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THE EDGE STEINER DOMINATION NUMBER OF A GRAPH

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Abstract

A subset D of vertices in G is called dominating set if every vertex not in D has at least one neighbor in D. An edge Steiner set of G is a set $W \subseteq V(G)$ such that every edge of G is contained in a Steiner W-tree of G. The edge Steiner number $s_e(G)$ of W is the minimum order of its edge Steiner set and any edge Steiner set of order $s_e(G)$ is an edge Steiner set of G. A set of vertices W in G is called an edge Steiner dominating set of G if W is both edge Steiner set and a dominating set of G. The minimum cardinality of an edge Steiner dominating set of G is its edge Steiner domination number and is denoted by $\gamma_{se}(G)$. An edge Steiner dominating set of size $\gamma_{se}(G)$ is said to be a γ_{se}- set of G. The edge Steiner domination number of certain classes of graphs is determined. Necessary conditions for connected graphs of order with edge Steiner domination number p or $p-1$ are given. It is shown that for every two integers a and b with

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$2 \leq a \leq b$, there exists a connected graph G such that $\gamma_s(G) = a$ and $\gamma_{se}(G) = b$, where $\gamma_s(G)$ is the Steiner domination number of a graph.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively.

The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. A set of vertices D in a graph G is a dominating set if each vertex of G is dominated by some vertex of D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. For a nonempty set W of vertices in a connected graph G, the Steiner distance $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a Steiner tree with respect to W or a Steiner W-tree It is to be noted that $d(W) = d(u, v)$, when $W = \{u, v\}$. The set of all vertices of G that lie on some Steiner W-tree is denoted by $S(W)$. If $S(W) = V$, then W is called a Steiner set for G. A Steiner set of minimum cardinality is a minimum Steiner set or simply a s-set of G and this cardinality is the Steiner number $s(G)$ of G. A set $W \subseteq V(G)$ is called an edge Steiner set of G if every edge of G is contained in a Steiner W-tree of G. The edge Steiner number $s_e(G)$ of G is the minimum cardinality of its edge Steiner sets and any edge Steiner set of cardinality $s_e(G)$ is a edge Steiner set of G. A set of vertices W in G is called a Steiner dominating set if W is both a Steiner and a dominating set. The minimum cardinality of a Steiner dominating set of G is its Steiner domination number and is denoted by $\gamma_s(G)$ is said to be a γ_s- set of G. The following theorem is used in sequel.

Theorem 1.1 [13]. *Each extreme vertex of G belongs to edge Steiner set of G.*

2. The Edge Steiner Domination Number of a Graph

Definition 2.1. Let G be a connected graph of order at least three vertices. A set of vertices W in G is called an edge Steiner dominating set of G if W is both an edge Steiner set and a dominating set of G. The minimum cardinality of an edge Steiner dominating set of G is its edge Steiner domination number and is denoted by $\gamma_{se}(G)$. An edge Steiner dominating set of G size $\gamma_{se}(G)$ is said to be a $\gamma_{se}(G)$- set of G.

Example 2.2. Consider the graph G given in Figure 2.1. The set $W = \{v_1, v_2, v_4\}$ is an edge Steiner dominating set for G so that $\gamma_{se}(G) = 3$.

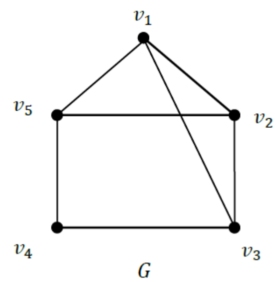

Figure 2.1.

Remark 2.3. Consider the graph G given in Figure 2.1. The set $W = \{v_2, v_4\}$ is a Steiner dominating set of G so that $\gamma_s(G) = 2$. Thus the Steiner domination number and the edge Steiner domination number of a graph are different.

Remark 2.4. There can be more than one edge Steiner domination set for a graph. Consider the graph G given in Figure 2.2, $W = \{v_1, v_2, v_5, v_8\}$ and $W_1 = \{v_1, v_4, v_5, v_6\}$ are two γ_{se}- sets.

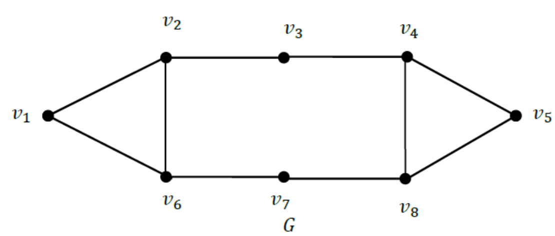

Figure 2.2.

Theorem 2.5. *Each extreme vertex G of belongs to every edge Steiner dominating set of G.*

Proof. This follows from Theorem 1.1.

Observation 2.6. Let G be a connected graph and v be a cut-vertex of G. Then every edge Steiner dominating set contains at least one element from each component of $G - \{v\}$.

Observation 2.7. If G is a connected graph of order p, then $2 \leq \max\{s_e(G), \gamma(G)\} \leq \gamma_{se}(G) \leq p$.

In the following we determine the edge Steiner dominating number of some standard graphs.

