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GENERALIZED FIXED POINT THEOREM FOR MULTI-VALUED CONTRACTIVE MAPPING IN CONE *b*-METRIC SPACE

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Abstract

In this paper, we prove some fixed point results for multi-valued contractive mappings in cone b-metric space which is generalization of the fixed point theorems for contractive mappings in cone b-metric space. We conclude examples to support our main results. Hence, our results unify, generalize and complement the comparable results from the current literature.

1. Introduction

A very popular tool to solve existence of fixed point problems is the Banach Contraction Theorem [1] which plays an important role in several branches of mathematics. Bakhtin [2] gave the concept of b-metric spaces and by generalizing the famous Banach contraction principle in metric spaces

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proved the contraction mapping principle in b-metric spaces. Recently, the fixed point in non-convex analysis, especially in an ordered normed space, occupies a prominent place in many aspects (see [3-8]), the author defines an ordering by using a cone, which naturally induces a partial ordering in Banach spaces. Huang and Zhang [7] introduced cone metric spaces as a generalization of metric spaces. Moreover, some fixed point theorems were proved for contractive mappings expanding certain results of fixed points in metric spaces (see [9, 10, 11, 12 and 13]). Later some fixed point theorems for b-metric spaces were given by Xie and Wang [14]. Hussain and Shah [11] introduced cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. Throughout this paper, we have proved a generalization of fixed point theorem for multi-valued contractive mapping in cone b-metric space by using triangular inequality.

II. Preliminaries and Definitions

Definition 2.1 [5]**.** Let E be a real Banach space and P be a subset of E. By θ we denote zero element of E. The subset P of E is called a cone if and only if.

(i) P is closed, nonempty and $P \neq \{\theta\}$;

(ii) $a, b \in R,\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$;

(iii) $P \cup (-P) = \{\theta\}$.

Definition 2.2 [13]**.** Let Z be a nonempty set. Suppose that a mapping $d : Z \times Z \to E$ satisfies.

(i) $\theta \preccurlyeq d(u, v)$ for all $u, v \in Z$ with $u \neq v$ and $d(u, v) = \theta$ if and only if $u = v$;

(ii) $d(u, v) = d(v, u)$ for all $u, v \in Z$;

(iii) $d(u, v) \preccurlyeq [d(u, w) + d(w, v)]$ for all $u, v, w \in Z$.

Then d is called a cone metric on Z and (Z, d) is called a cone metric space.

Definition 2.3 [8]**.** Let Z be a nonempty set and $r \geq 1$ be a given real number. A function $d : Z \times Z \to E$ is said to be a cone b-metric if the following conditions hold.

(i) $\theta \preccurlyeq d(u, v)$ for all $u, v \in Z$ and $d(u, v) = \theta$ if only if $u = v$;

(ii) $d(u, v) = d(v, u)$ for all $u, v \in Z$;

(iii) $d(u, w) \preccurlyeq r[d(u, v) + d(v, w)]$ for all $u, v, w \in Z$.

The pair (Z, d) is called a cone b-metric space.

Definition 2.4 [8]. Let (Z, d) be a cone b-metric space, $z \in Z$ and $\{z_n\}$ be a sequence in Z, then

(i) $\{z_n\}$ converges to z whenever, for every $c \in E$ with $\theta << c$, there is a natural number N such that $d(z_n, z) << c$ for all $n \geq N$. We denote this by $\log_{n \to \infty} Z_n = z$ or $z_n \to z(n \to \infty)$.

(ii) $\{z_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $\theta << c$, there is a natural number N such that $d(z_n, z_m) << c$ for all $n, m \geq N$.

(iii) (Z, d) is a complete cone b-metric space if every Cauchy sequence is convergent.

Lemma 2.5 [8]. *Let (Z, d) be a cone b-metric space. The following properties are often used while dealing with cone b-metric spaces in which the cone is not necessarily normal.*

(i) *If $u \preccurlyeq v$ and $v \preccurlyeq w$, then $u << w$;*

(ii) *If $\theta \preccurlyeq u << c$ for each $c \in \text{int}\, P$, then $u = \theta$;*

(iii) *If $a \preccurlyeq b + c$ for each $c \in \text{int}\, P$, then $a \preccurlyeq b$;*

(iv) *If $\theta \preccurlyeq d(z_n, z) \preccurlyeq z_n$ and $b_n \to \theta$, then $z_n \to z$;*

(v) *If $a \preccurlyeq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$;*

(vi) *If $c \in \text{int}\, P$, $\theta \preccurlyeq a_n$ and $a_n \to \theta$, then exists $n_0 \in N$ such that $a_n << c$ for all $n > n_0$.*

Theorem 2.6. *Let (Z, d) be a complete cone b-metric space with $k \geq 1$ and let $R : Z \to Z$ be a continuous mapping satisfying the contractive* *condition*

$$d(Ru, Rv) \leq \psi_1 \frac{d(u, Ru)d(u, Rv) + (v, Rv)d(v, Ru)}{d(u, Rv) + d(v, Ru)}.$$

