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FUZZY QUOTIENT- 3 CORDIAL LABELING ON SOME SUBDIVISION GRAPHS

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Abstract

Consider a non-trivial, simple, undirected, connected, and finite graph G with p vertices and q edges. G's vertex and edge sets are $V(G)$ and $E(G)$ respectively. Let the function $\sigma : V(G) \to [0, 1]$ defined by $\sigma(\alpha) = \frac{\alpha}{10}, \alpha \in Z_4 - \{0\}$ and for each $\alpha\beta \in E(G)$, the induced function $\mu : E(G) \to [0, 1]$ defined by $\mu(\alpha\beta) = \frac{1}{10}\left\lceil \frac{3\sigma(\alpha)}{\sigma(\beta)} \right\rceil$, where $\sigma(\alpha) \leq \sigma(\beta)$. σ is called fuzzy quotient 3 cordial labeling if $|\nu_\sigma(\iota) - \nu_\sigma(\tau)| \leq 1$ and $|\varepsilon_\mu(\iota) - \varepsilon_\mu(\tau)| \leq 1$. For $\iota \in \left\{\frac{r}{10}, r \in Z_4 - \{0\}\right\}$, $\nu_\sigma[\iota]$ and $\varepsilon_\mu[\iota]$ represent the number of vertices and edges assigned the labels ι respectively, If a graph admit this labeling, then it is fuzzy quotient 3 cordial. The existence of above labeling on $S'(C_\eta[m])$, $S'(C_\eta[Km]A)$, $S'(C_\eta[a, d])$ and $S'(C_\eta[a, r])$ are examined and the results are provided in this paper.

1. Introduction

Labeling is a process of assigning values to vertices, edges, or both of a graph based on certain conditions. Rosa (1967) and Graham and Sloane (1967) were the first to use this technique (1980). The researchers are highly

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motivated and enthusiastic about labeling the graph. Joseph A. Gallian summarises a comprehensive discussion of graph labelling. As a result of these labelings, we investigated and analysed some graph families as fuzzy quotient 3 cordial [14]. This paper investigates fuzzy quotient-3 cordial labeling on several subdivision graphs and demonstrates that the graphs are naturally fuzzy quotient 3 cordial.

2. Definitions

Definition 2.1. A graph denoted by $C_\eta(m)$ is produced by linking a vertex of the cycle C_η with m leaves.

Definition 2.2. A graph results by connecting the m leaves to the non-adjacent vertices of a cycle $C_{2\eta}$ is denoted by $C_{2\eta}[m]A$, $\eta \geq 2$.

Definition 2.3. Attaching $a + (i-1)d$, a, $d \geq 1$ leaves to the i^{th} vertex of a cycle C_η yields the new graph and it is denoted by $C_\eta[a, d]$.

Definition 2.4. Attaching $\frac{a(r^i - 1)}{r-1}$, a, $r \geq 1$ leaves to the i^{th} vertex of a cycle C_η yields the new graph and it is denoted by $C_\eta[a, r]$.

Definition 2.5. A graph $S'(G)$ is formed by inserting a new vertex into each pendant edge of graph G.

Definition 2.6. Consider a non-trivial, simple, undirected, connected, and finite graph G with p vertices and q edges. G's vertex and edge sets are $V(G)$ and $E(G)$ respectively. Let the function $\sigma : V(G) \to [0, 1]$ defined by $\sigma(\alpha) = \frac{\alpha}{10}$, $\alpha \in Z_4 - \{0\}$ and for each $\alpha\beta \in E(G)$, the induced function $\mu : E(G) \to [0, 1]$ defined by $\mu(\alpha\beta) = \frac{1}{10}\left\lceil \frac{3\sigma(\alpha)}{\sigma(\beta)} \right\rceil$, where $\sigma(\alpha) \leq \sigma(\beta)$. σ is called fuzzy quotient 3 cordial labeling if $|\nu_\sigma(\iota) - \nu_\sigma(\tau)| \leq 1$ and $|\varepsilon_\mu(\iota) - \varepsilon_\mu(\tau)| \leq 1$. For $\iota \in \left\{\frac{r}{10}, r \in Z_4 - \{0\}\right\}$, $\nu_\sigma[\iota]$ and $\varepsilon_\mu[\iota]$ represent the number of vertices and edges assigned the labels ι respectively, if a graph admit this labeling, then it is fuzzy quotient 3 cordial.

3. Main Results

Theorem 1. $S'(C_\eta[m])$ *is fuzzy quotient* 3 *cordial graph.*

Proof. Let $V(S'(C_\eta[m])) = \{x_\iota : 1 \leq \iota \leq \eta\} \cup \{y_\tau : 1 \leq \tau \leq \eta m\}$ $\cup \{x_\tau : 1 \leq \tau \leq \eta m\}$ and $E(S'(C_\eta[m])) = \{x_\iota x_{\iota+1} : 1 \leq \iota \leq \eta - 1\} \cup \{x_1 x_\eta\}$ $\cup \{x_\iota y_\tau : 1 \leq \iota \leq \eta, (\iota - 1)m + 1 \leq \tau \leq m(\iota)\} \cup \{y_\tau z_\tau : 1 \leq \tau \leq \eta m\}$.

