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A STUDY ON VERTEX COLOURING

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Abstract

Many real-life situations can be described by means of a diagram consisting of a set of points with lines joining certain pairs of points. Loosely speaking, such a diagram is what we mean by a graph. Graphs lend themselves naturally as models for a variety of situations.

Graph Colouring

Perhaps the least obvious application of direct graph theory comes in the form of colouring maps. On any map it is most often the case that any two adjacent regions are colored with a different colour so as to help distinguish their geographical features. It turns out that the dual to this problem is to assign a colour to each vertex of a simple graph such that no two adjacent vertices share the same colour.

A Graph Colouring Algorithm

- Assign colour 1 to the vertex with highest degree.
- Also assign colour 1 to any vertex that is not connected to this vertex.
- Assign colour 2 to the vertex with the next highest degree that is not already coloured.
- Also assign colour 2 to any vertex not connected to this vertex and that is not already coloured.

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- If uncoloured vertices remain, assign colour 3 to the uncoloured vertex with next highest degree and other uncoloured, unconnected vertices.
- Proceed in this manner until all vertices are coloured.

Some Types of Graph Colouring

1. Vertex colouring
2. Edge colouring
3. Total colouring
4. List colouring
5. Multi colouring
6. Minimum sum colouring
7. Harmonious colouring
8. Map colouring

Vertex Colouring

A vertex colouring of a graph G is a mapping $XC : V(G) \rightarrow S.$ Then elements of S are called colours the vertices of one colour form a colour class. If $[S^1] = k,$ we say that c is a K-colouring (often we used $S = \{1, 2, \ldots, k\}.$ A colouring is proper if adjacent vertices have different colours. A graph is k-colourable if it has a proper k-colouring.

The chromatic number $\chi(G)$ is the least k such that G is k-colourable. Obviously $\chi(G)$ exists as assigning distinct colours to vertices yields a proper $|v(G)|$-colouring. An optimal colouring of G is a $\chi(G)$-colouring.

A graph G is k-chromatic if $\chi(G) = k.$

Obviously, the complete graph k_n requires n colours,

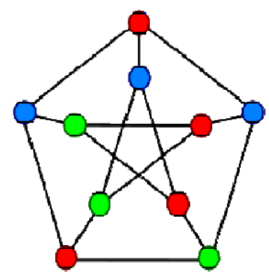
$$\text{So } \chi(k_n) = n, \text{ then } \chi(G) = \omega(G)$$

This bound can n be tight, but it can also be very loose. Indeed for any given integers $K \leq 1,$ there are graphs with clique number k and chromatic number l.

In a proper colouring, each colour class is a stable set,

So $\chi(G) \leq K_n$

Example of vertex colouring.



Theorem 2.2. (*BROOK'S*) *Let G be a connected graph. Then* $\chi(G) \leq G$ *unless G is either a complete graph or an odd cycle.*

Proof. In order to prove this theorem; we need the following case result.

Result. Let G be a connected graph which is not a complete graph. Then there exist three vertices u, v and w, such that $uv \in v(G)$

$$vw \in v(G) \text{ and } uw \notin v(G).$$

Proof. Since G is not complete there exist two vertices u and u' which are not linked by an edge.

Since G is connected there is a path between u and u'

Let P be a shortest (u, u') path and let v and w be respectively, the second and third vertices on P.

Then uv and vw are edges of the paths and uw is not an edge. Otherwise it word shortcut P.

Proof of the Theorem

We may assume that $n = (G) \geq 3$.

Since G is complete. If $n \leq 1$ and G is an odd cycle or bipartite when $n = 2$. In which cases the bound holds.

We shall find an ordering of the vertices so that the greedy colouring relative to it yields the desired bound.

Assume first that G is not-regular.

Let v_n be a vertex of degree less than.

Since G is connected, one can grow a spanning tree of G from v_n, assigning indices is decreasing order as we reach vertices.

We obtain an ordering $v_1, v_2, \ldots, v_n$ such that every vertex other than v_n has a higher-indexed neighbor.

We obtain an ordering $v_1, v_2, \ldots, v_n$ such that every vertex other than v_n has a higher-indexed neighbor.

Therefore the greedy-n colouring uses at most colours.

Now,

Assume that G is-regular. If G has a cut-vertex of we may apply the above method on each component of $G - x$ plus x.

Then, permuting the names of the colours, one can make the colouring agree on x, to complete a proper colouring of G.

Hence we may assume that G is 2-conneced.

In such a case, for G is not, by the result some vertex v_n has neighbors v_1 and v_2 Such that $v_1 v_2 \notin v(G)$.

