Advances and Applications in Mathematical Sciences
Volume 18, Issue 11, September 2019, Pages 1309-1326

HOMOMORPHISM OF FUZZY FILTERS OF LATTICES

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Abstract

The image of a fuzzy filter and the pre-image of a fuzzy filter under homomorphism are also fuzzy filters, are established. Further the effect of a homomorphism on join, meet and intersection of two fuzzy filters is studied.

Introduction

The notion of fuzzy sets was introduced by Lofti A. Zadeh in 1965. Zadeh had initiated fuzzy set theory as a modification of the ordinary set theory. In 1971, Rosenfeld introduced fuzzy sets in the realm of group theory and formulated the concept of a fuzzy subgroup of a group. Since then, researchers in various disciplines of mathematics have been trying to extend their ideas to the broader framework of the fuzzy setting. In 1982, Liu developed the concept of fuzzy subrings as well as fuzzy filters in lattice. Then the literature of these fuzzy algebraic concepts has been developed by many other mathematicians. One of the structures that are most extensively used and discussed in the crisp mathematical theory is certainly the lattice structure. As it is well known, it is considered as a relational, ordered structure, on one hand and as an algebra on the other. The concept of lattice was first defined by Dedekind in 1897, and then developed by Birkhoff G. in 1933. Here we made an attempt to study the algebraic nature of homomorphism of fuzzy filters of a lattice, by using the ideas of fuzzy theory and lattice theory.

2010 Mathematics Subject Classification: 06D72.
Keywords: fuzzy filter, lattice, homomorphism of fuzzy filters.
Received July 16, 2019; Accepted September 23, 2019

1. Preliminaries

Some well-known definitions and preliminary results are recalled.

Definition 1.1. A relation defined on a set S which is reflexive, anti symmetric and transitive is called a partial ordering on S. A set S with a partial ordering ρ defined on it is called a partially ordered set or a poset and is denoted by (S, ρ).

Definition 1.2. Let $(P, \leq)$ be a poset. Let A be a non-empty subset of P. An element $u \in P$ is called an upper bound of A if $a \leq u$ for all $a \in A$. An element $u \in P$ is called the least upper bound $(l.u.b)$ of A if

(i) u is an upper bound of A.

(ii) if v is any other upper bound of A, then $u \leq v$.

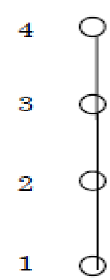
An element $l \in P$ is called the lower bound of A if $l \leq a$ for all $a \leq A$. An element $l \in P$ is called the greatest lower bound $(g.l.b)$ of A if

(i) l is an lower bound of A.

(ii) if m is any other lower bound of A, then $m \leq l$.

Definition 1.3. A lattice is a poset in which any two elements have a $g.l.b$ and a $l.u.b$. We denote the $l.u.b$ of $a \vee b$ and $g.l.b$ by $a \wedge b$.

Example 1.4. Consider the poset $\{1, 2, 3, 4\}$ with the usual $\leq$. Here $1 \leq 2 \leq 3 \leq 4$ and 2 covers 1 and 3 covers 2 and 4 covers 3. Hence we obtain the diagram for this poset. This is also a lattice.



Definition 1.5. Let $(L, \vee, \wedge)$ be a lattice. A non-empty subset S of L is called a filter of L if it satisfies the following conditions:

(i) $x, y \in S \Rightarrow x \wedge y \in S$.

(ii) $x \in S$ and $r \in L$ with $r \geq x \Rightarrow r \in S$.

Example 1.6. Now $(L = \{a, b, c, d\}, \vee, \wedge)$ is a ring under the operations $\vee$ and $\wedge$ defined by the following tables:

$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	d	d
c	c	b	c	d
d	d	d	d	d

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

Example 1.7. $(S = \{m, n\}, \vee_1, \wedge_1)$ is a ring, under the operations $*$ and $\bullet$ defined by the following table:

$\vee_1$	m	n
m	m	n
n	n	n

$\wedge_1$	m	n
m	m	m
n	m	n

Definition 1.8. Let X be a non-empty set. A mapping $\mu : X \to [0, 1]$ is called a fuzzy subset of X.

Definition 1.9. Let μ be any fuzzy subset of a set X and $\mu = \{(x_i, t_i)/i = 1$ to n and $t_i \in [0, 1]\}$. Then, $\{t_i/i = 1$ to $n\}$ is called the image set of μ and is denoted by $\operatorname{Im} \mu$.