$p = \eta + 2\eta m$ and $q = \eta + 2\eta m$.

The following cases must be considered while defining

$\sigma : V(S'(C_\eta[m])) \to [0, 1]$.

For $m = 1$

$\sigma(x_\iota) = 0.3 \quad 1 \leq \iota \leq \eta$

$\sigma(y_\tau) = 0.1 \quad 1 \leq \tau \leq \eta$

$\sigma(z_\tau) = 0.2 \quad 1 \leq \tau \leq \eta$

For $m \geq 2$, labeling of $x_\iota y_\tau$ and z_τ are as follows.

Case 1. $\eta = 3\xi, \xi \geq 1$.

Subcase 1.1. $m = 3\xi, \xi \geq 1$.

$\sigma(x_\iota) = 0.3 \quad 1 \leq \iota \leq \eta$

$$\sigma(y_\tau) = 0.3 \quad 1 \leq \tau \leq \left(\frac{\eta m - \eta}{3}\right)$$

$$\sigma(z_\tau) = 0.3 \quad 1 \leq \tau \leq \left(\frac{\eta m - \eta}{3}\right)$$

$$\sigma(y_\tau) = 0.1 \quad \left(\frac{\eta m - \eta}{3}\right) + 1 \leq \tau \leq \eta m$$

$$\sigma(y_\tau) = 0.2 \quad \left(\frac{\eta m - \eta}{3}\right) + 1 \leq \tau \leq \eta m$$

Subcase 1.2. $m = 3\xi + 1, \xi \geq 0$.

labeling of $x_\iota y_\tau$ and z_τ is same as in subcase 1.1

Subcase 1.3. $m = 3\xi + 2,\ \xi \geq 0.$

labeling of x_ι, y_τ and z_τ is same as in subcase 1.1

Case 2. $\eta = 3\xi + 1,\ \eta \geq 1.$

Subcase 2.1. $m = 3\xi,\ \eta \geq 1.$

$\sigma(x_\iota) = 0.3 \quad 1 \leq \iota \leq \eta$

$\sigma(y_\tau) = 0.3 \quad 1 \leq \tau \leq \left(\frac{\eta m - \eta + 1}{3}\right)$

$\sigma(z_\tau) = 0.3 \quad 1 \leq \tau \leq \left(\frac{\eta m - \eta + 1}{3}\right)$

$\sigma(y_\tau) = 0.1 \quad \left(\frac{\eta m - \eta + 1}{3}\right) + 1 \leq \tau \leq \eta m$

$\sigma(y_\tau) = 0.2 \quad \left(\frac{\eta m - \eta + 1}{3}\right) + 1 \leq \tau \leq \eta m$

Subcase 2.2. $m = 3\xi + 1,\ \xi \geq 0.$

labeling of x_ι, y_τ and z_τ is same as in subcase 1.1

Subcase 2.3. $m = 3\xi + 2,\ \xi \geq 0.$

labeling of x_ι, y_τ and z_τ is same as in subcase 3.1

Case 3. $\eta = 3\xi + 2,\ \eta \geq 1.$

Subcase 3.1. $m = 3\xi,\ \xi \geq 1.$

$\sigma(x_\iota) = 0.3 \quad 1 \leq \iota \leq \eta$

$\sigma(y_\tau) = 0.3 \quad 1 \leq \tau \leq \left(\frac{\eta m - \eta - 1}{3}\right)$

$\sigma(z_\tau) = 0.3 \quad 1 \leq \tau \leq \left(\frac{\eta m - \eta - 1}{3}\right)$

$$\sigma(y_\tau) = 0.1 \quad \left(\frac{\eta m - \eta - 1}{3}\right) + 1 \leq \tau \leq \eta m$$

$$\sigma(y_\tau) = 0.2 \quad \left(\frac{\eta m - \eta - 1}{3}\right) + 1 \leq \tau \leq \eta m$$

labeling of x_ι, y_τ and z_τ are same as in subcase 1.1

Subcase 3.3. $m = 3\xi + 2$, $\xi \geq 0$.

labeling of x_ι, y_τ and z_τ are same as in subcase 2.1

Taking $\frac{p}{3} = \delta$, $\nu_\sigma[\iota]$ and $\varepsilon_\mu[\iota]$, where $\iota \in \left\{\frac{r}{10}, r \in Z_4 - \{0\}\right\}$ is shown in the table below.

Table 1. $\nu_\sigma(\iota)$ and $\varepsilon_\mu(\iota)$ for $S'(C_\eta[m])$.

