge \gamma) \neq \wedge_{i=1}^{\infty}(\theta_i)$, where (θ_i)'s are fuzzy open sets in (X, T), implies that (X, T) is not a fuzzy Oz-space.

The following proposition gives a condition for a fuzzy Oz-space to become a fuzzy Moscow space.

Proposition 4.4. *If* $cl(\lambda) = \vee_{i=1}^{\infty}(\delta_i)$, *where* (δ_i)*'s are fuzzy regular closed sets, for a fuzzy open set* λ *in a fuzzy Oz-space* (X, T), *then* (X, T) *is a fuzzy Moscow space.*

Proof. Suppose that for a fuzzy open set λ in (X, T), $cl(\lambda) = \vee_{i=1}^{\infty}(\delta_i)$, where (δ_i)'s are fuzzy regular closed sets in (X, T). Since (X, T) is a fuzzy Oz-space, the fuzzy regular closed sets (δ_i)'s are fuzzy G_δ-sets in (X, T). Thus, for a fuzzy open set λ in (X, T), $cl(\lambda) = \vee_{i=1}^{\infty}(\delta_i)$ where (δ_i)'s are fuzzy G_δ-sets in (X, T) implies that (X, T) is a fuzzy Moscow space.

The following proposition shows that fuzzy semi-open sets are fuzzy pre-open sets in fuzzy Oz and fuzzy P-spaces.

Proposition 4.5. *If* λ *is a fuzzy semi-open set in a fuzzy Oz and fuzzy P-space* (X, T), *then* λ *is a fuzzy pre-open set in* (X, T).

Proof. Let λ be a fuzzy semi-open set in (X, T). Since (X, T) is a fuzzy Oz and fuzzy P-space (X, T), by Proposition 4.2, (X, T) is a fuzzy extremally disconnected space. By Theorem 2.7, the fuzzy semi-open set λ is a fuzzy pre-open set in (X, T).

Proposition 4.6. *If a fuzzy topological space* (X, T) *is a fuzzy Oz and fuzzy P-space, then for each fuzzy closed set* μ *in* (X, T), $\mathrm{int}(\mu)$ *is a fuzzy closed set in* (X, T).

Proof. Let μ be a fuzzy closed set in (X, T). Then, by Theorem 2.1, $\operatorname{int}(\mu)$ is a fuzzy regular open set in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.1, the fuzzy regular open set $\operatorname{int}(\mu)$ is a fuzzy F_σ-set in (X, T). Since (X, T) is a fuzzy P-space, the fuzzy F_σ-set $\operatorname{int}(\mu)$ is a fuzzy closed set in (X, T).

Proposition 4.7. *If a fuzzy topological space (X, T) is a fuzzy Oz and fuzzy P-space, then for every pair of fuzzy open sets λ and μ in (X, T) with $cl(\lambda) = 1 - \mu$, then $cl(\lambda) = 1 - cl(\mu)$, in (X, T).*

Proof. Let λ and μ be fuzzy open sets with $cl(\lambda) = 1 - \mu$, in (X, T). Then, $cl(\lambda) + \mu = 1$, in (X, T). Since (X, T) is a fuzzy Oz and fuzzy P-space, by Proposition 4.2, (X, T) is a fuzzy extremally disconnected space. Then, by Theorem 2.4 (d), for the pair of fuzzy open sets λ and μ in (X, T), $cl(\lambda) + cl(\mu) = 1$ and hence $cl(\lambda) = 1 - cl(\mu)$, in (X, T).

Proposition 4.8. *If λ is a fuzzy open set in a fuzzy Oz and fuzzy P-space (X, T), then there exists a fuzzy open set μ in (X, T) such that $\lambda \leq \mu$.*

Proof. Let λ be a fuzzy open set in (X, T). Since (X, T) is a fuzzy Oz and fuzzy P-space, by Proposition 4.2, (X, T) is a fuzzy extremally disconnected space. Then, $cl(\lambda)$ is a fuzzy open set in (X, T). Let $\mu = cl(\lambda)$. Now $\lambda \leq cl(\lambda)$, implies that $\lambda \leq \mu$, in (X, T). Thus, for the fuzzy open set λ, there exists a fuzzy open set μ in (X, T) such that $\lambda \leq \mu$.