For all $u, v \in Z$ *and* $\psi_1 \in [0, 1]$. *Then R has a unique fixed point in Z.*

III. Main Results

Theorem 3.1. *Let* (Z, d) *be a complete cone b-metric space with* $k \geq 1$ *and let* $R : Z \to Z$ *be a continuous mapping satisfying the contractive condition*

$$d(Ru, Rv) \leq \psi_1 \frac{d(u, Ru)d(u, Rv) + d(v, Rv)d(v, Ru)}{d(u, Rv) + d(v, Ru)} + \psi_2 \frac{[d(Ru, v) + d(u, v)][1 + d(v, Rv)]}{1 + d(u, v)} + \psi_3 d(u, v). \quad (1)$$

For all and $u, v \in Z$ then $\psi_1, \psi_2, \psi_3 \in [0, 1]$ with $k(2\psi_1 + \psi_2 + \psi_3) < 1$ then R has a unique fixed point in Z.

Proof. Let z_0 be an arbitrary point in Z. Define a sequence $\{z_n\}$ in Z such that $z_1 = R(z_0), z_1 = R(z_1), \ldots$. Replace u by z_{n-1} and v by z_n in (1), we have

$$\begin{aligned} d(Z_n, Z_{n+1}) &= d(Rz_{n-1}, Rz_n) \\ &\leq \psi_1 \frac{d(z_{n-1}, Rz_{n-1})d(z_{n-1}, Rz_n) + d(z_n, Rz_n)d(z_n, Rz_{n-1})}{d(z_{n-1}, Rz_n) + d(z_n, Rz_{n-1})} \\ &\quad + \psi_2 \frac{[d(Rz_{n-1}, z_n) + d(z_{n-1}, z_n)][1 + d(z_n, Rz_n)]}{1 + d(z_{n-1}, z_n)} \\ &\quad + \psi_3 d(z_{n-1}, z_n) \\ &\leq \psi_1 \frac{d(z_{n-1}, z_n)d(z_{n-1}, z_{n+1}) + d(z_n, z_{n+1})d(z_n, z_n)}{d(z_{n-1}, z_{n+1}) + d(z_n, z_n)} \\ &\quad + \psi_2 \frac{[d(z_n, z_n) + d(z_{n-1}, z_n)][1 + d(z_n, z_{n+1})]}{1 + d(z_{n-1}, z_n)} \\ &\quad + \psi_3 d(z_{n-1}, z_n). \end{aligned}$$

Using triangular inequality

$$\leq \psi_1 \frac{d(z_{n-1}, z_n)k(d(z_{n-1}, z_n) + d(z_n, z_{n+1})) + d(z_n, z_{n+1})k(d(z_{n-1}, z_n) + d(z_n, z_{n+1}))}{k(d(z_{n-1}, z_n) + d(z_n, z_{n+1}))}$$

$$+ \psi_2 \frac{[d(z_n, z_n) + d(z_{n-1}, z_n)][1 + d(z_n, z_{n+1})]}{1 + k(d(z_{n-1}, z_n))} + \psi_3 d(z_{n-1}, z_n)$$

$$\leq \psi_1(d(z_{n-1}, z_n) + d(z_n, z_{n+1}))$$

$$+\psi_2(d(z_n, z_{n-1}) + d(z_{n+1}, z_n))$$

$$+\psi_3 d(z_{n-1}, z_n).$$

Therefore

$$d(z_n, z_{n+1}) \leq \frac{\psi_1 + \psi_2 + \psi_3}{1 - \psi_1} d(z_{n-1}, z_n)$$

$$= hd(z_{n-1}, z_n) \tag{2}$$

Where $h = \dfrac{\psi_1 + \psi_2 + \psi_3}{1 - \psi_1} < 1 \, as \, k(2\psi_1 + \psi_2 + \psi_3) < 1$ we have

$$d(z_{n-1}, z_n) \leq hd(z_{n-2}, z_{n-1}).$$

By (2) we get,

$$d(z_n, z_{n+1}) \leq h^2 d(z_{n-2}, z_{n-1}).$$

Continue this process, we get

$$d(z_n, z_{n+1}) \leq h^n d(z_n, z_0).$$

Since $0 \leq h < 1 \, as \, n \to \infty$, $h^n \to 0$. Thus $\{z_n\}$ is cone b-metric space in Z such that $T(u = \lim T(z_n) = \lim z_{n+1} = u$. Thus u is a fixed point of R.

Uniqueness.