Value of η and m	$\nu_\sigma[0.1]$	$\nu_\sigma[0.2]$	$\nu_\sigma[0.3]$	$\varepsilon_\mu[0.1]$	$\varepsilon_\mu[0.2]$	$\varepsilon_\mu[0.3]$
$\eta = 3\xi$, $\xi \geq 1$ $m \geq 1$	δ	δ	δ	δ	δ	δ
$\eta = 3\xi + 1$, $\xi \geq 1$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$
$\eta = 3\xi + 1$, $\xi \geq 1$	δ	δ	δ	δ	δ	δ
$\eta = 3\xi + 1$, $\xi \geq 1$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$
$\eta = 3\xi + 2$, $\xi \geq 1$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$
$\eta = 3\xi + 2$, $\xi \geq 1$	δ	δ	δ	δ	δ	δ

$\eta = 3\xi + 2$, $\xi \geq 1$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$

It can be seen from the table 2, that $| \nu_\sigma(\iota) - \nu_\sigma(\tau) | \leq 1$ and $| \varepsilon_\mu(\iota) - \varepsilon_\mu(\tau) | \leq 1$. Where $\iota \neq \left\{\frac{r}{10}, r \in Z_4 - \{0\}\right\}$. Thus, the theorem is established.

Theorem 2. *$S'(C_{2\eta}[m]A)$ is fuzzy quotient-3 cordial graph.*

Proof. Let $V(S'(C_{2\eta}[m]A)) = \{x_\iota : 1 \leq \iota \leq \eta\} \cup \left\{y_\tau : 1 \leq \tau \leq \frac{\eta m}{2}\right\}$ $\cup \left\{z_\kappa : 1 \leq \kappa \leq \frac{\eta m}{2}\right\}$ and $E(S'(C_{2\eta}[m]A)) = \{x_\iota x_{\iota+1} : 1 \leq \iota \leq \eta - 1\} \cup \{x_\iota x_\eta\}$ $\cup \left\{x_{2\iota} z_\kappa : 1 \leq \iota \leq \frac{\eta}{2}, 1 + (\iota - 1)m \leq \kappa \leq \iota m\right\} \cup \left\{z_\kappa y_\kappa : 1 \leq \kappa \leq \frac{\eta m}{2}\right\}$.

$p = n + \eta m$ and $q = n + \eta m$. The following cases must be considered while defining

$\sigma : V(S'(C_{2\eta}[m]A)) \to [0, 1]$.

Case 1. If $m = 1$

$\sigma(x_\iota) = 0.1 \quad \iota \equiv 3, 4 (\text{mod } 6) \quad 1 \leq \iota \leq \eta$

$\sigma(x_\iota) = 0.2 \quad \iota \equiv 0, 1 (\text{mod } 6) \quad 1 \leq \iota \leq \eta$

$\sigma(x_\iota) = 0.3 \quad \iota \equiv 2, 5 (\text{mod } 6) \quad 1 \leq \iota \leq \eta.$

Subcase 1.1. $\eta = 6s$, $\xi \geq 1$

$\sigma(y_\tau) = 0.1 \quad \tau \equiv 0, 2 (\text{mod } 3) \quad 1 \leq \tau \leq \frac{\eta m}{2}$

$\sigma(z_\kappa) = 0.1 \quad \kappa \equiv 1, 2 (\text{mod } 3) \quad 1 \leq \kappa \leq \frac{\eta m}{2}$

$\sigma(z_\kappa) = 0.1 \quad \kappa \equiv 0 (\text{mod } 3) \quad 1 \leq \kappa \leq \frac{\eta m}{2}$

Subcase 1.2. $\eta = 6s + 2,\ \xi \geq 1$

$$\sigma(y_\tau) = 0.1 \quad \tau \equiv 0,\ 2 (\mathrm{mod}\ 3) \quad 1 \leq \tau \leq \frac{\eta m}{2} - 1$$

$$\sigma(y_\tau) = 0.2 \quad \tau \equiv 1 (\mathrm{mod}\ 3) \quad 1 \leq \tau \leq \frac{\eta m}{2} - 1$$

$$\sigma(y_{\frac{\eta m}{2}}) = 0.1$$

Labeling of z_κ for $1 \leq \kappa \leq \frac{\eta m}{2}$ is same as in subcase 1.1.

Subcase 1.3. $\eta = 6s + 4,\ \xi \geq 0$

Labeling of y_τ for $1 \leq \tau \leq \frac{\eta m}{2} - 1$ is same as in subcase 1.2 and $\sigma(y_{\frac{\eta m}{2}}) = 0.2.$

Labeling of z_κ for $1 \leq \kappa \leq \frac{\eta m}{2} - 2$ is same as in subcase 1.1 $\sigma(z_{\frac{\eta m}{2}-1}) = 0.1$ and $\sigma(z_{\frac{\eta m}{2}}) = 0.2.$