Remark 3.3. In view of the above proposition one will have the following result: "If λ is a fuzzy open set in a fuzzy Oz and fuzzy P-space (X, T), then there exist an increasing sequence of fuzzy open sets $\mu_i (i = 1 \text{ to } \infty)$ in (X, T) such that $\lambda \leq \mu_1 \leq \mu_2 \leq \mu_3 \leq \mu_4 \leq \ldots$."

Proposition 4.9. *If δ is a fuzzy closed set in a fuzzy Oz and fuzzy P-space (X, T), then there exists a fuzzy closed set η in (X, T) such that $\eta \leq \delta$.*

Proof. Let δ be a fuzzy closed set in (X, T). Then, $1 - \delta$ is a fuzzy open set in (X, T). Since (X, T) is a fuzzy Oz and fuzzy P-space, by Proposition 4.8, there exists a fuzzy open set μ in (X, T) such that $1 - \delta \leq \mu$ and then

$1-\mu \leq \delta$. Let $\eta = 1-\mu$. Then, η is a fuzzy closed set in (X, T). Hence for the fuzzy closed set δ, there exists a fuzzy closed set η in (X, T) such that $\eta \leq \delta$.

Remark 4.4. In view of the above proposition one will have the following result: "If δ is a fuzzy closed set in a fuzzy Oz and fuzzy P-space (X, T), then there exist a decreasing sequence of fuzzy closed sets $\eta_i (i = 1$ to $\infty)$ in (X, T) such that $\delta \geq \eta_1 \geq \eta_2 \geq \eta_3 \geq \eta_4 \geq \ldots$."

The following proposition shows that fuzzy Oz-spaces are not fuzzy hyperconnected spaces.

Proposition 4.10. *If a fuzzy topological space (X, T) is a fuzzy Oz-space then (X, T) is not a fuzzy hyperconnected space.*

Proof. Let λ be a fuzzy open set in (X, T) such that $cl(\lambda) \neq 1$. Since (X, T) is a fuzzy Oz-space, by Proposition 3.3, $cl(\lambda)$ is a fuzzy G_δ-set in (X, T). Then, $cl(\lambda) = \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$. This implies that $cl(\lambda) \leq \lambda_i$ for each $i \in I$ and $cl(\lambda) \leq \lambda_i \leq cl(\lambda_i)$ implies that $cl(\lambda) \neq 1$, in (X, T) and hence (X, T) is not a fuzzy hyperconnected space.

Proposition 4.11. *If a fuzzy topological space (X, T) is a fuzzy Oz and fuzzy P-space, then (X, T) is a fuzzy basically disconnected space.*

Proof. Let λ be a fuzzy open F_σ-set in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.3, $cl(\lambda)$ is a fuzzy G_δ-set, for the fuzzy open set λ in (X, T). Also since (X, T) is a fuzzy P-space, the fuzzy G_δ-set $cl(\lambda)$ is a fuzzy open set in (X, T). Hence, for the fuzzy open F_σ-set λ in (X, T), $cl(\lambda) \in T$, implies that (X, T) is a fuzzy basically disconnected space.

Proposition 4.12. *If λ is a fuzzy G_δ-set in a fuzzy Oz and fuzzy P-space (X, T) such that $cl(\lambda) \neq 1$, then $cl(\lambda)$ is a fuzzy G_δ-set in (X, T).*

Proof. Let λ be a fuzzy G_δ-set in (X, T) such that $cl(\lambda) \neq 1$. Since (X, T) is a fuzzy P-space, λ is a fuzzy open set in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.3, $cl(\lambda)$ is a fuzzy G_δ-set in (X, T).

Corollary 4.1. *If* μ *is a fuzzy* F_σ*-set in a fuzzy Oz and fuzzy P-space* (X, T) *such that* $\operatorname{int}(\mu) \neq 0$, *then* $\operatorname{int}(\mu)$ *is a fuzzy* F_σ*-set in* (X, T).

Proof. Let μ be a fuzzy F_σ-set in (X, T) such that $\operatorname{int}(\mu) \neq 0$. Then, $1 - \mu$ is a fuzzy G_δ-set in (X, T) such that $cl(1 - \mu) \neq 1$. Since (X, T) is a fuzzy Oz and fuzzy P-space, by Proposition 4.12, $cl(1 - \mu)$ is a fuzzy G_δ-set in (X, T). Now $cl(1 - \mu) = 1 - \operatorname{int}(\mu)$ [1], implies that $1 - \operatorname{int}(\mu)$ is a fuzzy G_δ-set in (X, T). Thus, $\operatorname{int}(\mu)$ is a fuzzy F_σ-set in (X, T).

