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g-IDEMPOTENCY OF LINEAR COMBINATIONS OF TWO IDEMPOTENT MATRICES

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Abstract

The aim of this paper is to study g-idempotency of linear combinations of two idempotent matrices. A complete solution is established to the problem of characterizing all situations, where a linear combination $C = \alpha M + \beta N$ of an idempotent matrix M and a tripotent matrix N is g-idempotent.

1. Introduction

In 2000, Baksalary J. K. and Baksalary O. M. [1] studied the linear combinations of two idempotent matrices and they have listed a set of conditions for a linear combination of two idempotent matrices to be idempotent. The idempotency of linear combinations of an idempotent matrix and a tripotent matrix was studied in [2].

The concept of g-idempotent matrices was introduced in [4], and the spectral theory of such matrices were obtained in [5].

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The purpose of this paper is to establish a complete solution to the problem of characterizing all situations, where the operation of combining linearly *M*, an idempotent matrix, and *N*, a tripotent matrix, preserves the *g*-idempotentcy property.

2. Preliminaries

Let the space of $n \times n$ complex matrices be denoted by $\mathbb{C}^{n\times n}$. Let $\mathbb{C}^n$ be the space of complex *n*-tuples. Let $u = (u_0, u_1, u_2, \ldots, u_{n-1}) \in \mathbb{C}^n$. Let G be the Minkowski metric tensor defined by $Gu = (u_0, -u_1, -u_2, \ldots, -u_{n-1})$. Then the Minkowski metric matrix G is given by $G = \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}$ and $G^2 = I_n$. Minkowski inner product on $\mathbb{C}^n$ is defined by $(u, v) = \langle u, Gv \rangle$, where $\langle \cdot, \cdot \rangle$ is the conventional Hilbert Space inner product. A space with Minkowski inner product is called a Minkowski space, which has been studied by physicists in optics. With respect to the Minkowski inner product the adjoint of a matrix $M \in \mathbb{C}^{n\times n}$ is given by $M^{\sim} = GM^*G$, where M^* is the usual Hermitian adjoint.

A matrix $M \in \mathbb{C}^{n\times n}$, that satisfies the relation $M^2 = M$ is called an idempotent matrix. If $M^3 = M$, then *M* is called tripotent matrix.

It is well known that a tripotent matrix *N* can uniquely be represented as a difference of two idempotent matrices, say N_1 and N_2 (i.e., $N = N_1 - N_2$), which are disjoint, in the sense that $N_1 N_2 = N_2 N_1 = 0$ (cf. Lemma 5.6.6 of [6]).

If N_1 and N_2 are non zero, then N is called an essentially tripotent matrix, otherwise, *N* reduces to a scalar multiple of an idempotent matrix.

A matrix $X \in \mathbb{C}^{n\times n}$, which satisfies $MXM = M$, $XMX = X$ and $MX = XM$, is called group inverse of *M*, and it is denoted by $M^{\#}$.

A complex matrix $M \in \mathbb{C}^{n\times n}$ satisfying $M^{\#} = M^{k-1}$ for $k = 2, 3, \ldots$ is called a $\{k\}$-group periodic matrix [3].

A complex matrix $M \in \mathbb{C}^{n \times n}$ is said to be g-idempotent, if $M = GM^2G$, where G is the Minkowski metric matrix, $G = \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}$.

For example, let $G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $M = \begin{pmatrix} -1/2 & i/2 \\ 3i/2 & -1/2 \end{pmatrix}$. Then M is g-idempotent matrix.

It was proved in [4] that if M is g-idempotent, then it is quadripotent (i.e., $M^4 = GMGGMG = GM^2G = M$).

3. Main Results

In this section we study the g-idempotency of linear combinations of two commuting idempotent matrices. A set of necessary conditions are given in Theorem 3.4 for linear combinations of a commuting idempotent matrix and an essentially tripotent matrix to be g-idempotent.

Lemma 3.1. *All g-idempotent matrices are* $\{3\}$*-group periodic.*

Proof. Let M be g-idempotent matrix. Then M is quadripotent (i.e., $M^4 = M$).

Since $MM^2M = M^4 = M$, $M^2MM^2 = M^2$ and $MM^2 = M^2M$, we have $M^\# = M^2$.