Case 2. If $m \geq 2$

$$\sigma(x_\iota) = 0.3 \quad 1 \leq \iota \leq \eta.$$

Subcase 2.1. $\eta = 6\xi,\ \xi \geq 1$ and $m \geq 2$

$$\sigma(z_\kappa) = 0.1 \quad 1 \leq \kappa \leq \left(\frac{\eta m + \eta}{2}\right)$$

$$\sigma(y_\tau) = 0.2 \quad 1 \leq \tau \leq \left(\frac{\eta m + \eta}{2}\right)$$

$$\sigma(z_\kappa) = 0.3 \quad \left(\frac{\eta m + \eta}{2}\right) + 1 \leq \kappa \leq \left(\frac{\eta m}{2}\right)$$

$$\sigma(y_\tau) = 0.3 \quad \left(\frac{\eta m + \eta}{2}\right) + 1 \leq \tau \leq \left(\frac{\eta m}{2}\right).$$

Subcase 2.2. $\eta = 6\xi + 2,\ \xi \geq 1$ and $m = 3\xi,\ \xi \geq 1$

$$\sigma(z_\kappa) = 0.1 \quad 1 \leq \kappa \leq \left(\frac{\eta m + \eta + 1}{3}\right)$$

$$\sigma(y_\tau) = 0.2 \quad 1 \leq \tau \leq \left(\frac{\eta m + \eta + 1}{3}\right)$$

$$\sigma(z_\kappa) = 0.3 \quad \left(\frac{\eta m + \eta + 1}{3}\right) + 1 \leq \kappa \leq \left(\frac{\eta m}{2}\right)$$

$$\sigma(y_\tau) = 0.3 \quad \left(\frac{\eta m + \eta + 1}{3}\right) + 1 \leq \tau \leq \left(\frac{\eta m}{2}\right).$$

Subcase 2.3. $\eta = 6\xi + 2, \xi \geq 1$ and $m = 3\xi + 1, \xi \geq 1$

$$\sigma(z_\kappa) = 0.1 \quad 1 \leq \kappa \leq \left(\frac{\eta m + \eta - 1}{3}\right)$$

$$\sigma(y_\tau) = 0.2 \quad 1 \leq \tau \leq \left(\frac{\eta m + \eta - 1}{3}\right)$$

$$\sigma(z_\kappa) = 0.3 \quad \left(\frac{\eta m + \eta - 1}{3}\right) + 1 \leq \kappa \leq \left(\frac{\eta m}{2}\right)$$

$$\sigma(y_\tau) = 0.3 \quad \left(\frac{\eta m + \eta - 1}{3}\right) + 1 \leq \tau \leq \left(\frac{\eta m}{2}\right).$$

Subcase 2.4. $\eta = 6\xi + 2, \xi \geq 1$ and $m = 3\xi + 2, \xi \geq 1$

labeling of y_τ and z_κ is same as in subcase 2.1

Subcase 2.5. $\eta = 6\xi + 4, \xi \geq 1$ and $m = 3\xi, \xi \geq 1$

labeling of y_τ and z_κ is same as in subcase 2.3

Subcase 2.6. $\eta = 6\xi + 2, \xi \geq 1$ and $m = 3\xi + 1, \xi \geq 1$

labeling of y_τ and z_κ is same as in subcase 2.2

Subcase 2.7. $\eta = 6\xi + 2, \xi \geq 1$ and $m = 3\xi + 2, \xi \geq 1$

labeling of y_τ and z_κ is same as in subcase 2.1

Taking $\frac{p}{3} = \delta$, $\nu_\sigma[\iota]$ and $\varepsilon_\mu[\iota]$, where $\iota \in \left\{\frac{r}{10}, r \in Z_4 - \{0\}\right\}$ is shown in the table below.

Table 2. $\nu_\sigma(\iota)$ and $\varepsilon_\mu(\iota)$ for $S'(C_{2\eta}[m]A)$.

Value of η and m	$\nu_\sigma[0.1]$	$\nu_\sigma[0.2]$	$\nu_\sigma[0.3]$	$\varepsilon_\mu[0.1]$	$\varepsilon_\mu[0.2]$	$\varepsilon_\mu[0.3]$
$\eta = 6\xi,\ \xi \geq 1$ $m \geq 1,\ m \geq 1$	δ	δ	δ	δ	δ	δ
$\eta = 6\xi + 2,\ \xi \geq 1$ (or) $\eta = 6\xi + 4,\ \xi \geq 0$ $m = 3\xi + 2,\ \xi \geq 0$	δ	δ	δ	δ	δ	δ
$\eta = 6\xi + 2,\ \xi \geq 1$ $m = 3\xi + 1,\ \xi \geq 0$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$
$\eta = 6\xi + 4,\ \xi \geq 0$ $m = 3\xi,\ \xi \geq 1$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$
$\eta = 6\xi + 2,\ \xi \geq 1$ $m = 3\xi,\ \xi \geq 1$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$
$\eta = 6\xi + 4,\ \xi \geq 0$ $m = 3\xi + 1,\ \xi \geq 1$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$

It can be seen from the table 2, that $|\nu_\sigma(\iota) - \nu_\sigma(\tau)| \leq 1$ and $|\varepsilon_\mu(\iota) - \varepsilon_\mu(\tau)| \leq 1$, where $\iota \neq \tau \in \left\{\frac{r}{10},\ r \in Z_4 - \{0\}\right\}$. Thus, the theorem is established.