Theorem 2.8. *For the complete graph* $K_p (p \geq 2)$, $\gamma_{se}(K_p) = p$.

Proof. Since every vertex of the complete graph $K_p (p \geq 2)$ is an extreme vertex by Observation 2.5, the vertex set of K_p is the unique edge Steiner dominating set of K_p. Thus $\gamma_{se}(K_p) = p$.

Theorem 2.9. *For a connected graph G of order* $P \geq 2$, $\gamma_{se}(G) = 2$ *if and only if there exists an edge Steiner dominating set* $W = \{u, v\}$ *of G such that* $d(u, v) \leq 3$.

Proof. First, assume that $\gamma_{se}(G) = 2$. Let $W = \{u, v\}$ be an edge Steiner dominating set of G. Suppose that $d(u, v) \geq 4$. Then consider the diametrical path, say P contains at least three internal vertices and hence $\gamma_{se}(G) \geq 3$, which is a contradiction. Therefore $d(u, v) \leq 3$. The converse is clear.

Theorem 2.10. *Let G be a connected graph of order* $p \geq 3$ *with at least one universal vertex. Then* $N(v)$ *is a subset of every edge Steiner dominating set of G.*

Proof. Let v be a vertex of degree $p - 1$ and $N(v) = \{v_1, v_2, \ldots, v_{p-1}\}$. Let W be an edge Steiner set of G. Suppose that $v_1 \notin W$. Then the edge vv_1 lies on a Steiner W-tree of G, say T. Since $v_1 \notin W$, v_1 is not an end vertex of T. Let T_1 be a tree obtained from T by removing the vertex v_1 in T and joining all the neighbors of v_1 other than v in T to v. Then T_1 is a Steiner W-tree such that $|V(T_1)| = |V(T_1)| - 1$ which is a contradiction to T_1 a Steiner W-tree. Therefore $N(v)$ is a Steiner subset of every edge dominating set.

Corollary 2.11. *Let G be a connected graph G with at least one Universal vertex. Then* $\gamma_{se}(G) \geq p - 1.$

Proof. This follows from Theorem 2.10.

Theorem 2.12. *For a connected graph G with v a Universal vertex,*

(i) *if v is a cut vertex of G, then* $\gamma_{se}(G) = p - 1.$

(ii) *if v is not a cut vertex of G, then* $\gamma_{se}(G) = p.$

Proof. By Corollary 2.11 $\gamma_{se}(G) \geq p - 1.$

(i) Let $W = N(v)$. Then W is an edge Steiner dominating set of G so that $\gamma_{se}(G) = p - 1.$

(ii) Since v is not a cut vertex of G, $\langle W \rangle$ is connected and so $s_e(G) \geq p.$ Therefore $W = V$ is the unique γ_{se} set of G so that $\gamma_{se}(G) = p.$

Corollary 2.13. *For a connected graph G with at least two universal vertices,* $\gamma_{se}(G) = p.$

Proof. Since G contains at least two universal vertices, G has not cut vertex. Then the result follows form Theorem 2.12 (ii).

Corollary 2.14. *For a connected graph G of order* $p \geq 3$ *such that* $G = K_1 + \cup m_j K_j$, *where* $\sum m_j \geq 2.$ *Then* $\gamma_{se}(G) = p - 1.$

Proof. Since G is connected graph with exactly one cut-vertex of degree $p - 1,$ the result follows from Theorem 2.12(i).

The edge Steiner domination number and the Steiner domination number of a graph

Theorem 2.15. *Every edge Steiner dominating set of a connected graph G is a Steiner dominating set of G.*

Proof. Let W be an edge Steiner dominating set of G. Then W is an edge Steiner set of G and a dominating set of G which implies W is a Steiner set of G and a dominating set of G. Therefore W is a Steiner dominating set of G.

Corollary 2.16. *For any connected graph* G, $\gamma_s(G) \leq \gamma_{se}(G).$

Proof. Let W be a γ_{se}-set of G. Then $|W| = \gamma_{se}(G)$. By Theorem 2.15, W is a Steiner dominating set of G so that $\gamma_{se}(G) \leq |W| \leq \gamma_{se}(G)$.

Remark 2.17. For the graph given in Figure 2.1, $\gamma_s(G) = 2$ and $\gamma_{se}(G) = 3$ so that $\gamma_s(G) < \gamma_{se}(G)$. Also, for any non-trivial tree T, $\gamma_s(T) = \gamma_{se}(T)$.

The following theorem gives a realization for the Steiner domination number and the edge Steiner domination number of a graph.

Theorem 2.18. *For any positive integers* $2 \leq a \leq b$, *there exists a connected graph G such that* $\gamma_s(G) = a$ *and* $\gamma_{se}(G) = b$.

