**Advances and Applications in Mathematical Sciences**
Volume 21, Issue 11, September 2022, Pages 6345-6350


# EQUITABLE EDGE COLORING OF PRISM GRAPH FAMILIES

**K. KUMAR[1] and G. JAYARAMAN[2]**

[1,2]Department of Mathematics
Vels Institute of Science, Technology
and Advanced Studies, Chennai, 600 117, India

## Abstract

An equitable edge coloring of $G$ is an assignment of colors to all the edge of a graph $G$ for which no two adjacent edges got the same colors and difference for any two color classes by at most one. The minimum number of colors required for such coloring is called as the equitable edge chromatic number. In this paper, we prove the results on equitable edge chromatic number for prism graph $D_n$, anti-prism graph $A_n$, and $n$-crossed prism graph $R_n$.

## 1. Introduction and Preliminaries

Let us consider all graphs are finite simple and undirected graph $G$. An edge coloring of a graph $G$ is an assignment of colors to the set of edges of $G$ in which the adjacent edges received different colors. The minimum number of colors required for such a proper edge coloring is called as edge chromatic number and denoted by $\chi'(G)$. Clearly $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of a graph $G$.

In 1964, Vizing [8] given the tight bound for edge coloring that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

In 1973, the concept of equitable coloring was defined by Meyer [4] and also given the conjectured that for any connected graph $G$, $\chi'(G) = \Delta(G)$.

After few years, as an extension of equitable coloring, the concept of

2020 Mathematics Subject Classification: 05C15.
Keywords: Prism graph, anti-prism graph, $n$-crossed prism graph, equitable edge coloring.
[2]Corresponding author; E-mail: jayaram07maths@gmail.com
Received January 12, 2022; Accepted March 5, 2022.

equitable edge coloring was introduced by Hilton and de Werra [2] in 1994. Veninstine Vivik et al. [6] proved the equitable edge chromatic number of splitting of helm and sunlet graph and Veninstine Vivik et al. [7] proved that equitable edge chromatic number of $S_n, W_n, H_n$ and $G_n$. Also given the algorithm. In 2020, Manikandan et al. [3] proved that an equitable edge coloring of strong product of pat and cycle.

In day to day life there are many problems on scheduling, optimization, network designing problems are linked to edge coloring.

The prism $D_n, n \geq 3$, is defined as the Cartesian-product $C_m \square K_2$ of a cycle on $m$ vertices with a complete graph on two vertices. Let the vertex set and the edge set of $D_n$ suchthat $V(D_n) = \{x_a, y_a : a = 1, 2, \ldots, n\}$ and $E(D_n) = \{x_a x_{a+1(\text{mod } n)}, y_a y_{a+1(\text{mod } n)}, x_a y_a : a = 1, 2, \ldots, n\}$.

The antiprism graph $A_n, n \geq 3$, [1], is a 4-regular graph consists of outer and inner cycle $C_n$, for which two cycles joined by an edges $v_a u_a$ and $u_a v_{a+1(\text{mod } n)}$ for $a = 1, 2, \ldots, n$.

Then $n$-crossed prism graph [5] for $n \geq 4$, positive even number is a graph obtained by taking two disjoint cycles say $C'_n$ and $C''_n$ and adding edges $(v_a, u_{a+1})$ for $a \in \{1, 3, \ldots, n-1\}$ and $(v_b, u_{b+1})$ for $b = \{2, 4, \ldots, n\}$. It is denoted by $R_n$.

An equitable edge coloring of graph $G$ is a function $c : E(G_n) \to \{1, 2, \ldots, \Delta\}$, the colors satisfying the following conditions.

(i) $c(e) \neq c(e')$, for any two adjacent edges $e, e' \in E(G)$

(ii) $||E_i| - |E_j|| \leq 1$, for every $(i, j)$

The minimum number of colors needed to color the graph is called the equitable edge chromatic number of $G$ and is denoted by $\chi'_e$.

## 2. Main Results

**Theorem 2.1.** *Let $D_n$ be the prism graph, then* $\chi'_e(D_n) = 3$, $n \geq 3$.

**Proof.** Let $V(D_n) = \{u_k, v_k : 1 \leq k \leq n\}$ and

Let $E(A_n) = \{e_k, s_k, f_k : 1 \leq k \leq n\}$, where the edges $\{e_k : 1 \leq k \leq n\}$ represents the edge $\{v_k v_{k+1(\text{mod } n)} : 1 \leq k \leq n\}$, the edges $\{s_k : 1 \leq k \leq n\}$ represents the edge $\{u_k v_k : 1 \leq k \leq n\}$ and the edges $\{f_k : 1 \leq k \leq n\}$ represents the edge $\{u_k v_{k+1(\text{mod } n)} : 1 \leq k \leq n\}$.