Theorem 3. *$S'(C_\eta[a,\ d])$ is fuzzy quotient-3 cordial graph.*

Proof. Let $V(S'(C_\eta[a,\ b])) = \{x_\iota : 1 \leq \iota \leq \eta\} \cup \left\{y_\tau : 1 \leq \tau \leq \frac{\eta}{2}[2a + (\eta - 1)d]\right\}$ $\cup \left\{z_\kappa : 1 \leq \kappa \leq \frac{\eta}{2}[2a + (\eta - 1)d]\right\}$ and $E(S'(C_\eta[a,\ b])) = \{x_\iota x_{\iota+1} : 1 \leq \iota \leq \eta - 1\}$

$$\cup \{x_\iota x_\eta\} \cup \left\{x_\iota y_\tau : 1 \leq \iota \leq \eta; (\iota - 1)a + \frac{(\iota-1)(\iota-2)d}{2} + 1 \leq \tau \leq \iota a + \frac{(\iota-1)d}{2}\right\}$$

$$\cup \left\{y_\tau z_\tau : 1 \leq \tau \leq \frac{\eta}{2}[2a + (\eta - 1)d]\right\} p = \eta[2a + (\eta - 1)d + 1] = q.$$

The following cases must be considered while defining

$\sigma : V(S'(C_\eta[a, b])) \to [0, 1].$

Taking $t = \frac{\eta}{2}[2a + (\eta - 1)d]$

Case 1. $\eta = 3\xi, \xi \geq 1$

Subcase 1.1. $a = 3\xi, \xi \geq 1$ and $r \geq 1$

$\sigma(x_\iota) = 0.3 \quad 1 \leq \iota \leq \eta$

$\sigma(y_\tau) = 0.1 \quad 1 \leq \tau \leq \frac{p}{3}$

$\sigma(y_\tau) = 0.3 \quad \frac{p}{3} + 1 \leq \tau \leq t$

$\sigma(z_\kappa) = 0.2 \quad 1 \leq \kappa \leq \frac{p}{3}$

$\sigma(z_\kappa) = 0.3 \quad \frac{p}{3} + 1 \leq \kappa \leq t.$

Case 2. $\eta = 3\xi + 1, \xi \geq 1$

Subcase 2.1. $a = 3\xi, \xi \geq 1$ and $d \geq 1$

$\sigma(x_\iota) = 0.3 \quad 1 \leq \iota \leq \eta$

$\sigma(y_\tau) = 0.1 \quad 1 \leq \tau \leq \frac{p-1}{3}$

$\sigma(y_\tau) = 0.3 \quad \frac{p-1}{3} + 1 \leq \tau \leq t$

$\sigma(z_\kappa) = 0.2 \quad 1 \leq \kappa \leq \frac{p-1}{3}$

$\sigma(z_\kappa) = 0.3 \quad \frac{p-1}{3} + 1 \leq \kappa \leq t.$

Subcase 2.2. $a = 3\xi + 1, \xi \geq 0$ and $d \geq 1$

The labeling x_ι, y_τ and z_κ are same as in subcase 1.1.

Subcase 2.3. $a = 3\xi + 2, \xi \geq 0$ and $d \geq 1$

$\sigma(x_\iota) = 0.3 \quad 1 \leq \iota \leq \eta$

$\sigma(y_\tau) = 0.1 \quad 1 \leq \tau \leq \frac{p+1}{3}$

$\sigma(y_\tau) = 0.3 \quad \frac{p+1}{3} + 1 \leq \tau \leq t$

$\sigma(z_\kappa) = 0.2 \quad 1 \leq \kappa \leq \frac{p+1}{3}$

$\sigma(z_\kappa) = 0.3 \quad \frac{p+1}{3} + 1 \leq \kappa \leq t.$

Case 3. $\eta = 3\xi + 2, \xi \geq 1.$

Subcase 3.1. $a = 3\xi + 1, \xi \geq 1$ and $d = 3\xi + 2, \xi \geq 0$ (or)

$a = 3\xi + 1, \xi \geq 0$ and $d = 3\xi, \xi \geq 1$ (or) $a = 3\xi + 2, \xi \geq 0$ and

$d = 3\xi + 1, \xi \geq 0.$

Labeling x_ι, y_τ and z_κ are same as in subcase 1.1.

Subcase 3.2. $a = 3\xi, \xi \geq 1$ and $d = 3\xi + 1, \xi \geq 0$ (or)

$a = 3\xi + 1, \xi \geq 0$ and $d = 3\xi + 2, \xi \geq 0$ (or) $a = 3\xi + 2, \xi \geq 0$ and $d = 3\xi, \xi \geq 1.$

Labeling x_ι, y_τ and z_κ are same as in subcase 2.1.