The following proposition shows that fuzzy super-connected spaces are not fuzzy Oz-spaces.

Proposition 4.13. *If* (X, T) *is a fuzzy super-connected space, then* (X, T) *is not a fuzzy Oz-space.*

Proof. Let λ be a fuzzy open set in (X, T). Then, by Theorem 2.1, $cl(\lambda)$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy super connected space, by Theorem 2.5, $cl(\lambda) = 1$, in (X, T). Then, $cl(\lambda) \neq \wedge_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy open sets in (X, T). Hence the fuzzy regular closed set $cl(\lambda)$ is not a fuzzy G_δ-set in (X, T). Hence (X, T) is not a fuzzy Oz-space.

Proposition 4.14. *If* (λ_i)*'s* $(i = 1$ *to* $\infty)$ *are fuzzy residual sets in a fuzzy Oz-space and fuzzy Moscow space* (X, T), *then there exists a fuzzy* G_δ*-set* γ *in* (X, T) *such that* $\gamma \leq \vee_{i=1}^{\infty}(\lambda_i)$.

Proof. Let (λ_i)'s $(i = 1$ to $\infty)$ be fuzzy residual sets in (X, T). Since (X, T) is a fuzzy Moscow space, by Theorem 2.8, there exists a fuzzy regular closed set γ in (X, T) such that $\gamma \leq \vee_{i=1}^{\infty}(\lambda_i)$. Also since (X, T) is a fuzzy Oz-space, the fuzzy regular closed set γ is a fuzzy G_δ-set in (X, T). Hence there exists a fuzzy G_δ-set γ in (X, T) such that $\gamma \leq \vee_{i=1}^{\infty}(\lambda_i)$.

Proposition 4.15. *If* (μ_i)*'s* $(i = 1$ *to* $\infty)$ *are fuzzy first category sets in the fuzzy Oz and fuzzy Moscow space* (X, T), *then there exists a fuzzy* F_σ*-set* η *in* (X, T) *such that* $\wedge_{i=1}^{\infty}(\mu_i) \leq \eta$.

Proof. Let (μ_i)'s ($i = 1$ to ∞) be fuzzy first category sets in (X, T). Then, $[1 - (\mu_i)]$' are fuzzy residual sets in (X, T). Since (X, T) is a fuzzy Oz and fuzzy Moscow space, by Proposition 4.14, there exists a fuzzy G_δ-set γ in (X, T) such that $\gamma \leq \vee_{i=1}^{\infty}(1 - \mu_i)$. This implies that $\gamma \leq 1 - \vee_{i=1}^{\infty}(\mu_i)$ and then $\wedge_{i=1}^{\infty}(\mu_i) \leq 1 - \gamma$. Let $\eta = 1 - \gamma$. Hence there exists a fuzzy F_σ-set η in (X, T) such that $\wedge_{i=1}^{\infty}(\mu_i) \leq \eta$.

Proposition 4.16. *If λ and μ are fuzzy open sets in a fuzzy F-space and fuzzy Oz-space (X, T) such that $\lambda \leq 1 - \mu$, there exist fuzzy F_σ-sets δ and η such that $\lambda \leq \delta, \mu \leq \eta$ and $\lambda \leq cl(\delta) \leq 1 - cl(\eta) \leq 1 - \mu$, in (X, T).*

Proof. Let λ and μ be fuzzy open sets in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.13, there exist fuzzy F_σ-sets δ and η in (X, T) such that $\lambda \leq \delta, \mu \leq \eta$ and $\delta \leq 1 - \eta$. Also since (X, T) is a fuzzy F-space, for the fuzzy F_σ-sets δ and η with $\delta \leq 1 - \eta$, in (X, T) and $\lambda \leq cl(\delta) \leq 1 - cl(\eta) \leq 1 - \mu$, in (X, T).