Definition 1.10. Let μ be any fuzzy subset of a set X and $t \in \operatorname{Im} \mu$. Then the set $\mu_t = \{x \in X/\mu(x) \geq t\}$ is called the level subset of μ. Clearly, $\mu_t \subseteq \mu_s$ whenever $t > s$.

Definition 1.11. A fuzzy subset μ of a lattice L is called a fuzzy sublattice of L, if the following conditions are satisfied:

For all $x, y \in L$,

(i) $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\}$.

Definition 1.12. A fuzzy subset μ of a lattice L is called a fuzzy lattice filter or fuzzy filter of L if, for all $x, y \in L$ the following conditions are satisfied:

(i) $\mu(x \vee y) \geq \max\{\mu(x), \mu(y)\}$

(ii) $\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\}$.

Theorem 1.13 [Characterization Theorem]**.** *A fuzzy subset* μ *of a lattice* L, *is a fuzzy filter of L if and only if*, *the level subsets* μ_t, $t \in \operatorname{Im}\mu$ *are filters of* L.

Definition 1.14. Let L and L, be two lattices. A function $x, y \in L$, is called an homomorphism if, for all $x, y \in L$,

(i) $f(x + y) = f(x) \vee f(y)$

(ii) $f(x \wedge y) = f(x) \wedge f(y)$.

Example 1.15. Now consider the lattice L defined in Example 1.6 and the lattice S defined in Example 1.7. Consider the function $f : L \to S$, defined by $f(a) = m$; $f(b) = n$; $f(c) = m$ and $f(d) = n$. Then, f is an onto homomorphism.

Remark 1.16. Let f be any function from a lattice L to a lattice L'. Then $x \in f^{-1}[f(x)]$ but in general, $x \neq f^{-1}[f(x)]$ where $x \in L$.

Proof. By example, Consider Example 1.15. Here, let $x = a \in L$. Then $x \Rightarrow f^{-1}[f(x)] = f^{-1}[m] = \{a, c\}$. Then $x \in f^{-1}[f(x)]$ and $x \neq f^{-1}[f(x)]$,

Remark 1.17. Let f be any function from a lattice L onto a lattice L'. Then $y = f[f^{-1}(y)]$, where $y \in L'$.

Proof. By example, consider Example 1.15. Here, let $y = m$ and $f^{-1}(y) = f^{-1}(m) = \{a, c\} \Rightarrow f[f^{-1}(y)] = f[\{a, c\}] = m.$ Hence $y = f[f^{-1}(y)]$.

Remark 1.18. A mapping $f : L \to L'$ is called as an isotone mapping if a $a \leq b \Rightarrow f(a) \leq f(b)$ for all $a, b \in L$.

2. Homomorphism of Fuzzy Filters

Here we define the image and pre-image of fuzzy filters and showed that they are also fuzzy filters. Some of their properties are also studied.

Definition 2.1. Let *f* be any function from a lattice L to a lattice L'; μ be any fuzzy subset of L and σ be any fuzzy subset of L'.

The image of μ under *f*, denoted by $f(\mu)$, is a fuzzy subset of L', defined by, whenever, $y \in L'$,

$$[f(\mu)](y) = \begin{cases} \underset{x \in f^{-1}(y)}{Sup} \; \mu(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

The preimage of σ under *f*, symbolized by $f^{-1}(\sigma)$ is a fuzzy subset of L defined by

$$[f^{-1}(\sigma)](x) = \sigma[f(x)], \text{ for all } x \in L.$$

Definition 2.2. Let *f* be any function from a lattice L to a lattice L' and let μ be any fuzzy subset of L. Then μ is called *f*-invariant if for all $x, y \in L$,

$$f(x) = f(y) \Rightarrow \mu x = \mu(y).$$

Example 2.3. Consider Example 1.15.

Let μ and θ be two fuzzy filters of L, where μ and θ are defined by,

$$\mu(x) = \begin{cases} 0.3 & \text{if } x \in \{a, c\} \\ 0.7 & \text{if } x \in \{b, d\} \end{cases}$$

and

$$\sigma(x) = \begin{cases} 0.4 & \text{if } x \in \{a, b\} \\ 0.4 & \text{if } x \in \{c, d\}. \end{cases}$$

Here $f(a) = f(c) = n$ and $f(b) = f(c) = m$. Then μ is f-invariant, since $\mu(a) = \mu(c) = 0.3$ and $\mu(b) = \mu(d) = 0.7$ and σ is not f-invariant, since $0.4 = \sigma(a) \neq \sigma(c) = 0.2$.