Moreover considering such vertices for which the component of v_n is $G - \{v_1, v_2\}$ has maximum size, the easily shows that $G - \{v_1, v_2\}$ is connected,

Then, indexing the vertices of a spanning free rooted in v_n in a decreasing order, with $\{3 \ldots n\}$ we obtain an ordering $v_1 \ldots v_n$ such that every vertex other than v_n has a higher-indexed neighbor.

Now,

The greedy colouring will assign colour 1 to both v_1 and v_2. So when colouring v_n at most -1 colours will be assigned to its neighbors.

Hence the greedy colouring will use at most colours.

Hence the proof

Chromatic Number in Vertex Colouring

Theorem. *The chromatic number of a graph G, is the minimum number of colours needed for a proper vertex colouring of G.*

If $\psi(G) = k_1$, G is said to be k-chromatic

For example, the chromatic number of the graph in figure 3.1 is 3.

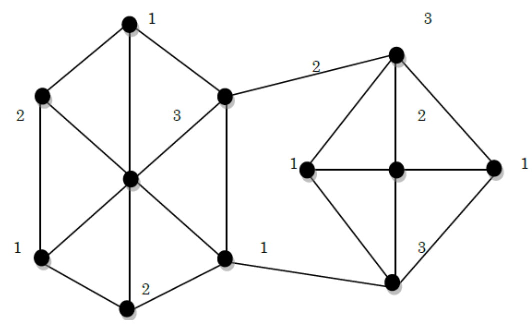


Figure 3.1. A 3-Chromatic graph.

Proposition. *A k-colouring of a graph G is a vertex colouring of G that uses k colors.*

Proposition. *A graph G is said to be k-colourable if G admits a proper vertex colouring using k-colours.*

Thus $\psi(G) = k$ if graph G is k-colourable but not $(k-1)$ colourable.

In considering the chromatic number of a graph only the adjacency of vertices is taken into account. A graph with a self-loop is regarded as un colourable, since the endpoint of the self-loop is adjacent to itself.

Moreover, a multiple adjacency has no more effect on the colours of its end point then single adjacency. As a consequence we may restrict ourselves to simple graphs when dealing with chromatic number.

It's clear that $\psi(G) = 1$ if and only if G has no edges and $\psi(G) = 2$ if and only if G is bipartite.

Basic Principles for Calculating Chromatic Number

Although the chromatic number is one of the most studies parameters in graph theory, no formula exist for the chromatic number of are arbitrary graph. Thus for the most part, one must be content with supplying bounds for the chromatic number of graphs.

A few basic principles recur in many chromatic number calculations. Now, we will try to find upper and lower bound to provide a direct approach to the chromatic number of a given graph.

Upper Bound

Show $\psi(G) \leq k$ by exhibiting a proper k-colouring

Lower Bound

Show $\psi(G) \geq k$ by using properties of graph G most especially by finding a sub graph that requires k-colours.

Proposition. *Let H be a sub graph of G. then* $\psi(G) \geq \psi(H)$.

Proof. Whatever colour are useful on the vertices of sub graph H in a minimum colouring G can also be used in colouring H by itself.

Proposition. *Let G be any graph. Then* $\psi(G) \geq \frac{|V(G)|}{\alpha(G)}$.

Proof. Given a k-colouring of G, the vertices being coloured with the same colour from an independent set

Let G be a graph with n vertices and c a k-colouring of G. use define $V_i = \{V/c(v) = i\}$ for $i = 0, 1, \ldots, k$.

Each v_i is an independent set.

Let $\alpha(G)$ be the independent number of G,

We have $v_i \leq \alpha(G)$

$$n = |V(G)| = |V_1| + |V_2| + \ldots + |V_k| \leq k\alpha(G) = \psi(G)\alpha(G)$$

We have $\psi(G) \geq \frac{|V(G)|}{\alpha(G)}$.

Most upper bounds on the chromatic number come from algorithms that produce colorings.

For example, assigning distinct colour to the vertices yields $(G) \leq n(G)$. This bound is best possible.

Since $\psi(k_n) = n$ but it holds with equality only for complete graphs. We can improve a "best possible" bound by obtaining another bound that is always at least as good.

For example, $\psi(G) \leq n(G)$ uses nothing about the structure of G, we can do better by colouring there vertices in some order and always using the "least available" colour.

Conclusion

The main aim of this study is to know where the vertex colouring used in web Mathematica. This project is a surprising application of the vertex colouring techniques we developed in this project, and involves an important chromatic number in vertex coloring. This paper gives an overview of the applications of vertex colouring in chromatic polynomial fields to some extent applications that uses vertex colouring concepts.

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