Proposition 4.17. *If λ and μ are fuzzy open sets in a fuzzy F-space and fuzzy Oz-space (X, T) such that $\lambda \leq 1 - \mu$, then $cl(\lambda) \leq 1 - cl(\mu)$, in (X, T).*

Proof. Let λ and μ be fuzzy open sets in (X, T). Since (X, T) is a fuzzy F-space and fuzzy Oz-space, by Proposition 4.16, there exist fuzzy F_σ-sets δ and η such that $\lambda \leq \delta, \mu \leq \eta$ and $cl(\delta) \leq 1 - cl(\eta)$, in (X, T). Now $\lambda \leq \delta, \mu \leq \eta$ implies that $cl(\lambda) \leq cl(\delta)$ and $cl(\mu) \leq cl(\eta)$ and then $cl(\lambda) \leq cl(\delta) \leq 1 - cl(\eta) \leq 1 - cl(\mu)$. Hence, for the fuzzy open sets λ and μ with $\lambda \leq 1 - \mu$ in a fuzzy F-space and fuzzy Oz-space (X, T), $cl(\lambda) \leq 1 - cl(\mu)$ in (X, T).

Proposition 4.18. *If λ and μ are fuzzy open sets in a fuzzy perfectly disconnected and fuzzy Oz-space (X, T) such that $\lambda \leq 1 - \mu$, then $cl(\lambda) \leq 1 - cl(\mu)$, in (X, T).*

Proof. The proof follows from Proposition 4.17 and Theorem 2.10.

The following proposition shows that fuzzy perfectly disconnected and fuzzy Oz-spaces are not fuzzy hyperconnected spaces.

Proposition 4.19. *If a fuzzy topological space (X, T) is a fuzzy perfectly disconnected and fuzzy Oz-space, then (X, T) is not a fuzzy hyperconnected space.*

Proof. Let λ and μ be non-zero fuzzy open sets in (X, T) with $\lambda \leq 1 - \mu$. Since (X, T) is a fuzzy perfectly disconnected and fuzzy Oz-space, by Proposition 4.18, $cl\lambda \leq 1 - cl(\mu)$, in (X, T). If $cl(\lambda) = 1$, then $1 \leq 1 - cl(\mu)$ and $cl(\mu) = 0$ and this will result in $\mu = 0$, in (X, T), a contradiction. Also, if $cl(\mu) = 1$, then $cl(\mu) \leq 1 - 1 = 0$ and $cl(\lambda) = 0$ and this will result in $\lambda = 0$, in (X, T), a contradiction. Hence the fuzzy open sets λ and μ are not fuzzy dense sets in (X, T). Hence (X, T) is not a fuzzy hyperconnected space.

The following proposition ensures the existence of fuzzy G_δ-sets by means of fuzzy pre-closed sets in fuzzy perfectly disconnected and fuzzy Oz-spaces

Proposition 4.20. *If λ is a fuzzy pre-closed set in a fuzzy perfectly disconnected and fuzzy Oz-space, then there exists a fuzzy G_δ-set δ in (X, T) such that $\delta \leq \lambda$.*

Proof. Let λ be a fuzzy pre-closed set in (X, T). Since (X, T) is a fuzzy perfectly disconnected space, by Theorem 2.11, $\text{int}(\lambda)$ is a fuzzy regular closed set in (X, T). Let $\delta = \text{int}(\lambda)$ and then $\delta \leq \lambda$, in (X, T). Since (X, T) is a fuzzy Oz-space, the fuzzy regular closed set δ is a fuzzy G_δ-set in (X, T). Hence there exists a fuzzy G_δ-set δ in (X, T) such that $\delta \leq \lambda$.

Proposition 4.21. *If λ is a fuzzy regular open set in a fuzzy extremally disconnected space (X, T), then λ is a fuzzy closed F_σ-set in (X, T).*

Proof. Let λ be a fuzzy regular open set in (X, T). Then, $\text{int}\, cl(\lambda) = \lambda$ (A). Since (X, T) is a fuzzy extremally disconnected space, for the fuzzy open set $\text{int}\, cl(\lambda)$ in (X, T), $cl[\text{int}\, cl(\lambda)]$ is a fuzzy open set in (X, T). From (A), $cl[\text{int}\, cl(\lambda)] = cl(\lambda)$, in (X, T). Thus, $cl(\lambda)$ is a fuzzy open set in (X, T) and then $\text{int}\, cl(\lambda) = cl(\lambda)$ (B). From (A) and (B), $\lambda = \text{int}\, cl(\lambda) = cl(\lambda)$ and thus $\lambda = cl(\lambda)$, in (X, T). This implies that λ is a fuzzy closed set in (X, T). By

Proposition 4.1, the fuzzy extremally disconnected space (X, T) is a fuzzy Oz-space and thus by Proposition 3.1, the fuzzy regular open set λ is a fuzzy F_σ-set in (X, T). Hence the fuzzy regular open set λ is a fuzzy closed F_σ-set in (X, T).