Lemma 2.4. *Let f be any function from a lattice L to a lattice L'; μ, θ be any two fuzzy subsets of L and μ, θ be any two fuzzy subsets of L'. Then the following are true*:

(i) $f[f^{-1}(\mu')] = \mu'$

(ii) $\mu \subseteq f^{-1}[f(\mu)]$

(iii) $f^{-1}[f(\mu)] = \mu$ provided that μ is f-invariant

(iv) $\mu \subseteq \theta \Rightarrow f(\mu) \subseteq f(\theta)$

(v) $\mu' \subseteq \theta' \Rightarrow f^{-1}(\mu') \subseteq f^{-1}(\theta')$.

Proof. (i) Let $y \in L'$ be arbitrary.

Then,

$$[f[f^{-1}(\mu')]](y) = \sup_{x \in f^{-1}(y)} \{[f^{-1}(\mu')](x)\} = \sup_{x \in f^{-1}(y)} \{\mu'[f(x)]\}$$

$$= \mu'(f[f^{-1}(y)]) = \mu'(y).$$

Thus $[f[f^{-1}(\mu')]](y) = \mu'(y)$, for all $y \in L'$. Hence $f[f^{-1}(\mu')] = \mu'$.

(2) Let $x \in L$ be arbitrary.

Then,

$$\{f^{-1}[f(\mu')]\}(x)[f(\mu)](f(x)) = \sup_{x \in f^{-1}[f(x)]} \{\mu(z)\}. \tag{1}$$

Here $f^{-1}[f(x)] \neq x$; But $x \in f^{-1}[f(x)]$.

Hence (1) $\Rightarrow \{f^{-1}[f(\mu)]\}(x) \geq \mu(x)$, for all $x \in L$.

$\Rightarrow \mu \Rightarrow f^{-1}[f(\mu)]$.

(iii) Let $x \in L$ be arbitrary. Then,

$$\{f^{-1}[f(\mu)]\}(x) = [f(\mu)](f(x)) = \sup_{z \in f^{-1}[f(x)]} \{\mu(z)\}$$

$$= \sup_{z \in \{x\}} \{\mu(z)\}, \ \mu \text{ is } f\text{-invariant}$$

$$= \mu(x).$$

Thus $[f^{-1}[f(\mu)](x) = \mu(x)$, for all $x \in L$.

$$\Rightarrow f^{-1}[f(\mu)] = \mu.$$

(iv) Assume that $\mu \subseteq \theta$

$$\Rightarrow \mu(x) \leq \theta(x), \text{ for all } x \in L.$$

Then,

$$[f(\mu)](y) = \underset{x \in f^{-1}(y)}{Sup} [\mu(x)], \text{ where } y \in L'$$

$$\leq \underset{x \in f^{-1}(y)}{Sup} [\theta(x)]$$

$$= [f(\theta)](y)$$

$$\Rightarrow [f(\mu)](y) \leq [f(\theta)](y), \ \forall y \in L'$$

$$\Rightarrow f(\mu) \subseteq f(\theta).$$

(v) Assume that $\mu' \subseteq \theta$

$$\Rightarrow \mu'(y) \leq \theta'(y), \text{ for all } y \in L'.$$

Now,

$$[f^{-1}(\mu')](x) = \mu'[f(\mu)]$$

$$\leq \theta'[f(x)]$$

$$f^{-1}[\theta'(x)].$$

Thus, $[f^{-1}(\mu')](x) \leq [f^{-1}(\theta')](x)$, for all $x \in L$.

$$\Rightarrow f^{-1}(\mu') \subseteq f^{-1}(\theta').$$

Theorem 2.5. *If f is a homomorphism from a lattice L onto a lattice* L', *then for each fuzzy filter* μ *of* L, $f(\mu)$ *is a fuzzy filter of* L', *and for each fuzzy filter* μ' *of* L', $f^{-1}(\mu')$ *is a fuzzy filter of L.*

Proof. Let f be an homomorphism from a lattice L onto a lattice L'.