Define $c : E(D_n) \to \{1, 2, 3\}$, as follows. Let us partition the edge set of prism graph $E(D_n)$ as follows.

**Case (i).** When $n$ is even

$$E_1 = \{e_1, e_2, \ldots, e_{n-1}\} \cup \{f_1, f_2, \ldots, f_{n-1}\} \tag{2.1}$$

$$E_2 = \{e_2, e_4, \ldots, e_n\} \cup \{f_2, f_4, \ldots, f_n\} \tag{2.2}$$

$$E_3 = \{s_k : 1 \leq k \leq n\} \tag{2.3}$$

From equation (2.1) to (2.3), the prism graph $D_n$ is equitable edge colored with 4 colors. $E_1$, $E_2$ and $E_3$ are the color classes of $E(D_n)$. Also we observe that these color classes $E_1$, $E_2$ and $E_3$ are independent sets of $D_n$ and its satisfies the inequality $|\,|E_i| - |E_j|\,| \leq 1$, for $i \neq j$. Hence $\chi'_e(D_n) \leq 3$. Since $\Delta = 3$ and $\chi'_e(D_n) \geq \Delta = 3$. Therefore $\chi'_e(D_n) = 3$.

**Case (ii):** When $n$ is odd

$$E_1 = \{e_1, e_2, \ldots, e_{n-2}\} \cup \{f_1, f_2, \ldots, f_{n-2}\} \cup \{s_n\} \tag{2.4}$$

$$E_2 = \{e_2, e_4, \ldots, e_{n-1}\} \cup \{f_2, f_4, \ldots, f_{n-1}\} \cup \{s_1\} \tag{2.5}$$

$$E_3 = \{e_n\} \cup \{f_n\} \cup \{s_k : 2 \leq k \leq n-1\} \tag{2.6}$$

From equation (2.4) to (2.6), the prism graph $D_n$ is equitable edge colored with 3 colors. $E_1$, $E_2$ and $E_3$ are the color classes of $E(D_n)$. Also we observe that these color classes $E_1$, $E_2$ and $E_3$ are independent sets of $D_n$ and its holds the inequality $|| E_i | - | E_j || \leq 1$, for $i \neq j$. Hence $\chi'_e(D_n) \leq 3$. Since $\Delta = 3$ and $\chi'_e(D_n) \geq \Delta = 3$. Therefore $\chi'_e(D_n) = 3$. Thus $c$ is equitable edge colored with 3 colors.

**Theorem 2.2.** *Let* $A_n$ *be the antiprism graph of order* $n \geq 4$, *then* $\chi'_e(A_n) = 4$.

**Proof.** Let $V(A_n) = \{u_k, v_k : 1 \leq k \leq n\}$ and

$E(A_n) = \{e_k, e'_k, s_k, f_k : 1 \leq k \leq n\}$, where the edges $\{e_k : 1 \leq k \leq n\}$ represents the edge $\{v_k v_{k+1(\text{mod } n)} : 1 \leq k \leq n\}$, the edges $\{e'_k : 1 \leq k \leq n\}$ represents the edge $\{u_k u_{k+1(\text{mod } n)} : 1 \leq k \leq n\}$, the edges $\{s_k : 1 \leq k \leq n\}$ represents the edge $\{u_k v_{k+1(\text{mod } n)} : 1 \leq k \leq n\}$ and the edges $\{f_k : 1 \leq k \leq n\}$ represents the edge $\{u_k v_k : 1 \leq k \leq n\}$.

Define $c : E(A_n) \to \{1, 2, 3, 4\}$ as follows. Let us partition the edge set of anti-prism graph $E(A_n)$ as follows. Consider the following two cases