Proposition 4.22. *If* μ *is a fuzzy regular closed set in a fuzzy extremally disconnected space* (X, T), *then* μ *is a fuzzy open* G_δ*-set in* (X, T).

Proof. Let μ be a fuzzy regular closed set in (X, T). Then, $1 - \mu$ is a fuzzy regular open set in (X, T). Since (X, T) is a fuzzy extremally disconnected space, by Proposition 4.21, $1 - \mu$ is a fuzzy closed F_σ-set in (X, T). This implies that μ is a fuzzy open G_δ-set in (X, T).

5. Weak Fuzzy Oz-Spaces

Definition 5.1. A fuzzy topological space (X, T) is called a weak fuzzy Oz-space if for each fuzzy F_σ-set δ in (X, T), $cl(\delta)$ is a fuzzy G_δ-set in (X, T).

Example 5.1. Let $X = \{a, b, c\}$. Let $I = [0, 1]$. The fuzzy sets α, β, γ and δ are defined on X as follows:

$\alpha : X \to I$ is defined by $\alpha(a) = 0.5; \alpha(b) = 0.6; \alpha(c) = 0.4,$

$\beta : X \to I$ is defined by $\beta(a) = 0.4; \beta(b) = 0.5; \beta(c) = 0.6,$

$\gamma : X \to I$ is defined by $\gamma(a) = 0.6; \gamma(b) = 0.4; \gamma(c) = 0.5,$

$\delta : X \to I$ is defined by $\delta(a) = 0.5; \delta(b) = 0.5; \delta(c) = 0.5.$

Then, $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \beta \vee [\alpha \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \alpha \wedge [\beta \vee \gamma], \beta \wedge [\alpha \vee \gamma], \gamma \wedge [\alpha \vee \beta], \alpha \wedge \beta \wedge \gamma, \alpha \vee \beta \vee \gamma, 1\}$ is a fuzzy topology on X. Also by computation, one can find that $\delta = [1 - (\alpha \vee \beta)] \vee [1 - (\alpha \vee \gamma)] \vee [1 - (\beta \vee \gamma)] \vee [1 - (\alpha \vee [\beta \wedge \gamma])]$ and δ is a fuzzy F_σ-set in (X, T). By computation $cl(\delta) = [1 - (\alpha \wedge [\beta \vee \gamma])]$, in (X, T). Also $[1 - (\alpha \wedge [\beta \vee \gamma])] = (\alpha \vee \beta) \wedge (\beta \vee \gamma) \wedge (\alpha \vee \beta \vee \gamma)$, implies that $1 - (\alpha \wedge [\beta \vee \gamma])$

is a fuzzy G_δ-set in (X, T). Thus, for the fuzzy F_σ-set δ in (X, T), $cl(\delta)$ is a fuzzy G_δ-set in (X, T), implies that (X, T) is a weak fuzzy Oz-space. It is to be noted that δ is not a fuzzy open set in (X, T).

Proposition 5.1. *If* η *is a fuzzy* G_δ*-set in a weak fuzzy Oz-space* (X, T), *then* $\text{int}(\lambda)$ *is a fuzzy* F_σ*-set in* (X, T).

Proof. Let λ be a fuzzy G_δ-set in (X, T). Then, $1 - \eta$ is a fuzzy F_σ-set in (X, T). Since (X, T) is a weak fuzzy Oz-space, $cl(1 - \eta)$ is a fuzzy G_δ-set in (X, T). Now $cl(1 - \eta) = 1 - \text{int}(\eta)$, implies that $1 - \text{int}(\eta)$ is a fuzzy G_δ-set in (X, T) and thus $\text{int}(\eta)$ is a fuzzy F_σ-set in (X, T).

Proposition 5.2. *If* λ *is a fuzzy first category set in a weak fuzzy Oz-space* (X, T), *then there exists a fuzzy* G_δ*-set* γ *in* (X, T) *such that* $cl(\lambda) \leq \gamma$.