Assume that μ is a fuzzy filter of L and μ' is a fuzzy filter of L'.

To prove:

(i) $f(\mu)$ is a fuzzy filter of L'.

(ii) $f^{-1}(\mu')$ is a fuzzy filter of L.

For (i). It is enough to prove $[f(\mu)]_s$ is an filter of L', for all $s \in \operatorname{Im} f(\mu)$.

Let $t \in \operatorname{Im} f(\mu)$ be arbitrary.

Then for some $y \in L'$, we have,

$[f(\mu)](y) = \sup\limits_{x \in f^{-1}(y)} \mu(x) = t \leq \mu(1)$, since $\mu(x) \leq \mu(1)$ for all $x \in L$.

Now, if $t = 0$, then $[f(\mu)]_t = L'$.

Assume that $t > 0$.

Let $\varepsilon > 0$ be any real number such that $\alpha = t - \varepsilon$.

Let $z \in [f(\mu)]_t$ be arbitrary.

$$\Rightarrow [f(\mu)](z) \geq t$$

$$\Rightarrow \sup_{y \in f^{-1}(z)} \mu(y) \geq \alpha$$

$$\Rightarrow \mu(y) \geq \alpha, \text{ for some } y \in f^{-1}(z)$$

$$\Rightarrow y \in \mu_\alpha$$

$$\Rightarrow f(y) \in f(\mu_\alpha)$$

$$\Rightarrow z \in f(\mu_\alpha).$$

Hence $[f(\mu)]_t \subseteq f(\mu_\alpha)$. (1)

And let, $z \in f(\mu_\alpha)$ be arbitrary.

$\Rightarrow z = f(y)$ for some y such that $\mu(y) \geq \alpha$.

$\Rightarrow [f(\mu)](z) \geq \alpha$, since $y \in f^{-1}(z)$

$\Rightarrow [f(\mu)](z) \geq t$, since $\varepsilon > 0$ be arbitrary.

$\Rightarrow z \in [f(\mu)]_t$.

Hence $f(\mu_\alpha) \subseteq [f(\mu)]_t$ (2)

From (1) and (2), we get $[f(\mu)]_t = f(\mu_\alpha)$ (3)

Clearly $\alpha \leq \mu(0)$

$\Rightarrow \mu_\alpha$ is an filter of L.

$\Rightarrow f(\mu_\alpha)$ is an filter of L'.

Now (3) $\Rightarrow [f(\mu)]_t$ is an filter of L'.

$\Rightarrow f(\mu)$ is a fuzzy filter of L'.

Thus $f(\mu)$ is a fuzzy filter of L'.

For (ii).

To prove $f^{-1}(\mu')$ is a fuzzy filter of L.

Here

$$f^{-1}(\mu'(x-y)) = \mu'[f(x-y)] = \mu'[f(x) - f(y)] \geq \min\{\mu'[f(x)], \mu'[f(y)]\}$$

$$= \min\{f^{-1}[\mu'(x)], f^{-1}[\mu'(y)]\}$$

and

$$f^{-1}(\mu'(xy)) = \mu'[f(xy)] = \mu'[f(x)f(y)] \geq \min\{\mu'[f(x)], \mu'[f(y)]\}$$

$$= \min\{f^{-1}[\mu'(x)], f^{-1}[\mu'(y)]\}.$$

Hence $f^{-1}(\mu')$ is a fuzzy filter of L.

Theorem 2.6. *Let f be a homomorphism from a lattice L onto a lattice* L'. *If* μ *and* σ *are fuzzy filters of L, then the following are true*:

(i) $f(\mu \vee \sigma) = f(\mu) \vee f(\sigma)$

(ii) $f(\mu \wedge \sigma) = f(\mu) \wedge f(\sigma)$

(iii) $f(\mu \cap \sigma) = f(\mu) \cap f(\sigma)$.

Proof. Let $y \in L'$ and let $\varepsilon > 0$ be given.

(i) Let $\alpha = (f(\mu \vee \sigma))(y)$ and $\beta = (f(\mu) \vee f(\sigma))(y)$.

Now $\alpha - \varepsilon < \underset{x \in f^{-1}(y)}{Sup} (\mu \vee \sigma)(x)$.

$\Rightarrow \alpha - \varepsilon < (\mu \vee \sigma)(x_0)$ for some $x_0 \in L$ such that $f(x_0) = y$.