Subcase 3.3. $a = 2\xi, \xi \geq 1$ and $d = 3\xi, \xi \geq 1$ (or)

$a = 3\xi + 1, \xi \geq 0$ and $d = 3\xi + 1, \xi \geq 0$ (or) $a = 3\xi + 2, \xi \geq 0$ and $d = 3\xi + 2, \xi \geq 0$

Labeling x_ι, y_τ and z_κ are same as in subcase 2.3

Taking $\frac{p}{3} = \delta$, $\nu_\sigma[\iota]$ and $\varepsilon_\mu[\iota]$, where $\iota \in \left\{\frac{r}{10}, r \in Z_4 - \{0\}\right\}$ is shown in the table below.

Table 3. $\nu_\sigma(\iota)$ and $\varepsilon_\mu(\iota)$ for $S'(C_\eta[a, d])$.

Value of η and m	$\nu_\sigma[0.1]$	$\nu_\sigma[0.2]$	$\nu_\sigma[0.3]$	$\varepsilon_\mu[0.1]$	$\varepsilon_\mu[0.2]$	$\varepsilon_\mu[0.3]$
$\eta = 3\xi, \xi \geq 1$ $a, d \geq 1$	δ	δ	δ	δ	δ	δ
$\eta = 3\xi + 1, \xi \geq 1$ $a = 3\xi, \xi \geq 1$ $d \geq 1$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$
$\eta = 3\xi + 1, \xi \geq 1$ $a = 3\xi + 1, \xi \geq 0$ $d \geq 1$	δ	δ	δ	δ	δ	δ
$\eta = 3\xi + 1, \xi \geq 1$ $a = 3\xi + 2, \xi \geq 0$ $d \geq 1$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$
$\eta = 3\xi + 2, \xi \geq 1$ $a = 3\xi, \xi \geq 1$ $d = 3\xi + 2, \xi \geq 0$	δ	δ	δ	δ	δ	δ
$\eta = 3\xi + 2, \xi \geq 1$ $a = 3\xi + 1, \xi \geq 0$ $d = 3\xi, \xi \geq 1$	δ	δ	δ	δ	δ	δ
$\eta = 3\xi + 2, \xi \geq 1$ $a = 3\xi + 2, \xi \geq 0$	δ	δ	δ	δ	δ	δ

$d = 3\xi + 1,\ \xi \geq 0$						
$\eta = 3\xi + 2,\ \xi \geq 1$ $a = 3\xi,\ \xi \geq 1$ $d = 3\xi + 1,\ \xi \geq 0$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$
$\eta = 3\xi + 2,\ \xi \geq 1$ $a = 3\xi + 1,\ \xi \geq 0$ $d = 3\xi + 2,\ \xi \geq 0$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$
$\eta = 3\xi + 2,\ \xi \geq 1$ $a = 3\xi + 2,\ \xi \geq 0$ $d = 3\xi,\ \xi \geq 1$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$
$\eta = 3\xi + 2,\ \xi \geq 1$ $a = 3\xi,\ \xi \geq 1$ $d = 3\xi,\ \xi \geq 1$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$
$\eta = 3\xi + 2,\ \xi \geq 1$ $a = 3\xi + 1,\ \xi \geq 0$ $d = 3\xi + 1,\ \xi \geq 0$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$
$\eta = 3\xi + 2,\ \xi \geq 1$ $a = 3\xi + 2,\ \xi \geq 0$ $d = 3\xi + 2,\ \xi \geq 0$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$

It can be seen from the table 2, that $|\nu_\sigma(\iota) - \nu_\sigma(\tau)| \leq 1$ and $|\varepsilon_\mu(\iota) - \varepsilon_\mu(\tau)| \leq 1$. Where $\iota \neq \tau \in \left\{\frac{r}{10},\ r \in Z_4 - \{0\}\right\}$. Thus, the theorem is established.

Theorem 4. *$S'(C_\eta[a, r])$ is fuzzy quotient-3 cordial graph.*

Proof. Let $V(S'(C_\eta[a, r])) = \{x_\iota : 1 \leq \iota \leq \eta\} \cup \left\{y_\tau : 1 \leq \tau \leq \frac{a(r^n - 1)}{r - 1}\right\}$

$\cup\left\{z_\kappa : 1 \le \kappa \le \frac{a(r^n - 1)}{r - 1}\right\}$ and $E(S'(C_\eta[a, r])) = \{x_\iota x_{\iota+1} : 1 \le \iota \le \eta - 1\}$

$\cup\{x_\iota x_\eta\} \cup \left\{x_\iota y_\tau : 1 \le \iota \le \eta; \frac{a(r^{\iota-1} - 1)}{r - 1} + 1 \le \tau \le \frac{a(r^\eta - 1)}{r - 1}\right\}$

$\cup\left\{y_\tau z_\tau : 1 \le \tau \le \frac{a(r^\eta - 1)}{r - 1}\right\}$ $p = \eta + 2\frac{a(r^\eta - 1)}{r - 1} = q'$;

The following cases must be considered while defining

$\sigma : V(S'(C_\eta[a, r])) \to [0, 1]$.