Proof. If $a = b$, take $G = K_{1,a}$. Then it is easily verified that, $\gamma_s(G) = a = \gamma_{se}(G)$. If $a = 2$, $b = 3$, consider the graph G given in Figure 2.1. It is easily seen that $\gamma_s(G) = 2$ and $\gamma_{se}(G) = 3$. If $a = 2$, $b \geq 4$, let G be the graph in Figure 2.3. obtained from the path on three vertices $P : u_1, u_2, u_3$, by adding $b - 3$ new vertices $v_1, v_2, \ldots, v_{b-3}$ and joining each $v_i (1 \leq i \leq b - 3)$ with u_1, u_2, u_3. Let $W = \{u_1, u_3\}$. Then W is a Steiner dominating set of G so that $\gamma_s(G) = 2 = a$. Since u_2 is a Universal vertex such that it is not a cut-vertex of G, by Theorem 2.12 (ii) $\gamma_{se}(G) = b - 3 + 3 = b$.

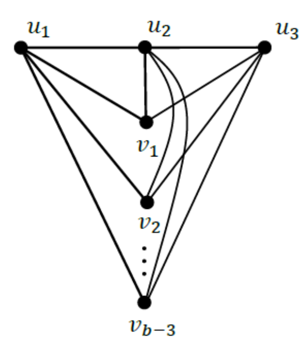

Figure 2.3.

If $a \geq 3$, $b \geq 4$ and $b \neq a + 1$, let G be the graph obtained in Figure 2.4 from the path on three vertices $P : u_1, u_2, u_3$, by adding the new vertices

$v_1, v_2, \ldots, v_{b-a-1}$ and $w_1, w_2, \ldots, w_{a-2}$ and joining each $v_i (1 \le i \le b - a - 1)$ with u_1, u_2, u_3 and also joining each $w_i (1 \le i \le a - 2)$ with u_1 and u_2. Since each $w_i (1 \le i \le a - 2)$ is an extreme vertex of G, each $w_i (1 \le i \le a - 2)$ belongs to every Steiner dominating set of G. Let $W = \{w_1, w_2, \ldots, w_{a-2}\}$. Then W is not a Steiner dominating set of G. Also, it is easily verified that $W \cup \{v\}$, where $v \notin W$, is not a Steiner dominating set of G. Now, it is clear that $W \cup \{u_1, u_3\}$ is a Steiner dominating set of G and so $\gamma_s(G) = a$. Since u_2 is an Universal vertex of G such that it is not a cut-vertex of G, it follows from Theorem 2.12 (ii) that $\gamma_{se}(G) = b - a - 1 + a - 2 + 3 = b$.

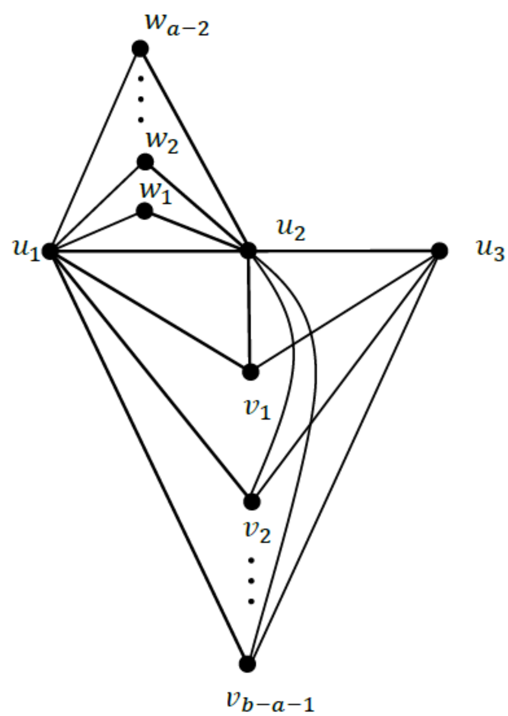

Figure 2.4.

If $a \ge 3$, $b \ge 4$ and $b = a + 1$, consider the graph G given in Figure 2.5. It is easily observed that, $w_i (1 \le i \le a - 2)$ belongs to every Steiner dominating set of G. Let $W = \{w_1, w_2, \ldots, w_{a-2}\}$. Then W is not a Steiner domination set of G. Also, it is easily verified that $W \cup \{v\}$, where $v \notin W$ is not a Steiner dominating set of G. However, it is clear that $W \cup \{v_1, v_2\}$ is a Steiner dominating set of G so $\gamma_s(G) = a$.

By Theorem 2.5, $w_i (1 \leq i \leq a-2)$ belongs to every edge Steiner dominating set of G and then W is not an edge Steiner dominating set of G. Also, it is easily verified that neither $W \cup \{v\}$ nor $W \cup \{u, v\}$, where $u, v \notin W$ is an edge Steiner dominating set of G. However, $W \cup \{v_2, v_7, v_8\}$ is an edge Steiner dominating set of G and so $\gamma_{se}(G) = a - 2 + 3 = a + 1 = b$. Thus the proof is complete.

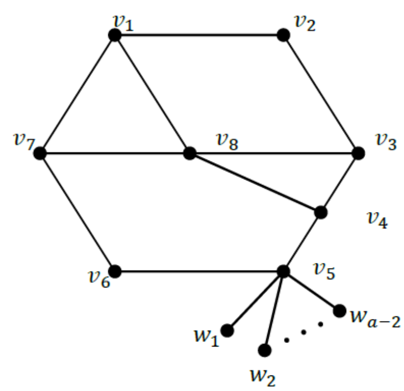

Figure 2.5

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