Let $u \in U$ is a fixed point of R then by (1),

$$\begin{aligned} d(u, u) &= d(Ru, Ru) \\ &\le \psi_1 \frac{d(u, u)d(u, u) + d(u, u)d(u, u)}{d(u, u) + d(u, u)} \\ &\quad + \psi_2 \frac{[d(u, u) + d(u, u)][1 + d(u, u)]}{1 + d(u, u)} \\ &\quad + \psi_3 d(u,u) \\ &\le (\psi_1 + \psi_2 + \psi_3)d(u, u). \end{aligned}$$

Which is true only if $d(u, u) = 0,$ since $0 \le k(2\psi_1 + \psi_2 + \psi_3) < 1$ and $d(u, u) \ge 0.$ Thus $d(u, u) \ge 0$ if Z is a fixed point of R then we have,

$$d(u, v) = d(Ru, Rv) \le \psi_2 d(u, v).$$

Which gives $d(u, u) \ge 0,$ since $0 \le \psi_2 < 1$ and $d(u, u) \ge 0.$ Thus fixed point of R is unique.

Example 3.2. Let $Z = [0, 1].$ Define $d : Z \times Z \to \mathbb{R}^+$ by

$$d(u, v) = | u + v |^2 + | u - v |^2.$$

For all $u, v \in Z.$ Define $F(u, v) = \frac{uv}{4}.$

Example 3.3. Let Z and $Z = \{p, q, r, s, t, u\}, E = \mathbb{R}^2$ and $p = \{(u, v) : u, v \ge 0\}$ is a cone in $E.$ Define $d : Z \times X \to E$ as follows

$$d(u, u) \ge 0, \forall z \in Z$$

$$d(p, q) = d(q, p) = (6, 36)$$

$$\begin{aligned} d(p, r) &= d(r, p) = d(r, s) = d(s, r) = d(q, r) = d(r, q) \\ &= d(q, s) = d(s, q) = d(p, t) = d(t, p) \\ &= d(p, u) = d(u, p) = (1, 6) \end{aligned}$$

$$\begin{aligned} d(p, u) &= d(u, p) = d(q, t) = d(t, q) = d(r, u) = d(u, r) \\ &= d(s, u) = d(u, s) = d(t, u) = d(u, t) \\ &= (7, 42). \end{aligned}$$

Then $d(Z, d)$ is a complete cone b-metric space.

IV. Conclusion

Hence in this paper we have proved a fixed point theorem for multi-valued contractive mapping in cone b-metric space by using triangular inequality, which is generalization and extension of the results due to Azam and Mehmood [15], Chu and Huang [5] and Kutbi and Karapinar [16].

V. Future Scope

Fixed point for multivalued contractive mapping in Cone b-metric space is an interesting concept. There is scope to examine the applicability of this space in different branches to study.

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Conflict of Interest

There is no conflict of interest.

References

[1] S. Banach, Sur les operations dans les ensembles abstrait et leur application aux equations, Integrals. Fundam. Math. 3 (1922), 133-181.

[2] I. A. Bakhtin, The contraction mapping principle in atmost metric spaces. Funct. Anal. Gos. Ped. Inst. Unianowsk 30 (1989), 26-37.

[3] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Semin.Mat Fis. Univ. Modena 46, 263-276.

[4] S-H. Cho, J-S. Bae, Common fixed point theorems for mappings satisfying property (E.A.) on cone metric space, Math. Comput. Model. 53 (2011), 945-951.

[5] Z. Chu, X. Huang and X. Liu, New multivalued fixed point results in cone b-metric spaces; J. Nonlinear Sci. Appl. 9 (2016), 3504-3510.

[6] K. Deimling, Nonlinear Functional Analysis. Springer, Berlin, (1985).

[7] L-G. Huang and X. Zhang, Cone metric spaces and fixed point theorem of contractive mappings, J. Math. Anal. Appl. 332(2) (2007), 1468-1476.

[8] N. Hussain and M. H. Shah, KKM mappings in cone b-metric spaces, a survey Nonlinear Anal. 4(7) (2011), 2591-2601.

[9] M. Boriceanu, M. Bota and A. Petrusel Multivalued fractals in b-metric spaces, Cent. Eur. J. Math. 8(2) (2010), 367-377.

[10] M. Bota A. Molnar and V. Csaba, On Ekeland's variational principle in b-metric spaces, Fixed Point Theory 12 (2011), 21-28.

[11] S. Jankovic, Z. Kadelburg and S. Radenovic, On cone metric spaces: a survey, Nonlinear Anal. 4(7) (2011), 2591-2601.

[12] S. Rezapour and R. Hamlbarani, Some notes on the paper 'Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 345 (2008), 719-724.

[13] N. Sharma, Fixed point theorem in cone b-metric space using contractive mappings, Global journal of pure and applied mathematics 13(7) (2017), 2997-3004.

[14] S. Xie, Y. Wang and L. Zhong, Some Fixed Point Theorems In b-metric spaces, Intr. Journal of pure and applied mathematics 10(4) (2016), 571-580.

[15] A. Azam, N. Mehmood, J. Ahmed and S. Radenovic, Multivalued fixed point theorems in cone b-metric spaces, Journal of inequalities and applications (2013), 528-532.

[16] M. A. Kutbi, E. Karapinar, J. Ahmad and A. Azam, Some fixed point results for multivalued mappings in b-metric spaces, Journal of inequalities and applications 126 (2014), 1-11.