Taking $t = \frac{\eta}{2}[2a + (\eta - 1)d]$

Case 1. $\eta = 3\xi,\ \xi \ge 1$

Subcase 1.1. $a = 3\xi,\ \xi \ge 1$ and $r \ge 1$

$\sigma(x_\iota) = 0.3 \quad 1 \le \iota \le \eta$

$\sigma(y_\tau) = 0.1 \quad 1 \le \tau \le \frac{p}{3}$

$\sigma(y_\tau) = 0.3 \quad \frac{p}{3} + 1 \le \tau \le t$

$\sigma(z_\kappa) = 0.2 \quad 1 \le \kappa \le \frac{p}{3}$

$\sigma(z_\kappa) = 0.3 \quad \frac{p}{3} + 1 \le \kappa \le t.$

Subcase 1.2. $a = 3\xi + 1,\ \xi \ge 0$ and $r = 3\xi,\ \xi \ge 1$ (or) $r = 3\xi + 2,\ \xi \ge 0$

$\sigma(x_\iota) = 0.3 \quad 1 \le \iota \le \eta$

$\sigma(y_\tau) = 0.1 \quad 1 \le \tau \le \frac{p+1}{3}$

$\sigma(y_\tau) = 0.3 \quad \frac{p+1}{3} + 1 \le \tau \le t$

$\sigma(z_\kappa) = 0.2 \quad 1 \le \kappa \le \frac{p+1}{3}$

$\sigma(z_\kappa) = 0.3 \quad \frac{p+1}{3} + 1 \le \kappa \le t.$

Subcase 1.3. $a = 3\xi + 1, \xi \ge 0$ and $r = 3\xi + 1, \xi \ge 0$

Labeling x_ι, y_τ and z_κ are same as in subcase 1.1.

Subcase 1.4. $a = 3\xi + 2, \xi \ge 0$ and $r = 3\xi, \xi \ge 1$ (or) $r = 3\xi + 2, \xi \ge 0$

$\sigma(x_\iota) = 0.3 \quad 1 \le \iota \le \eta$

$\sigma(y_\tau) = 0.1 \quad 1 \le \tau \le \frac{p-1}{3}$

$\sigma(y_\tau) = 0.3 \quad \frac{p-1}{3} + 1 \le \tau \le t$

$\sigma(z_\kappa) = 0.2 \quad 1 \le \kappa \le \frac{p-1}{3}$

$\sigma(z_\kappa) = 0.3 \quad \frac{p-1}{3} + 1 \le \kappa \le t.$

Subcase 1.5. $a = 3\xi + 2, \xi \ge 0$ and $r = 3\xi + 1, \xi \ge 0$

Labeling x_ι, y_τ and z_κ are same as in subcase 1.1

Case 2. $\eta = 3\xi + 1, \xi \ge 1.$

Subcase 2.1. $a = 3\xi, \xi \ge 1$ and $r \ge 1$

Labeling x_ι, y_τ and z_κ are same as in subcase 1.4.

Subcase 2.2. $a = 3\xi + 1, \xi \ge 0$ and $r = 3\xi, \xi \ge 1$ (or) $r = 3\xi + 1, \xi \ge 0$

Labeling x_ι, y_τ and z_κ are same as in subcase 1.1.

Subcase 2.3. $a = 3\xi + 1, \xi \ge 0$ and $r = 3\xi + 2, \xi \ge 0$

Labeling x_ι, y_τ and z_κ are same as in subcase 1.4.

Subcase 2.4. $a = 3\xi + 2, \xi \ge 0$ and $r = 3\xi, \xi \ge 1$ (or) $r = 3\xi + 1, \xi \ge 0$

Labeling x_ι, y_τ and z_κ are same as in subcase 1.2.

Subcase 2.5. $a = 3\xi + 2, \xi \geq 0$ and $r = 3\xi + 2, \xi \geq 0$

Labeling x_ι, y_τ and z_κ are same as in subcase 1.4

Case 3. $\eta = 3\xi + 2, \xi \geq 1$.

Subcase 3.1. $a = 3\xi, \xi \geq 1$ and $r \geq 1$.

Labeling x_ι, y_τ and z_κ are same as in subcase 1.2.

Subcase 3.2. $a = 3\xi + 1, \xi \geq 0$ and $r = 3\xi, \xi \geq 1$ (or) $r = 3\xi + 2, \xi \geq 0$

Labeling x_ι, y_τ and z_κ are same as in subcase 1.4.