**Case (i).** When $n$ is odd

$$E_2 = \{e_1, e_3, \ldots, e_{n-2}\} \cup \{e'_2, e'_4, \ldots, e'_{n-3}\} \cup \{s_{n-1}\} \cup \{e'_n\} \tag{2.7}$$

$$E_2 = \{e_2, e_4, \ldots, e_{n-1}\} \cup \{e'_1, e'_3, \ldots, e'_{n-2}\} \cup \{s_n\} \tag{2.8}$$

$$E_3 = \{e_n\} \cup \{e'_{n-1}\} \cup \{s_k : 1 \leq k \leq n-2\} \tag{2.9}$$

$$E_4 = \{f_k : 1 \leq k \leq n\} \tag{2.10}$$

From the equation (3.7) to (3.10), the anti-prism graph $A_n$ is equitable edge colored with 4 colors. The color classes are $E(A_n) = \{E_1, E_2, E_3, E_4\}$. Also we observe that these color classes $E_1, E_2, E_3$ and $E_4$ are independent sets of $A_n$ and its satisfies the inequality $|| E_i | - | E_j || \leq 1$, for $i \neq j$. Hence $\chi'_e(A_n) \leq 4$. Since $\Delta = 4$ and $\chi'_e(A_n) \geq \Delta = 4$. Therefore $\chi'_e(A_n) = 4$.

**Case(ii).** When $n$ is even

$$E_1 = \{e_1, e_3, \ldots, e_{n-1}\} \cup \{e'_2, e'_4, \ldots, e'_n\} \tag{2.11}$$

$$E_2 = \{e_2, e_4, \ldots, e_n\} \cup \{e'_1, e'_3, \ldots, e'_{n-1}\} \tag{2.12}$$

$$E_3 = \{s_k : 1 \le k \le n\} \tag{2.13}$$

$$E_4 = \{f_k : 1 \le k \le n\} \tag{2.14}$$

From the equation (2.11) to (2.14), the anti-prism graph $A_n$ is equitable edge colored with 4 colors. $E_1$, $E_2$, $E_3$ and $E_4$ are color classes of $E(A_n)$. Also we observe that these color classes $E_1$, $E_2$, $E_3$ and $E_4$ are independent sets of $A_n$ and its satisfies the inequality $||E_i| - |E_j|| \le 1$, for $i \ne j$. Hence $\chi'_e(A_n) \le 4$. Since $\Delta = 4$ and $\chi'_e(A_n) \ge \Delta = 4$. Therefore $\chi'_e(A_n) = 4$.

**Theorem 2.3**. *Let $n \ge 4$, be a positive even number and let $R_n$ be the n-crossed prism graph, then $\chi'_e(R_n) = 3$.*

**Proof.** Let $V(R_n) = \{x_k, w_k : k \in \{1, 2, \ldots, n\}$ and

$$E(R_n) = \begin{cases} \left\{s_l = w_k x_{k+1} : k \in \{1, 3, \ldots, n-1\}, l \in \left\{1, 2, \ldots, \frac{n}{2}\right\}\right\} \cup \\ \left\{t_l = x_{k-1} w_k : k \in \{2, 4, \ldots, n\}, l \in \left\{1, 2, \ldots, \frac{n}{2}\right\}\right\} \cup \\ \{e_l = x_k x_{k+1(\bmod n)} : l, k \in \{1, 2, 3, \ldots, n\}\} \cup \\ \{f_l = w_k w_{k+1(\bmod n)} : l, k \in \{1, 2, 3, \ldots, n\}\} \end{cases}$$

Define an edge coloring $c : E(R_n) \to \{1, 2, 3\}$ as follows. Let us partition the edge set of $n$-crossed prism graph $R_n$ as follows:

$$E_1 = \{e_1, e_3, \ldots, e_{n-1}\} \cup \{f_1, f_2, \ldots, f_{n-1}\} \tag{2.15}$$

$$E_1 = \{e_2, e_4, \ldots, e_n\} \cup \{f_2, f_4, \ldots, f_n\} \tag{2.16}$$

$$E_3 = \left\{s_l, t_l : l \in \left\{1, 2, \ldots, \frac{n}{2}\right\}\right\} \tag{2.17}$$

From equation (2.15) to (2.17), the $n$-crossed prism graph $R_n$ is equitable edge colored with 3 colors. $E_1$, $E_2$ and $E_3$ are color classes of $E(R_n)$. Also we observe that these color classes $E_1$, $E_2$ and $E_3$ are independent sets of $R_n$ and its satisfies the inequality $||E_i| - |E_j|| \le 1$, for $i \ne j$. Hence $\chi'_e(R_n) \le 3$. Since $\Delta = 3$ and $\chi'_e(R_n) \ge \chi'_e(R_n) \ge \Delta(R_n) \ge 3$. This implies that $\chi'_e(R_n) = 3$.

## 3. Conclusion

In this paper, we investigated an equitable edge coloring and equitable edge chromatic number for prism graph $D_n$, anti-prism graph $A_n$, and $n$-crossed prism graph $R_n$.

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