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MATHEMATICAL MODELLING OF MOISTURE CONTENT IN UNSATURATED HETEROGENEOUS SOIL AND ITS SOLUTION BY USING METHOD OF FUNCTIONAL SEPARABLE

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Abstract

This paper discusses the mathematical model of moisture content in unsaturated heterogeneous soil. The governing nonlinear partial differential equation is solved by using Method of Functional Separable with suitable initial and boundary conditions with aqueous diffusivity coefficients. The functional separable solution is obtained as quadratic polynomial form. The numerical results and graphical presentation of the solution is given via MATLAB.

Introduction

Groundwater is extremely vital to our lifestyle. Groundwater is derived from downpour and softening snow that falls down from the surface; it occurs among soil particles in open pore spaces or in rock cracks and cracks. Infiltration is the mechanism by which precipitation or water flows through subsurface soils and passes into rocks through cracks and pore spaces. Most

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drinking water supplies and frequently irrigation water for horticultural requirements are drawn from underground sources. More than 90 percentage of the liquid fresh water available on or near the surface of earth is groundwater. The water infiltration performs an important position to manipulate salinity of water; water pollution and agriculture purpose. This is moreover worthwhile in the fields of chemical and nuclear waste management issues.

Groundwater is appeared usually in two areas. The unsaturated zone, immediately beneath earth's surface, consisting of water and air within the open spaces, or pores. Beneath the unsaturated zone is the saturated region, in which both pores and rock fractures are filled with water. The boundary between the unsaturated zone and the underground saturated zone is known as the water table. The water table might be simply underneath or several feet underneath the earth's surface.

Moisture content is the amount of water present in a substance, for example, soil (known as soil moisture), rock, ceramics, natural product or wood. It is used in a wide variety of scientific and technological fields and is defined as a ratio that can vary from zero (totally dry) to the porosity value of the products at saturation. The value of the moisture content in dry soil is zero in the unsaturated porous form of media and its value is one if the porous medium is entirely saturated with water. The moisture content boundary is from zero to one. It may be volumetric or (gravimetric) dependent [11]. The Volumetric content of moisture is interpreted to mathematically as:

$$\theta = \frac{V_W}{V_T}$$

Here V_W is the volume of water and $V_T = V_S + V_W + V_A$ is the total volume (i.e. soil volume + water volume + air space).

In different practical situations, the water flow through the soil has been found to be unsteady and slightly saturated due to changes in the moisture content as a function of depth and time. It is partially saturated since not all pore spaces are completely filled with flowing liquid.

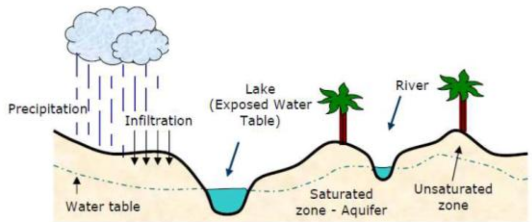

Figure 1. Moisture zones [2].

The phenomenon of one-dimensional movement of ground water is of great importance for hydrologists, agriculturists and people related to the science of water resources. This phenomenon has been well-tried to make clear according to many researchers from various points of view. Here are a few examples. A solution of groundwater flow equation using Homotopy decomposition method is explained by Atangana A. and Botha J. [1]. A numerical model to simulate water flow via unsaturated zone is studied by Prasad H. et al. [4]. An approximate solution using singular perturbation method is found by Mehta M. [7]. A pressure head in unsaturated soil through infiltration using Homotopy analysis method is determined by Patel K. et al. [12]. The Burger's equation for one-dimensional vertical groundwater recharge in unsaturated homogeneous porous media through q-Homotopy analysis approach is solved by Shah K. and Singh T. [16]. A mathematical model of ground water flow using Laplace homotopy perturbation method is discussed by Tawfiq L. and Jabber A. [18]. The problem on hydrological circumstance is verified by Verma A. [19].

The Method of Functional Separable has given an approximate solution to the problem of vertical groundwater in unsaturated heterogeneous soil.

Assumption and mathematical description of problem

We have assumed for this model that groundwater recharge occurs over a large reservoir of such a geological location as shown in Figure 2, which is the vertical cross-section taken from Figure 1.