Thus M is $\{3\}$-group periodic.

Remark 3.2. Let M and N be two non zero idempotent matrices. If $M = \alpha N$ for some $\alpha \in \mathbb{C}$ (the set of all complex numbers), then $\alpha = 1$.

Proof. Since $M^2 = (\alpha N)^2 = \alpha^2 N^2$

$$= \alpha(\alpha N) \; [\because N^2 = N]$$

$$= \alpha M \; [\because M = \alpha N]$$

$$(1 - \alpha)M = 0$$

Since $M \neq 0$, we have $\alpha = 1$.

Theorem 3.3. *Let M and N be two non zero, commuting idempotent matrices. Then a linear combination $C = \alpha M + \beta N$ with non zero complex scalars α and β, is g-idempotent if and only if one of the following conditions* (1) *and* (2) *holds.*

(1) $M - N = 0$ *holds along with either one of the following sets of conditions*:

(i) $\alpha + \beta = 0$

(ii) $\alpha + \beta = 1$ *and* $MG = GM$

(2) $M - N \neq 0$ *holds along with either one of the following sets of conditions*:

(i) $\alpha = 1, \beta = -1, MN = N$ *and* $G(M - N)G = M - N$

(ii) $\alpha = \omega, \beta = \omega^2 - \omega, MN = N$ *and* $G(M - N)G = N$

(iii) $\alpha = \omega^2, \beta = \omega - \omega^2, MN = N$ *and* $G(M - N)G = N$

(iv) $\alpha = \omega, \beta = \omega^2, MN = 0$ *and* $GNG = M$

(v) $\alpha = 1, \beta = 1, MN = 0$ *and* $G(M + N)G = M + N$

(vi) $\alpha = -1, \beta = 1, MN = M$ *and* $G(N - M)G = N - M$

(vii) $\alpha = \omega^2 - \omega, \beta = \omega, MN = M$ *and* $G(N - M)G = M$

(viii) $\alpha = \omega - \omega^2, \beta = \omega^2, MN = M$ *and* $G(N - M)G = M$

(ix) $\alpha = \omega^2, \beta = \omega, MN = 0$ *and* $GMG = N$.

Proof. Let us assume that $C = \alpha M + \beta N$ is g-idempotent.

(1) If $M - N = 0$, then $C = (\alpha + \beta)M$.

(2) Since C is g-idempotent, we have

$$G[(\alpha + \beta)M]^2 G = (\alpha + \beta)M$$

i.e., $(\alpha+\beta)^2 GMG = (\alpha+\beta)M$ [As M is idempotent]

If $\alpha+\beta = 0$, then $C = 0$, which is situation (i).

If $\alpha+\beta \neq 0$, then $(\alpha+\beta)GMG = M$.

Since M and GMG are two non zero idempotents, we have $\alpha+\beta = 1$ by remark 3.2 and hence $GM = MG$, which is situation (ii).

Conversely,

If $M - N = 0$ holds along with $\alpha+\beta = 0$, then we have $C = 0$, which is trivially *g*-idempotent.

If $M - N = 0$ holds along with $\alpha+\beta = 1$, then we have $C = M$.

Now

$GC^2G = GM^2G = GMG = MGG = M = C$, which implies C is *g*-idempotent.

(2) If $M - N \neq 0$.

Since C is *g*-idempotent, we have

$$G[\alpha M + \beta N]^2 G = \alpha M + \beta N$$

i.e., $G(\alpha^2 M + \beta^2 N + 2\alpha\beta MN)G = \alpha M + \beta N$ (3.1)

By Lemma 3.1, $C = \alpha M + \beta N$ is {3}-group periodic. Hence the choice of α and β must be necessarily one among the following cases by Theorem 3.1 of [3].

Case 1. $\alpha \in \Omega_3$, the set of all cube roots of unity, and $\alpha+\beta = 0$.

By the corresponding sub case 1(b) of Theorem 3.1 of [3], we have $MN = N$. It follows from (3.1) that $\alpha G(M-N)G = M - N$, which implies $\alpha = 1$. Therefore $G(M-N)G = M - N$, which is situation (i).