$= \underset{x_0 = a+b}{Sup} (\min\{\mu(a), \sigma(b)\})$, where $a, b \in L$.

$\Rightarrow \alpha - \varepsilon < \min\{\mu(a_0), \sigma(b_0)\}$, for some $a_0, b_0 \in L$ such that $x_0 = a_0 \vee b_0$.

Then $\beta = \underset{y = y_1 + y_2}{Sup} (\min\{f(\mu))(y_1), f(\sigma)(y_2)\})$ (1), where $y_1, y_2 \in L'$.

$\Rightarrow \beta \geq \min\{(f(\mu))(f(a_0)), (f(b_0))\}$, since $y = f(x_0) = f(a_0) \vee f(b_0)$

$\Rightarrow \beta = \min\{(f^{-1}(f(\mu))(a_0), f^{-1}(f(\sigma))(b_0)$

$\geq \min\{\mu(a_0), \sigma(b_0)\}$, by Lemma 2.4

$> \alpha - \varepsilon$

$\Rightarrow \beta \geq \alpha$, since $\varepsilon > 0$ is arbitrary (2)

Now (1) $\Rightarrow \beta - \varepsilon < \underset{y = y_1 + y_2}{Sup} (\min \{(f(\mu)(y_1), (f(\sigma)(y_2)\})$, where $y_1, y_2 \in L'$.

$\Rightarrow \beta - \varepsilon < (f(\mu))(y_1)$ and $\beta - \varepsilon < (f(\sigma))(y_2)$, for some $y_1, y_2 \in L$ such that $y = y_1 \vee y_2$.

$\Rightarrow \beta - \varepsilon < \mu(x_1)$ and $\beta - \varepsilon < \sigma(x_2)$, for some $x_1, x_2 \in L$, such that $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$ by Definition 2.1

$\Rightarrow \beta - \varepsilon < \min \{\mu(x_1), \sigma(x_2)\}$

$\leq (\mu \vee \sigma)(x_1 \vee x_2)$, by Proposition 4.8

$\leq \underset{x \in f^{-1}(y)}{Sup} ((\mu \vee \sigma)(x))$, since $x_1 \vee x_2 \in f^{-1}(y)$

$= (f(\mu \vee \sigma))(y) = \alpha$.

Hence $\beta \leq \alpha$ (3)

From (2) and (3), we have, $\alpha = \beta$.

$\Rightarrow [f(\mu \vee \sigma)](y) = (f(\mu) \vee f(\sigma))(y)$, for all $y \in L'$.

$\Rightarrow f(\mu \vee \sigma) = f(\mu) \vee f(\sigma)$.

(ii) Let $\alpha = [f(\mu \wedge \sigma)](y)$ and $\beta = (f(\mu) \wedge f(\sigma))(y)$.

Now $\alpha - \varepsilon < \underset{z \in f^{-1}(y)}{Sup} (\mu \wedge \sigma)(z)$

$< (\mu \wedge \sigma)(x)$ for some $x \in f^{-1}(y)$

$\Rightarrow \alpha - \varepsilon < \min \{\mu(x_1), \sigma(x_2)\}$ for some $x_1, x_2 \in L$ such that $x = x_1 \wedge x_2$

$\leq \min \{(f^{-1}(f(\mu)))(x_1), (f^{-1}(f(\sigma)))(x_2)\}$, by Lemma 2.4

$= \min \{(f(\mu)) \wedge (f(x_1)(f(\sigma)) \wedge (f(x_2))\}$

$\leq (f(\mu) \wedge f(\sigma))(f(x_1), f(x_2))$

$\Rightarrow \alpha - \varepsilon(f(\mu) \wedge f(\sigma))(f(x))$, since f is an homomorphism

$= \beta$

$\Rightarrow \alpha \leq \beta. (4)$

Again $\beta - \varepsilon < (f(\mu) \wedge f(\sigma))(y)$

$= \underset{y = y_1 y_2}{Sup} (\min \{(f(\mu))(y_1), (f(\sigma))(y_2)\})$, where $y_1, y_2 \in L'$.

$$= \underset{y = y_1 y_2}{Sup} \left(\min \left\{ \underset{z \in f^{-1}(y_1)}{Sup} \mu(z), \underset{z \in f^{-1}(y_2)}{Sup} \sigma(z) \right\} \right)$$