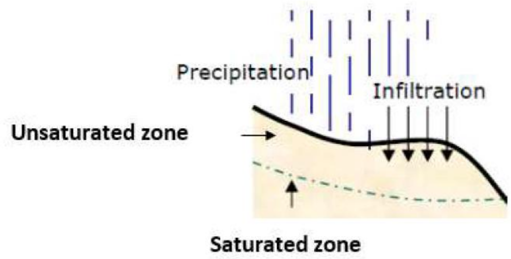

Figure 2. Moisture content in unsaturated zone.

The following assumptions have been considered for this model.

- The medium is heterogeneous soil and the porosity and permeability are therefore chosen to be variable.
- At atmospheric pressure, the air seems stagnant in the void space.
- The flowing of water below the ground surface considered to be continuous in vertically downward direction, neglecting spreading in other direction.
- The rule of Darcy is valid [2].
- The coefficient of aqueous diffusivity is preferred as being directly proportional to the moisture content [8].

The moisture content (θ_w) is considered to be variable, however Depth (Z) and Time (T) are regarded to be independent variables.

Mathematical Construction

In this model studied, when water or rainfall takes place in a downward direction on the surface of the soil, Darcy's law [4] on the motion of water in a porous medium is complied with.

$$\tilde{v} = -k \cdot \Delta\mu \tag{1}$$

where $\tilde{v}$: Volume flux of moisture,

k : Coefficient of aqueous variable conductivity and

$\Delta\mu$: Gradient of the whole moisture potential.

As such flow takes place, the equation of continuity for unsaturated soil must be satisfied.

$$\frac{\partial(\rho_s PS)}{\partial T} = -\Delta m = -\nabla(\rho \tilde{v}) \tag{2}$$

where ρ_s : Bulk density of the heterogeneous medium on dry weight basis,

m : Mass of flux of the water at any time $T > 0$,

ρ : Flux density,

S : Saturation of water and

P : Variable porosity of heterogeneous porous media.

Taking variation laws for the porosity and permeability of the uniform heterogeneous medium described as a function of Z [9].

$$P(Z) = \frac{1}{a - bZ} \tag{3}$$

$$k(Z) = k_c(1 + dZ) \tag{4}$$

Here a, b and d are constants.

By the incompressibility of water and considering the relation $\theta_w = P(Z)\, S(Z, T)$ [5] in equation (2), we get

$$\frac{\partial(\rho_s \theta_w)}{\partial T} = -\nabla(\rho \tilde{v}) \tag{5}$$

Using equation (1), we find

$$\frac{\partial(\rho_s \theta_w)}{\partial T} = \nabla(\rho k \nabla \mu) \tag{6}$$

The flow is considered to actually happen only in a vertical downward direction then, $\nabla \mu = \frac{\partial \mu}{\partial Z}$ [17], equation (6) reduced to,

$$\rho_s \frac{\partial \theta_w}{\partial T} = \rho \frac{\partial}{\partial Z}\left(k \frac{\partial \mu}{\partial Z}\right) \tag{7}$$

However, given the relationship $\psi_c = \mu - gZ$ [5], in equation (7), we get equation (8), which is usually known as Richard's equation.

$$\frac{\partial \theta_w}{\partial T} = \frac{\partial}{\partial Z}\left(\frac{\rho}{\rho_s} k \frac{\partial \psi_c}{\partial Z} + \frac{\rho}{\rho_s} kg\right) \tag{8}$$

where ψ_c : Capillary pressure potential,

g : Gravitational constant and

Z : Direction of flow

It is well widely known that the capillary pressure potential ψ_c and the moisture content θ_w are connected to a single valued function, so equation (8) will be as follows.

$$\frac{\partial \theta_w}{\partial T} = \frac{\partial}{\partial Z}\left(D(\theta_w) \frac{\partial \theta_w}{\partial Z} + \frac{\rho}{\rho_s} kg\right) \tag{9}$$

where $D(\theta_w) = \frac{\rho}{\rho_s} k \frac{\partial \psi_c}{\partial \theta_w}$ is referred to as the aqueous diffusivity coefficient, which is directly proportional to the moisture content [8].

The equation (9) will be,

$$\frac{\partial \theta_w}{\partial T} = \varepsilon_0 \frac{\partial}{\partial Z}\left(\theta_w \frac{\partial \theta_w}{\partial Z}\right) + \frac{\rho g}{\rho_s} \frac{\partial k}{\partial Z} \tag{10}$$

where ε_0 is proportionality constant.