Proof. Let λ be a fuzzy first category set in (X, T). By Theorem 2.12, there is a fuzzy F_σ-set η in (X, T) such that $\lambda \leq \eta$. Then, $cl(\lambda) \leq cl(\eta)$. Since (X, T) is a weak fuzzy Oz-space, $cl(\lambda)$ is a fuzzy G_δ-set in (X, T). Let $\gamma = cl(\eta)$. Then, γ is a fuzzy G_δ-set in (X, T) such that $cl(\lambda) \leq \gamma$.

Proposition 5.3. *If* μ *is a fuzzy residual set in a weak fuzzy Oz-space* (X, T), *then there exists a fuzzy* F_σ*-set* δ *in* (X, T) *such that* $\delta \leq \text{int}(\mu)$.

Proof. Let μ be a fuzzy residual set in (X, T). Then, $1 - \mu$ is a fuzzy first category set in (X, T). Since (X, T) is a weak fuzzy Oz-space, by Proposition 5.2, there exists a fuzzy G_δ-set γ in (X, T) such that $cl(1 - \mu) \leq \gamma$. Then, $1 - \text{int}(\mu) \leq \gamma$ and $1 - \gamma \leq \text{int}(\mu)$. Let $\delta = 1 - \gamma$. Then, δ is a F_σ-set in (X, T) such that $\delta \leq \text{int}(\mu)$.

Proposition 5.4. *If* λ *is a fuzzy* σ*-boundary set in a weak fuzzy Oz-space* (X, T), *then* $cl(\lambda)$ *is a fuzzy* G_δ*-set in* (X, T).

Proof. Let λ be a fuzzy σ-boundary set in (X, T). By Theorem 2.3, λ is a fuzzy F_σ-set in (X, T). Since (X, T) is a weak fuzzy Oz-space, $cl(\lambda)$ is a fuzzy G_δ-set in (X, T).

Proposition 5.5. *If* λ *is a fuzzy* σ*-boundary set in a weak fuzzy Oz-space* (X, T), *then* $\operatorname{int} cl(\lambda)$ *is a fuzzy* F_σ*-set in* (X, T).

Proof. Let λ be a fuzzy σ-boundary set in (X, T). Since (X, T) is a weak fuzzy Oz-space, by Proposition 5.4, $cl(\lambda)$ is a fuzzy G_δ-set in (X, T). By Proposition 5.1, $\operatorname{int} cl(\lambda)$ is a fuzzy F_σ-set in (X, T).

Proposition 5.6. *If* λ *is a fuzzy* σ*-boundary set in a weak fuzzy Oz-space* (X, T), *then* λ *is a fuzzy somewhere dense set in* (X, T).

Proof. Let λ be a fuzzy σ-boundary set in (X, T). Since (X, T) is a weak fuzzy Oz-space, by Proposition 5.5, $\operatorname{int} cl(\lambda)$ is a fuzzy F_σ-set in (X, T) and then $\operatorname{int} cl(\lambda) \neq 0$, in (X, T). Hence λ is a fuzzy somewhere dense set in (X, T).

Proposition 5.7. *If* $\lambda \leq 1 - \mu$, *where* λ *is a fuzzy open set and* μ *is a fuzzy* F_σ*-set in a weak fuzzy Oz-space* (X, T), *then there exist fuzzy* F_σ*-sets* δ *and* λ *in* (X, T) *such that* $\lambda \leq \delta$, $\mu \leq \eta$.

Proof. Suppose that $\lambda \leq 1 - \mu$, where λ is a fuzzy open set and μ is a fuzzy F_σ-set in (X, T). Now $\lambda \leq 1 - \mu$, implies that $\operatorname{int}(\lambda) \leq \operatorname{int}(1 - \mu)$ and then $\operatorname{int}(\lambda) \leq 1 - cl(\mu)$, in (X, T). Since $\lambda = \operatorname{int}(\lambda), \lambda \leq 1 - cl(\mu)$, in (X, T). Since (X, T) is a weak fuzzy Oz-space, $cl(\mu)$ is a fuzzy G_δ-set in (X, T). Then, $1 - cl(\mu)$, is a fuzzy F_σ-set in (X, T). Let $\delta = 1 - cl(\mu)$ and then δ is a fuzzy F_σ-set in (X, T) such that $\lambda \leq \delta$. (A).