Subcase 3.3. $a = 3\xi + 1, \xi \geq 0$ and $r = 3\xi + 1, \xi \geq 0$

Labeling x_ι, y_τ and z_κ are same as in subcase 1.1.

Subcase 3.4. $a = 3\xi + 2, \xi \geq 0$ and $r = 3\xi, \xi \geq 1$ (or) $r = 3\xi + 2, \xi \geq 0$

Labeling x_ι, y_τ and z_κ are same as in subcase 1.1.

Subcase 3.5. $a = 3\xi + 2, \xi \geq 0$ and $r = 3\xi + 1, \xi \geq 0$

Labeling x_ι, y_τ and z_κ are same as in subcase 1.4.

Taking $\frac{p}{3} = \delta$, $\nu_\sigma[\iota]$ and $\varepsilon_\mu[\iota]$, where $\iota \in \left\{\frac{r}{10}, r \in Z_4 - \{0\}\right\}$ is shown in the table below.

Table 4. $\nu_\sigma(\iota)$ and $\varepsilon_\mu(\iota)$ for $S'(C_\eta[a, r])$.

Value of η, a and r	$\nu_\sigma[0.1]$	$\nu_\sigma[0.2]$	$\nu_\sigma[0.3]$	$\varepsilon_\mu[0.1]$	$\varepsilon_\mu[0.2]$	$\varepsilon_\mu[0.3]$
$\eta = 3\xi, \xi \geq 1$ $a = 3\xi, \xi \geq 1$ $r = 3\xi, \xi \geq 1$ $r = 3\xi + 2, \xi \geq 0$	δ	δ	δ	δ	δ	δ
$\eta = 3\xi, \xi \geq 1$	δ	δ	δ	δ	δ	δ

$a = 3\xi + 1, \xi \geq 0$ $a = 3\xi + 2, \xi \geq 0$ $r = 3\xi + 1, \xi \geq 0$						
$\eta = 3\xi, \xi \geq 1$ $a = 3\xi + 1, \xi \geq 0$ $r = 3\xi, \xi \geq 1$ $r = 3\xi + 2, \xi \geq 0$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$
$\eta = 3\xi, \xi \geq 1$ $a = 3\xi + 2, \xi \geq 0$ $r = 3\xi, \xi \geq 1$ $r = 3\xi + 2, \xi \geq 0$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$
$\eta = 3\xi + 1, \xi \geq 1$ $a = 3\xi, \xi \geq 0$ $r \geq 1$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$
$\eta = 3\xi + 1, \xi \geq 1$ $a = 3\xi + 1, \xi \geq 0$ $r = 3\xi, \xi \geq 1$ $r = 3\xi + 1, \xi \geq 0$	δ	δ	δ	δ	δ	δ
$\eta = 3\xi + 1, \xi \geq 1$ $a = 3\xi + 1, \xi \geq 0$ $a = 3\xi + 2, \xi \geq 0$ $r = 3\xi + 2, \xi \geq 0$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$
$\eta = 3\xi + 1, \xi \geq 1$ $a = 3\xi + 1, \xi \geq 0$ $r = 3\xi + 2, \xi \geq 1$ $r = 3\xi + 1, \xi \geq 0$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$

$\eta = 3\xi + 2, \xi \geq 1$ $a = 3\xi, \xi \geq 0$ $r \geq 1$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$	$\delta + \frac{1}{3}$	$\delta + \frac{1}{3}$	$\delta - \frac{2}{3}$
$\eta = 3\xi + 1, \xi \geq 1$ $a = 3\xi + 1, \xi \geq 0$ $r = 3\xi + 1, \xi \geq 0$	δ	δ	δ	δ	δ	δ
$\eta = 3\xi + 1, \xi \geq 1$ $a = 3\xi + 2, \xi \geq 0$ $r = 3\xi, \xi \geq 1$ $r = 3\xi + 2, \xi \geq 0$	δ	δ	δ	δ	δ	δ
$\eta = 3\xi + 1, \xi \geq 1$ $a = 3\xi + 1, \xi \geq 0$ $r = 3\xi, \xi \geq 1$ $r = 3\xi + 2, \xi \geq 0$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$
$\eta = 3\xi + 1, \xi \geq 1$ $a = 3\xi + 2, \xi \geq 0$ $r = 3\xi + 1, \xi \geq 0$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$	$\delta - \frac{1}{3}$	$\delta - \frac{1}{3}$	$\delta + \frac{2}{3}$

It can be seen from the table 2, that $|\nu_\sigma(\iota) - \nu_\sigma(\tau)| \leq 1$ and $|\varepsilon_\mu(\iota) - \varepsilon_\mu(\tau)| \leq 1$. Where $\iota \neq \tau \in \left\{\frac{r}{10}, r \in Z_4 - \{0\}\right\}$. Thus, the theorem is established.

4. Conclusion

The presence of fuzzy quotient 3 labelling on some subdivision graphs is discussed and established in this study. Our next step will be to investigate this concept in different graph families and identify applications for it.

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