Now the aqueous conductivity coefficient $k - k_0 \theta_w$, $k_0 = 0.232$ [7], then the equation (10) becomes,

$$\frac{\partial \theta_w}{\partial T} = \varepsilon_0 \frac{\partial}{\partial Z}\left(\theta_w \frac{\partial \theta_w}{\partial Z}\right) + A \frac{\partial \theta_w}{\partial Z} \tag{11}$$

where $A = \frac{\rho g k_0}{\rho_s} =$ constant

The equation (11) is a governing a non-linear second order partial differential equation, which represents the one-dimensional unsteady flow in unsaturated heterogeneous soil in vertically downward direction, over a large basin of geological configuration considered as model.

The following are the associated initial and boundary conditions:

The initial moisture content of water is given by

$$\theta_w(Z, 0) = \theta_{w0}, \text{ where } Z > 0 \tag{12}$$

At soil surface $z = 0$, the initial moisture content of the water is given by

$$\theta_w(0, T) = \theta_{w1}, \text{ where } T > 0 \tag{13}$$

where $\theta_{w0} \neq \theta_{w1}$.

Solution by Method of Functional Separable

Suppose Equation (11) admits functional separable solution [10, 15] of the form

$$\theta_w(Z, T) = \gamma_1(Z)\lambda_1(T) + \gamma_2(Z)\lambda_2(T) + \lambda_3(T) \quad \text{where} \quad \gamma_1(Z) = Z^2 \quad \text{and} \quad \gamma_2(Z) = Z \tag{14}$$

By substituting (14) into (11), we get

$$\begin{aligned} Z^2\lambda_1'(T) + Z\lambda_2'(T) + \lambda_3'(T) = {} & \varepsilon_0[Z^2\lambda_1(T) + Z\lambda_2(T) + \lambda_3(T)]2\lambda_1(T) \\ & + \varepsilon_0[4Z^2\lambda_1^2(T) + 4Z\lambda_1(T) + \lambda_2(T) + \lambda_2^2(T)] \\ & + A[2Z\lambda_1(T) + \lambda_2(T)] \end{aligned}$$

By Simplification, we get

$$\begin{aligned} Z^2\lambda_1'(T) + Z\lambda_2'(T) + \lambda_3'(T) = {} & 6\varepsilon_0 Z^2\lambda_1^2(T) + 6\varepsilon_0 Z\lambda_1(T)\lambda_2(T) + \varepsilon_0\lambda_2^2(T) \\ & + 2AZ\lambda_1(T) + 2\varepsilon_0\lambda_1(T)\lambda_3(T) + A\lambda_2(T) \end{aligned}$$

By comparing like power of Z, we get

$$\lambda_1'(T) = 6\varepsilon_0\lambda_1^2(T) \tag{15}$$

$$\lambda_2'(T) = 6\varepsilon_0\lambda_1(T)\lambda_2(T) + 2A\lambda_1(T) \tag{16}$$

$$\lambda_3'(T) = 6\varepsilon_0\lambda_2^2(T) + 2\varepsilon_0\lambda_1(T)\lambda_3(T) + A\lambda_2(T) \tag{17}$$

Using equation (15),

$$\frac{\partial \lambda_1(T)}{\partial T} = 6\varepsilon_0 \lambda_1^2(T)$$

Solution is given by separation of variables,

$$\lambda_1(T) = -\frac{1}{6\varepsilon_0 T + c_1} \tag{18}$$

Using equation (16),

$$\frac{\partial \lambda_2(T)}{\partial T} = 6\varepsilon_0 \lambda_1(T)\lambda_2(T) + 2A\lambda_1(T)$$

Using equation (18), we rewrite this equation as,

$$\frac{\partial \lambda_2(T)}{\partial T} = 6\varepsilon_0\left(\frac{-1}{6\varepsilon_0 T + c_1}\right)\lambda_2(T) + 2A\left(\frac{-1}{6\varepsilon_0 T + c_1}\right)$$

Solution is given by separation of variables,

$$\lambda_2(T) = \frac{c_2}{\varepsilon_0(6\varepsilon_0 T + c_1)} - \frac{A}{3\varepsilon_0} \tag{19}$$

Using equation (17),

$$\frac{\partial \lambda_3(T)}{\partial T} = \varepsilon_0 \lambda_2^2(T) + 2\varepsilon_0 \lambda_1(T)\lambda_3(T) + A\lambda_2(T)$$

Using equation (18) and (19), we get a first order linear differential equation.