Also $\lambda \leq 1 - \mu$, implies that $\mu \leq 1 - \lambda$ and $cl(\mu) \leq cl(1 - \lambda)$. Since $\lambda \in T, 1 - \lambda$ is fuzzy closed in (X, T) and $cl(1 - \lambda) = 1 - \lambda$ and then $cl(\mu) \leq (1 - \lambda)$. Now $1 - \lambda = \mu \vee cl(\mu) \vee (1 - \lambda)$, where μ is a fuzzy F_σ-set and $(1 - \lambda), cl(\mu)$ are fuzzy closed sets in (X, T) implies that $1 - \lambda$ is a fuzzy F_σ-set in (X, T). Let $\eta = 1 - \lambda$ and then η is a fuzzy F_σ-set in (X, T) such that $\mu \leq \eta$ (B). Hence, for the fuzzy open set λ and a fuzzy F_σ-set μ in (X, T) such that $\lambda \leq 1 - \mu$, from (A) and (B), there exist fuzzy F_σ-sets δ and η in (X, T) such that $\lambda \leq \delta$, $\mu \leq \eta$.

6. Weak Fuzzy Oz-Spaces and Other Fuzzy Topological Spaces

The following propositions give conditions under which fuzzy Oz-spaces become weak fuzzy Oz-spaces.

Proposition 6.1. *If each fuzzy F_σ-set is a fuzzy open set in a fuzzy Oz-space (X, T), then (X, T) is a weak fuzzy Oz-space.*

Proof. Let δ be a fuzzy F_σ-set in (X, T). By hypothesis, δ is a fuzzy open set in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.3, $cl(\delta)$ is a fuzzy G_δ-set in (X, T). Hence, for the fuzzy F_σ-set δ, $cl(\delta)$ is a fuzzy G_δ-set in (X, T), implies that (X, T) is a weak fuzzy Oz-space.

Proposition 6.2. *If the closure of each fuzzy F_σ-set is a fuzzy regular closed set in a fuzzy Oz-space (X, T), then (X, T) is a weak fuzzy Oz-space.*

Proof. Let δ be a fuzzy F_σ-set in (X, T). By hypothesis, $cl(\delta)$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.3, $cl(\delta)$ is a fuzzy G_δ-set in (X, T). Hence, for the fuzzy F_σ-set δ, $cl(\delta)$ is a fuzzy G_δ-set in (X, T), implies that (X, T) is a weak fuzzy Oz-space.

Proposition 6.3. *If the interior of each fuzzy G_δ-set is a fuzzy regular open set in a fuzzy Oz-space (X, T), then (X, T) is a weak fuzzy Oz-space.*

Proof. Let η be a fuzzy F_σ-set in (X, T). Then, $1 - \eta$ is a fuzzy G_δ-set in (X, T). By hypothesis, $\operatorname{int}(1 - \lambda)$ is a fuzzy regular open set in (X, T). Then, $1 - cl(\lambda)$ is a fuzzy regular open set in (X, T) and $cl(\eta)$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy Oz-space, by Proposition 3.3, $cl(\eta)$ is a fuzzy G_δ-set in (X, T). Hence, for the fuzzy F_σ-set η, $cl(\eta)$ is a fuzzy G_δ-set in (X, T), implies that (X, T) is a weak fuzzy Oz-space.

Proposition 6.4. *If each fuzzy F_σ-set is a fuzzy regular open set in a fuzzy Oz-space (X, T), then (X, T) is a weak fuzzy Oz-space.*

Proof. Let λ be a fuzzy F_σ-set in (X, T). By hypothesis λ is a fuzzy regular open set in (X, T). Since λ is a fuzzy regular open set, λ is a fuzzy open set in (X, T) and by Theorem 2.1, $cl(\lambda)$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy Oz-space, $cl(\lambda)$ is a fuzzy G_δ-set in (X, T). Thus, for the fuzzy F_σ-set λ in (X, T), $cl(\lambda)$ is a fuzzy G_δ-set in (X, T), implies that (X, T) is a weak fuzzy Oz-space.

The following proposition gives a condition under which weak fuzzy Oz-spaces become fuzzy Oz-spaces.

Proposition 6.5. *If (X, T) is a fuzzy fraction dense and weak fuzzy Oz-space, then (X, T) is a fuzzy Oz-space.*

Proof. Let μ be a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Theorem 2.13, $\mu = cl(\eta)$, where η is a fuzzy F_σ-set in (X, T). Since (X, T) is a weak fuzzy Oz-space, $cl(\eta)$ is a fuzzy G_δ-set in (X, T). Thus the fuzzy regular closed set μ is a fuzzy G_δ-set in (X, T). Hence (X, T) is a fuzzy Oz-space.