$\Rightarrow \beta - \varepsilon < \min \left\{ \underset{z \in f^{-1}(y_1)}{Sup} \mu(z), \underset{z \in f^{-1}(y_2)}{Sup} \sigma(z) \right\}$, for some $y_1, y_2 \in L'$ such that $y = y_1 \wedge y_2$

$\Rightarrow \beta - \varepsilon \min \{\mu(x_1), \sigma(x_2)\}$ for some $x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)$

$\leq (\mu \wedge \sigma)(x_1 \wedge x_2)$

$\Rightarrow \beta - \varepsilon \leq \underset{x \in f^{-1}(y_1)}{Sup} ((\mu \wedge \sigma)(x))$, since $y = y_1 y_2 = f(x_1 x_2)$

$= (f(\mu \wedge \sigma)(y) = \alpha.$

Hence $\beta \leq \alpha$. (5)

From (4) and (5), we have $\alpha = \beta$.

$$\Rightarrow [f(\mu \wedge \sigma)](y) = (f(\mu) \wedge f(\sigma))(y)$$

$$\Rightarrow f(\mu \wedge \sigma) = f(\mu) \wedge f(\sigma).$$

Clearly $\mu \cap \sigma \subseteq \mu$ and $\mu \cap \sigma \subseteq \sigma$.

Then by (iv) of Lemma 2.4, we have,

$f(\mu \cap \sigma) \subseteq f(\mu)$ and $f(\mu \cap \sigma) \subseteq f(\sigma)$

$$\Rightarrow f(\mu \cap \sigma) \subseteq f(\mu) \cap f(\sigma). (11)$$

It is enough to prove $f(\mu) \cap f(\sigma) \subseteq f(\mu \cap \sigma)$.

Let $y \in L'$ be arbitrary.

Now, $[f(\mu \cap \sigma)](y) = \underset{x \in f^{-1}(y)}{Sup} (\mu \cap \sigma)(x)$

$(\mu \cap \sigma)(u)$ (12), for some $u \in L$ such that $u \in f^{-1}(y)$.

and $[f(\mu) \cap f(\sigma)](y) = \min \{(f(\mu))(y), (f(\sigma))(y)\}$

$$= \min \left\{ \underset{x \in f^{-1}(y)}{Sup} \mu(x), \underset{x \in f^{-1}(y)}{Sup} \sigma(x) \right\}$$

$$= \min \{\mu(r), \sigma(s)\}, (13),$$

where $\mu(r) = \underset{x \in f^{-1}(y)}{Sup} \mu(x)$, for some $r \in f^{-1}(y)$ and $\sigma(s) = \underset{x \in f^{-1}(y)}{Sup} \sigma(x)$, for some $s \in f^{-1}(y)$.

$\leq (\mu \cap \sigma)(u)$, since $\underset{x \in f^{-1}(y)}{Sup} (\mu \cap \sigma)(x) = (\mu \cap \sigma)(u)$.

$= [f(\mu \cap \sigma)](y)$, by (12)

$\Rightarrow [f(\mu) \cap f(\sigma)](y) \leq [f(\mu \cap \sigma)](y)$, by (13)

$\Rightarrow [f(\mu) \cap f(\sigma)] \subseteq [f(\mu \cap \sigma)]$

Thus, $[f(\mu) \cap f(\sigma)] = [f(\mu \cap \sigma)]$.

Remark 2.7. Let f be a homomorphism from a lattice L onto a lattice L'. If μ and σ are fuzzy filters of L, then $[f(\mu \cap \sigma)] \subseteq (y) \leq [f(\mu) \cap f(\sigma)]$, with equality if at least one of μ or σ is f-invariant. But here we have $[f(\mu) \cap f(\sigma)] = [f(\mu \cap \sigma)]$ even if both μ and σ are not f-invariant. This is possible in homomorphism because homomorphism is an isotone mapping.

Theorem 2.8. *Let f be a homomorphism from a lattice L onto a lattice L'. If μ' and θ' are any two fuzzy filters of L' then*

(i) $[f^{-1}(\mu')] \vee [f^{-1}(\theta')] = f^{-1}(\mu' \vee \theta')$.

(ii) $[f^{-1}(\mu')] \wedge [f^{-1}(\theta')] = f^{-1}(\mu' \wedge \theta')$.