$$\frac{\partial \lambda_3(T)}{\partial T} + \frac{2\varepsilon_0}{6\varepsilon_0 T + c_1}\lambda_3(T) = \frac{c_2^2}{\varepsilon_0(6\varepsilon_0 T + c_1)} + \frac{Ac_2}{3\varepsilon_0(6\varepsilon_0 T + c_1)} - \frac{2A^2}{9\varepsilon_0}$$

Solution is given by,

$$\lambda_3(T) = \frac{-c_2^2}{4\varepsilon_0^2(6\varepsilon_0 T + c_1)} + \frac{c_2 A}{6\varepsilon_0^2} - \frac{A^2(6\varepsilon_0 T + c_1)}{36\varepsilon_0^2} + \frac{c_3}{(6\varepsilon_0 T + c_1)^{\frac{1}{3}}} \tag{20}$$

Here c_1, c_2 and c_3 are integration constants such that $\lambda_1(0)$, $\lambda_2(0)$ and $\lambda_3(0) \neq 0$.

For initial condition (12), substituting $T = 0$ in the solution (14), we find

$$\theta_{w0}(Z) = Z^2\lambda_1(0) + Z\lambda_2(0) + \lambda_3(0) \tag{21}$$

where $\lambda_1(0)$, $\lambda_2(0)$ and $\lambda_3(0) \neq 0$ constants.

Let $\lambda_1(0) = p, \lambda_2(0) = q, \lambda_3(0) = r$ where nonzero constants c_1, c_2 and c_3 are arbitrary.

Putting the values of $\lambda_1(0)$, $\lambda_2(0)$ and $\lambda_3(0)$ in (18), (19) and (20), we get

$$c_1 = -\frac{1}{p},\ c_2 = \left(-\frac{1}{p}\right)\alpha_0\left(q + \frac{A}{3\varepsilon_0}\right) \text{ and } c_3 = \left(-\frac{1}{p}\right)^{\frac{1}{3}}\left(r - \frac{q^2}{4p}\right)$$

Substituting these values of c_1, c_2 and c_3 in equations (18), (19) and (20), we get

$$\lambda_1(T) = -\frac{1}{6\varepsilon_0 T - \frac{1}{p}}$$

$$\lambda_2(T) = \frac{\left(-\frac{1}{p}\right)\varepsilon_0\left(q + \frac{A}{3\varepsilon_0}\right)}{\varepsilon_0\left(6\varepsilon_0 T - \frac{1}{p}\right)} - \frac{A}{3\varepsilon_0}$$

$$\lambda_3(T) = \frac{-\left(\left(-\frac{1}{p}\right)\varepsilon_0\left(q + \frac{A}{3\varepsilon_0}\right)\right)^2}{4\varepsilon_0^2\left(6\varepsilon_0 T - \frac{1}{p}\right)} + \frac{\left(-\frac{1}{p}\right)\varepsilon_0\left(q + \frac{A}{3\varepsilon_0}\right)A}{6\varepsilon_0^2} - \frac{A^2\left(6\varepsilon_0 T - \frac{1}{p}\right)}{36\varepsilon_0^2}$$

$$+ \frac{c_3}{\left(6\varepsilon_0 T - \frac{1}{P}\right)^{\frac{1}{3}}}$$

Now substituting above values of $\lambda_1(T)$, $\lambda_2(T)$ and $\lambda_3(T)$ in (14), we get

$$\theta_w(Z, T) = \left(-\frac{1}{6\varepsilon_0 T - \frac{1}{p}}\right)Z^2 + \left(\frac{\left(-\frac{1}{p}\right)\varepsilon_0\left(q + \frac{A}{3\varepsilon_0}\right)}{\varepsilon_0\left(6\varepsilon_0 T - \frac{1}{p}\right)} - \frac{A}{3\varepsilon_0}\right)Z$$

$$+\frac{-\left(\left(-\frac{1}{p}\right)\varepsilon_0\left(q+\frac{A}{3\varepsilon_0}\right)\right)^2}{4\varepsilon_0^2\left(6\varepsilon_0 T-\frac{1}{p}\right)}+\frac{\left(-\frac{1}{p}\right)\varepsilon_0\left(q+\frac{A}{3\varepsilon_0}\right)A}{6\varepsilon_0^2}-\frac{A^2\left(6\varepsilon_0 T-\frac{1}{p}\right)}{36\varepsilon_0^2}$$

$$+\frac{c_3}{\left(6\varepsilon_0 T-\frac{1}{p}\right)^{\frac{1}{3}}} \tag{22}$$

where $A=\frac{\rho g K_0}{\rho_s}=$ constant

Equation (22) is a desired solution of equation (11) describing moisture content in unsaturated heterogeneous soil. The numerical results and graphical presentation of solution (22) can be obtained through MATLAB.