Proposition 6.6. *If λ is a fuzzy open set in a fuzzy fraction dense and weak fuzzy Oz-space (X, T), then there exists a fuzzy G_δ-set η such that $\lambda \leq \eta$ in (X, T).*

Proof. Let λ be a fuzzy open set in (X, T). Since (X, T) is a fuzzy fraction dense space, $cl(\lambda) = cl(\mu)$, where μ is a fuzzy F_σ-set in (X, T). Now $\lambda \leq cl(\lambda)$, implies that $\lambda \leq cl(\mu)$, in (X, T). Since (X, T) is a weak fuzzy Oz-space, $cl(\mu)$ is a fuzzy G_δ-set in (X, T). Let $\eta = cl(\mu)$. Hence there exists a fuzzy G_δ-set η such that $\lambda \leq \eta$ in (X, T).

Corollary 6.1. *If μ is a fuzzy closed set in a fuzzy fraction dense and weak fuzzy Oz-space (X, T), then there exists a fuzzy F_σ-set δ such that $\delta \leq \mu$ in (X, T).*

Proof. Let μ be a fuzzy closed set in (X, T). Then, $1 - \mu$ is a fuzzy open set in (X, T). Since (X, T) is a fuzzy fraction dense and weak fuzzy Oz-space, by Proposition 6.6, there exists a fuzzy G_δ-set η such that $1 - \mu \leq \eta$, in (X, T). Then, $1 - \eta \leq \mu$. Let $\delta = 1 - \eta$ and then δ is a fuzzy

F_σ-set in (X, T). Hence, for the fuzzy closed set μ, there exists a fuzzy F_σ-set δ in (X, T) such that $\delta \leq \mu$.

Proposition 6.7. *If (X, T) is a fuzzy fraction dense and fuzzy Oz-space, then (X, T) is a weak fuzzy Oz-space.*

Proof. Let μ be a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Theorem 2.13, $\mu = cl(\eta)$, where η is a fuzzy F_σ-set in (X, T). Since (X, T) is a fuzzy Oz-space, the fuzzy regular closed set μ is a fuzzy G_δ-set in (X, T). Thus, for the fuzzy F_σ-set η in (X, T), $cl(\eta)$ is a fuzzy G_δ-set in (X, T), implies that (X, T) is a weak fuzzy Oz-space.

Remark 6.1. In view of the Propositions 6.5 and 6.7, one will have the following result: "Fuzzy fraction dense and fuzzy Oz-space are weak fuzzy Oz-spaces and fuzzy fraction dense and weak fuzzy Oz-space are fuzzy Oz-spaces".

Proposition 6.8. *If $\lambda \leq 1 - \mu$, where λ and μ are fuzzy σ-boundary sets in a fuzzy F-space and weak fuzzy Oz-space (X, T), then there exist fuzzy G_δ-sets δ and η in (X, T) such that $\lambda \leq \delta, \mu \leq \eta$ and $\delta \leq 1 - \eta$.*

Proof. Suppose that $\lambda \leq 1 - \mu$, where λ and μ are fuzzy σ-boundary sets in (X, T). Since (X, T) is a fuzzy F-space, by Theorem 2.14, $cl(\lambda) \leq 1 - cl(\mu)$, in (X, T). By Proposition 5.4, for the fuzzy σ-boundary sets λ and μ in the weak fuzzy Oz-space (X, T), $cl(\lambda)$ and $cl(\mu)$ are fuzzy G_δ-sets in (X, T). Let $\delta = cl(\lambda)$ and $\eta = cl(\mu)$. Then it follows that $\lambda \leq \delta, \mu \leq \eta$ and $\delta \leq 1 - \eta$, in (X, T).

Proposition 6.9. *If (X, T) is a weak fuzzy Oz-space and fuzzy P-space, then (X, T) is a basically disconnected space.*

Proof. Let λ be a fuzzy open F_σ-set in (X, T). Since (X, T) is a weak fuzzy Oz-space, for the fuzzy F_σ-set λ, $cl(\lambda)$ is a fuzzy G_δ-set in (X, T). Then the fuzzy G_δ-set $cl(\lambda)$ is a fuzzy open set in the fuzzy P-space (X, T). Hence for the fuzzy open F_σ-set λ, $cl(\lambda)$ is a fuzzy open set in (X, T), implies that (X, T) is a basically disconnected space.

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