Proof. For (i).

Let $x \in L$ and $\varepsilon > 0$.

Let $\alpha = [f^{-1}(\mu') \vee f^{-1}(\theta')](x)$ and $\beta = [f^{-1}(\mu' \vee \theta')](x)$

$\alpha - \varepsilon < \underset{x=x_1+x_2}{Sup} (\min \{(f^{-1}(\mu')(x_1), (f^{-1}(\theta')(x_2)\}),$ where $x_1, x_2 \in L$

$= \underset{x=x_1+x_2}{Sup} (\min \{\mu'(f(x_1)), (f^{-1}(x_2))\})$

$\Rightarrow \alpha - \varepsilon < \min \{\mu'(f^{-1}(a_1)), \theta'(f(a_2))\})$ for some $a_1, a_2 \in L$ such that $x = a_1 \vee a_2$.

$\leq (\mu' \vee \theta')(f(a_1 \vee a_2)) = (f^{-1}(\mu' \vee \theta'))(x) = \beta$.

$\Rightarrow \alpha \leq \beta$, since $\varepsilon > 0$ is arbitrary.

$\Rightarrow [f^{-1}(\mu') \vee f^{-1}(\theta')] \leq [f^{-1}(\mu' \vee \theta')](x)$

$\Rightarrow [f^{-1}(\mu')] \vee [f^{-1}(\theta')] \subseteq f^{-1}(\mu' \vee \theta')$ (1)

Now, $\beta = [f^{-1}(\mu' \vee \theta')](x)$

$\Rightarrow \beta - \varepsilon < (\mu' \vee \theta')[f(x)]$

$= \underset{f(x)=y+z}{Sup} \{\min \{\mu'(y), \theta'(z)\}\}$ where $f(x), y, z \in R'$

$\Rightarrow \beta - \varepsilon < \mu'(r)$ for some $r, s \in L'$ such that $f(x) = r \vee s$

$\Rightarrow \beta - \varepsilon < \mu'(r)$ and $\beta - \varepsilon < \theta'(s)$

$\Rightarrow \beta - \varepsilon < \mu'[f(x_1)]$ and $\beta - \varepsilon < \theta'[f(x_2)]$ for some $x_1, x_2 \in L$ such that $x_1 \in f^{-1}(r)$ and $x_2 \in f^{-1}(s)$

$\Rightarrow \beta - \varepsilon < \mu'[f(x_1)]$ and $\Rightarrow \beta - \varepsilon < \mu'[f(x_1)]$

$\Rightarrow \beta - \varepsilon < \min \{[f^{-1}\mu'](x_1)], [f^{-1}\theta'](x_2)\}$. (2)

Now we prove $x_1 \vee x_2 = x$.

Suppose $x_1 \vee x_2 \neq x.$

$\Rightarrow (x_1 \vee x_2) \neq f(x_1) \vee f(x_2)$ since f is an isotone mapping.

And $(x_1 \vee x_2) = f(x_1) \vee f(x_2),$ since f is homomorphism.

$= r \vee s.$

Hence $f(x) \neq r \vee s.$

This proves that $x_1 \vee x_2 = x.$

From (2), we have, $\Rightarrow (x_1 \vee x_2) \neq f(x_1) \vee f(x_2)$ with x1∨ x2 = x

$$\leq \underset{x=u+v}{Sup} \{\min \{f^{-1}[\mu'(u)],\ f^{-1}[\theta'(v)]\}\}$$

$$= [f^{-1}(\mu') \vee f^{-1}(\sigma')](x) = \alpha.$$

$\Rightarrow \beta \leq \alpha,$ since $\varepsilon > 0$ is arbitrary.

$$\Rightarrow [f^{-1}(\mu' \vee \theta')](x) \leq [f^{-1}(\mu') \vee f^{-1}(\theta')](x)$$

$$\Rightarrow [f^{-1}(\mu' \vee \theta')] \subseteq [f^{-1}(\mu') \vee f^{-1}(\theta')]. \tag{3}$$

From (1) and (3), we have, $f^{-1}(\mu') \vee f^{-1}(\theta') = f^{-1}(\mu' \vee \theta').$

For (ii).