Case 2. $\alpha \in \Omega_3$ and $\alpha+\beta \in \Omega_3$.

The possibility of $\beta \in \Omega_3$ is neglected by Note 2 of Section 3 of [3]. Hence we have $MN = N$ by sub case 2(b) of Theorem 3.1 of [3].

Pre and post multiplying (3.1) by NG and GN respectively, we have

$$(\alpha + \beta)^2 NG = \alpha NGM + \beta NGN \tag{3.2}$$

$$(\alpha + \beta)^2 GN = \alpha MGN + \beta NGN \tag{3.3}$$

Post multiplying (3.2) by M and N leads to

$$(\alpha + \beta)^2 NGM = \alpha NGM + \beta NGN \tag{3.4}$$

$$(\alpha + \beta) NGN = NGN$$

It follows that $\alpha + \beta = 1$ or $NGN = 0$.

If $\alpha + \beta = 1$, then it can be proved from (3.2), (3.3) and (3.4) that $GN = NG$.

Substituting this in (3.1), we have $\alpha G(M - N)G = M - N$. By Remark 3.2, we have $\alpha = 1$. This implies that $\beta = 0$, a contradiction. Therefore, we must have $NGN = 0$.

It follows from (3.4) that $(\alpha + \beta)^2 = \alpha$ or $NGM = 0$.

If $NGM = 0$, then it follows from (3.2) that $NG = 0$ and then $N = 0$, a contradiction.

Therefore, we must have $(\alpha + \beta)^2 = \alpha$. Substituting the value of β in (3.1) we get

$$\alpha G[M + (\alpha^2 - 1)N]G = M + (\alpha - 1)N \tag{3.5}$$

Cubing the above equation, we have $GMG = M$ [since $MN = N$ and $\in \Omega^2 \Rightarrow \alpha = 1$]

It follows from (3.5) that $G(M - N)G = N$. [$\alpha \neq 1$, otherwise $\beta = 0$]. Since $\alpha \in \Omega_3$, we have $\alpha = \omega$ or ω^2. Hence we have the situation (ii) or (iii).

Case 3. $\alpha \in \Omega_3$ and $\beta \in \Omega_3$.

It is clear that $\alpha + \beta \neq 0$. As before, the possibility of $\alpha + \beta \in \Omega_3$ is neglected by Note 2 of Section 3 of [3]. Hence we have $\alpha + \beta \notin \{0\} \cup \Omega_3$ and $MN = 0$ by sub case 3(b) of Theorem 3.1 of [3]. It follows from (3.1) that

$$G(\alpha^2 M + \beta^2 N)G = \alpha M + \beta N \tag{3.6}$$

Cubing (3.6), we have

$$G(M + N)G = M + N \tag{3.7}$$

Pre and post multiplying (3.6) by MG and M respectively leads to $\alpha^2 MGM = \alpha MGM$, which implies $\alpha = 1$ or $MGM = 0$. Similarly, we get $\beta = 1$ or $NGN = 0$. Hence we have at least one of the following situations.

a. $MGM = 0$ and $NGN = 0$

b. $\alpha = 1$ and $\beta = 1$

c. $\alpha = 1$ and $NGN = 0$

d. $\beta = 1$ and $MGM = 0$

(a) Post multiplying (3.7) by N and M leads respectively to

$$GMGN = N \text{ and } GNGM = M \tag{3.8}$$

Pre multiplying (3.7) by NG and MG leads respectively to

$$MG = MGN \text{ and } NG = NGM \tag{3.9}$$

It follows from (3.8) and (3.9) that $GMG = N$ and $GNG = M$

Post multiplying (3.6) by M and N leads respectively to $\alpha^2 = \beta$ and $\beta^2 = \alpha$.

Hence we have the following three possibilities.

$$\alpha = \beta = 1 : \alpha = \omega, \beta = \omega^2 : \alpha = \omega^2, \beta = \omega$$

Hence the situations (iv) and (v) are obtained.

(b) The condition (3.7) follows immediately from (3.6), which is situation (v).

(c) Post multiplying (3.6) and (3.7) by N, we have $GMGN = \beta N = N$.

Since $N \neq 0$, we have $\beta = 1$. It leads to the situation (v) again along with $NGN = 0$.