Numerical results and Graphical representation of the problem

For numerical values, the following values are considered:

$$\theta_{w0}(Z)=e^{z} \text{ for any } Z>0 \tag{23}$$

Now substituting $\theta_{w0}(Z)=e^{z}$, $\lambda_1(0)=p$, $\lambda_2(0)=q$ and $\lambda_3(0)=r$ in (21), we get

$$e^{z}=pZ^2+qZ+r$$

Using expansion of $e^{z}=1+z+\frac{z^2}{2}+\ldots$ and equating the coefficients of same powers of Z, we find

$p=0.5$, $q=1$ and $r=1$

From standard literature the values of certain constants are taken as follows:

$$A=\frac{\rho g K_0}{\rho_s}\approx 1$$

The numerical results and graphical presentation of solution (22) have been carried out via MATLAB. Table 1 represents the numerical results for

θ_w Versus T for given fix time $T = 0,\ 0.001, 0.002,\ 0.003,\ 0.004,\ 0.005,\ 0.006,\ 0.007,\ 0.008,\ 0.009,\ 0.01$ and Figure 3 displays the graphs of θ_w Versus Z for given fix time $T = 0,\ 0.001, 0.002,\ 0.003,\ 0.004,\ 0.005,\ 0.006,\ 0.007,\ 0.008,\ 0.009,\ 0.01$.

Table 1. θ_w Versus Z for fixed T.

T	0	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.01
Z						$\theta_w(Z, T)$					
0.1	0.0016	0.0062	0.0108	0.0155	0.0202	0.0250	0.0298	0.0346	0.0395	0.0444	0.0494
0.2	0.1166	0.1218	0.1271	0.1324	0.1377	0.1431	0.1486	0.1541	0.1596	0.1652	0.1709
0.3	0.2416	0.2475	0.2534	0.2594	0.2654	0.2715	0.2777	0.2839	0.2901	0.2964	0.3028
0.4	0.3766	0.3832	0.3898	0.3965	0.4033	0.4101	0.4170	0.4240	0.4310	0.4381	0.4452
0.5	0.5216	0.5289	0.5364	0.5438	0.5514	0.5590	0.5667	0.5744	0.5822	0.5901	0.5981
0.6	0.6766	0.6848	0.6930	0.7073	0.7096	0.7181	0.7266	0.7352	0.7439	0.7526	0.7615
0.7	0.8416	0.8506	0.8597	0.8688	0.8781	0.8874	0.8968	0.9063	0.9159	0.9255	0.9353

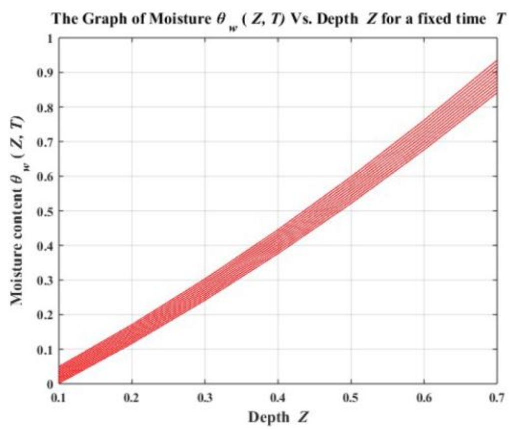

Figure 3. The 2-dimension graph represents the solution of equation (22).

Conclusion

In present study, we have discussed problem of moisture content in unsaturated heterogeneous soil through mathematical model. Equation (11) denotes governing equation of the phenomena that is nonlinear second order partial differential equation. The solution is obtained in equation (22) using Method of Functional Separable using exponential initial condition (23) in the solution (22), we have obtained the unknown coefficients. The solution is obtained in the form of second-degree polynomial that represents moisture content. Using MATLAB, the numerical results of moisture content at different depth Z are obtained at different time T. Figure 3 also describes the solution graphically, showing that the moisture content increases both in terms of depth and time, that is consistent with the physical nature of the problem.

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