Let $x \in L$ and $\varepsilon > 0.$

Let $\alpha = [f^{-1}(\mu') \wedge f^{-1}(\theta')](x)$ and $\beta = f^{-1}(\mu' \wedge \theta')(x)$

$\alpha - \varepsilon < \underset{x=x_1x_2}{Sup} (\min \{f^{-1}(\mu')(x_1),\ (f^{-1}(\theta')(x_2)\}),$ where $x_1,\ x_2 \in L$

$$= \underset{x=x_1x_2}{Sup} (\min \{\mu'(f(x_1)),\ \theta'(f(x_2))\})$$

$\Rightarrow \alpha - \varepsilon < \min \{\mu'(f(a_1)),\ \theta'(f(a_2))\}$ for some $a_1,\ a_2 \in L$ such that $x = a_1 \wedge a_2.$

$\leq (\mu' \wedge \theta')(f(a_1 \wedge a_2))$

$= (f^{-1}(\mu' \wedge \theta'))(x) = \beta.$

$\Rightarrow \alpha \leq \beta$, since $\varepsilon > 0$ is arbitrary.

$\Rightarrow [f^{-1}(\mu') \wedge f^{-1}(\theta')](x) \leq f^{-1}(\mu' \wedge \theta')(x).$

Hence, $[f^{-1}(\mu')] \wedge [f^{-1}(\theta')] \subseteq f^{-1}(\mu' \wedge \theta')$ (4).

Now, $\beta = [f^{-1}(\mu' \wedge \theta')](x)$

$\Rightarrow \beta - \varepsilon < (\mu' \wedge \theta')[f(x)]$

$= \underset{f(x)=y,\, z}{Sup} \{\min\{\mu'(y), \theta'(z)\}\}$ where $f(x), y, z \in R'$

$= \{\min\{\mu'(r), \theta'(s)\}\}$, for some $r, s \in L'$ such that $f(x) = r \wedge s$

$\Rightarrow \beta - \varepsilon < \mu'(r)$ and $\beta - \varepsilon < \theta'(s)$

$\Rightarrow \beta - \varepsilon < \mu'[f(x_1)]$ and $\beta - \varepsilon < \theta'[f(x_2)]$ for some $x_1, x_2 \in L$ such that $x_1 \in f^{-1}(r)$ and $x_2 \in f^{-1}(s)$

$\Rightarrow \beta - \varepsilon < [f^{-1}\mu'](x_1)$ and $\beta - \varepsilon < [f^{-1}\theta'](x_2)$

$\Rightarrow \beta - \varepsilon < \min\{[f^{-1}\mu'](x_1)], [f^{-1}\theta'](x_2)\}$ (5)

Now we prove $x_1 \wedge x_2 = x$.

Suppose $x_1 \wedge x_2 \neq x$. Then $f(x_1 \wedge x_2) \neq f(x)$, since f is an isotone mapping.

And $f(x_1 \wedge x_2) = f(x_1) \wedge f(x_2)$, since f is homomorphism.

$= r \wedge s.$

Hence $f(x) \neq r \wedge s$. This proves that $x_1 \wedge x_2 = x$.

From (5), we have, $\beta - \varepsilon < \min\{[f^{-1}\mu'](x_1)], [f^{-1}\theta'](x_2)\}$ with $x_1 \wedge x_2 = x$.

$$\leq \underset{x=u\cdot v}{Sup}\{\min\{f^{-1}[\mu'(u)],\ f^{-1}[\theta'(v)]\}\}$$

$$= [f^{-1}(\mu') \wedge f^{-1}(\sigma')](x)$$

$$= \alpha.$$

$\Rightarrow \beta \leq \alpha$, since $\varepsilon > 0$ is arbitrary.

$\Rightarrow [f^{-1}(\mu' \wedge \theta')](x) \leq [f^{-1}(\mu') \wedge f^{-1}(\theta')](x)$

$\Rightarrow [f^{-1}(\mu' \wedge \theta')] \subseteq [f^{-1}(\mu') \wedge f^{-1}(\theta')]$. (6)

From (4) and (6), we have, $f^{-1}(\mu') \wedge f^{-1}(\theta') = f^{-1}(\mu' \wedge \theta')$.

Remark 2.9. Let f be a homomorphism from a lattice L onto a lattice L'. If μ' and θ' are any two fuzzy filters of L' then the above equalities are replaced by '$\subseteq$'. These equalities are possible here because of the isotonicity of f.

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