(d) This is similar to the sub case (c) and it turns again to the situation (iv).

Interchanging α and β in cases 1 to 3 and also the role of M and N are interchanged, we see that the conditions (vi) to (ix) are obtained.

By substituting the corresponding sets of conditions (i) to (ix) in (3.1), the sufficiency follow.

Theorem 3.4. *Let $N \in \mathbb{C}^{n \times n}$ be an essentially tripotent matrix uniquely decomposed as $N = N_1 - N_2$, where N_1 and N_2 are non-zero idempotent matrices such that $N_1N_2 = 0 = N_1N_2$. Let $M \in \mathbb{C}^{n \times n}$ be a non-zero idempotent matrix such that $MN = NM$. If a linear combination $C = \alpha M + \beta N$ with non-zero $\alpha, \beta \in C$, is a g-idempotent matrix then at least one of the following sets of conditions necessarily hold.*

i. $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}, MN_1GN_1 = N_1GN_1M$ *and* $MN_2GN_2 = N_2GN_2M$,

ii. $\alpha = \frac{1}{4}, \beta = -\frac{1}{4}, MN_1GN_1 = N_1GN_1M$ *and* $N_2GN_2M = 0$,

iii. $\alpha = 2, \beta = -2, MN_1GN_1 = N_1GN_1M$ *and* $MN_2GN_2 = 0$,

iv. $\alpha + \beta = 0, MN_1GN_1 = N_1GN_1M$ *and* $MN_2GN_2 = 0 = N_2GN_2M$,

v. $\alpha = 2, \beta = 1, MN_1GN_1 = 0 = N_1GN_1M$ *and* $N_2GN_2 = N_2GN_2M$,

vi. $\alpha = \frac{5 \pm i\sqrt{7}}{8}, \beta = \frac{-3 \pm i\sqrt{7}}{8}, N_1GN_1M = 0$ *and* $MN_2GN_2 = N_2GN_2M$,

vii. $\alpha = 2 \pm \sqrt{2}, \beta = 1 \pm \sqrt{2}, MN_1GN_1 = 0$ *and* $MN_2GN_2 = N_2GN_2M$,

viii. $\alpha - \beta = 1, MN_1GN_1M = 0$ *and* $MN_2GN_2 = N_2GN_2M$

The remaining conditions are obtained from (i) *to* (viii) *by interchanging* β *with* $-\beta$ *and* N_1 *with* N_2. *If the choice of* α *and* β *differ from all the above conditions then*

$$MN_1GN_1M = 0 = MN_2GN_2M.$$

Proof. Since $N = N_1 - N_2$, we have $N^2 = N_1 + N_2$. So $N_1 = \frac{N + N^2}{2}$ and $N_2 = \frac{N^2 - N}{2}$.

Then it is easy to prove that $MN = NM$ holds if and only if $MN_1 = N_1M$ and $MN_2 = N_2M$.

Let the matrix $C = \alpha M + \beta N_1 - \beta N_2$ be g-idempotent. Then

$$G[\alpha^2 M + \beta^2 N_1 + \beta^2 N_2 + 2\alpha\beta MN_1 - 2\alpha\beta MN_2]G = \alpha M + \beta N_1 - \beta N_2 \quad (3.10)$$

Pre and post multiplying (3.10) by N_1G and N_1 respectively, we have

$$[\alpha^2 MN_1 + \beta^2 N_1 + 2\alpha\beta MN_1]GN_1 = N_1G[\alpha MN_1 + \beta N_1] \quad (3.11)$$

Pre and post multiplying (3.11) by M, we have

$$\alpha + \beta = 0 \text{ otherwise } \alpha + \beta = 1 \text{ or } MN_1GN_1M = 0 \quad (3.12)$$

Similarly, pre and post multiplying (3.10) by N_2G and N_2 respectively, we get

$$[\alpha^2 MN_2 + \beta^2 N_2 - 2\alpha\beta MN_2]GN_2 = N_2G[\alpha MN_2 - \beta N_2] \quad (3.13)$$

Pre and post multiplying (3.13) by M, we have

$$\alpha - \beta = 0 \text{ otherwise } \alpha - \beta = 1 \text{ or } MN_2GN_2M = 0 \quad (3.14)$$

The two sets of situations (3.12) and (3.14) will give the following different cases.

Case 1. $\alpha + \beta = 0$ and $\alpha - \beta = 0$

This case is not possible as it leads to $\beta = 0$, a contradiction.

Case 2. $\alpha + \beta = 0$ and $\alpha - \beta = 1$

This gives $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$. Now the equation (3.11) implies that

$$\frac{3}{4} N_1GN_1 = \frac{1}{2} N_1GN_1M + \frac{1}{4} MN_1GN_1 \tag{3.15}$$

Pre and post multiplying (3.15) by M, we have $MN_1GN_1 = N_1GN_1M$.

Similarly, we get $MN_2GN_2 = N_2GN_2M$, which is situation (i)

Case 3. $\alpha + \beta = 0$ and $MN_2GN_2M = 0$

It follows from (3.11) that

$$(\alpha + 1)N_1GN_1 = N_1GN_1M + \alpha MN_1GN_1 \tag{3.16}$$

Pre and post multiplying (3.16) by M, we have $MN_1GN_1 = N_1GN_1M$.

From (3.13), we have

$$3\alpha MN_2GN_2 + (\alpha - 1)N_2GN_2 = N_2GN_2M \tag{3.17}$$

Pre and post multiplying (3.17) by M, we have respectively,

$$(4\alpha - 1)MN_2GN_2 = 0$$

and

$$(\alpha - 2)N_2GN_2M = 0$$

If $\alpha = \frac{1}{4}$, then $\beta = -\frac{1}{4}$ and $N_2GN_2M = 0$, which is (ii).

If $\alpha = 2$, then $\beta = -2$ and $MN_2GN_2M = 0$, which is (iii).

Otherwise $MN_2GN_2 = 0 = N_2GN_2M$, which is (iv).

Case 4. $\alpha + \beta = 1$ and $\alpha - \beta = 1$

This leads to $\beta = 0$, a contradiction.

Case 5. $\alpha - \beta = 1$ and $MN_1GN_1M = 0$

It follows from (3.13) that

$$(2-\alpha)MN_2GN_2 + (\alpha-1)N_2GN_2 = N_2GN_2M \tag{3.18}$$

Pre and post multiplying the above by M, we get

$$(\alpha-2)N_2GN_2M = (\alpha-2)MN_2GN_2 \tag{3.19}$$

From (3.11), we see that

$$(\alpha^2-3\alpha+2)N_1GN_1 + (3\alpha^2-2\alpha)MN_1GN_1 = \alpha N_1GN_1M \tag{3.20}$$

Pre and post multiplying (3.20) by M, we get, respectively

$$(4\alpha^2-5\alpha+2)MN_1GN_1 = 0 \tag{3.21}$$

and

$$(\alpha^2-4\alpha+2)N_1GN_1M = 0 \tag{3.22}$$

Now

(a) Consider (3.19). If $\alpha = 2$, then $\beta = 1$. From (3.18), we have $N_2GN_2 = N_2GN_2M$. It follows from (3.21) and (3.22) that $MN_1GN_1 = 0 = N_1GN_1M$, which is (v).

(b) Considering (3.21), if $\alpha = \frac{5 \pm i\sqrt{7}}{8}$, then $\beta = \frac{-3 \pm i\sqrt{7}}{8}$. It follows from (3.19) and (3.22) that $MN_2GN_2 = N_2GN_2M$ and $N_1GN_1M = 0$, which is (vi).

(c) Considering (3.22), if $\alpha = 2 \pm \sqrt{2}$, then $\beta = 1 \pm \sqrt{2}$. It follows from (3.19) and (3.21) that $MN_2GN_2 = N_2GN_2M$ and $MN_1GN_1 = 0$, which is (vii).

(d) If α differs from all the above three sub cases (a) to (c), then by (3.19), we have $MN_2GN_2 = N_2GN_2M$, which is (viii).

The remaining sets of conditions are obtained from the above cases 1 to 5 by interchanging β with -β and N_1 with N_2. If α and β do not obey any of the above cases, then the only possibility is $MN_1GN_1M = MN_2GN_2M$.

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