Advances and Applications in Mathematical Sciences
Volume 21, Issue 8, June 2022, Pages 4661-4670

AN INITIAL VALUE PROBLEM FOR A SYSTEM OF 'N' SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS WITH ROBIN INITIAL CONDITIONS

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Abstract

In this paper, an initial value problem for a system of 'n' singularly perturbed delay differential equations with Robin initial conditions is considered. A Shishkin piecewise uniform mesh is implemented and combined with a classical finite difference method to create a numerical method for solving this problem. The numerical approximations obtained are essentially first order convergence uniformly with respect to the singular perturbation parameters. Numerical result is provided in support of the theory.

1. Introduction

In this section, the initial value problem for a linear system of n singularly perturbed first order delay differential equations with Robin initial conditions is considered

$$\vec{L}\vec{u}(x) = E\vec{u}'(x) + A(x)\vec{u}(x) + B(x)\vec{u}(x-1) = f(x),\ x \in (0, 2] \tag{1.1}$$

$$\beta\vec{u}(x) = \vec{u}(x) - E\vec{u}'(x) = l(x) \text{ where } x \in \Omega^* = [-1, 0], \tag{1.2}$$

For all $x \in [0, 2]$, $\vec{u}(x) = (u_1(x), u_2(x), \ldots, u_n(x))^T$ and $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T$. E, $A(x)$ and $B(x)$ are $n \times n$ matrices. $E = diag(\vec{\varepsilon})$, $\vec{\varepsilon} =$

2020 Mathematics Subject Classification: 65D25, 34A12, 34D15, 34D10.

Keywords: Singularly perturbed delay differential equations, Initial Value Problem, Robin initial condition, Finite difference schemes, Shishkin mesh, Parameter uniform convergence.

Received March 24, 2022; Accepted April 3, 2022

$(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ with $0 < \varepsilon_1 \le \varepsilon_2, \ldots, \le \varepsilon_n < 1$, $B(x) = diag(\vec{b})$, $\vec{b} = (b_1(x), b_2(x), \ldots, b_n(x))$. For all $x \in [0, 2]$, the components $a_{ij}(x)$ and $b_i(x)$ of $A(x)$ and $B(x)$ respectively, satisfy

$$b_i(x),\ a_{ij}(x) \le 0 \text{ for } 1 \le i \ne j \le n \text{ and } a_{ii}(x) > \sum_{i \ne j} | a_{ij}(x) + b_i(x) | \quad (1.3)$$

$$\text{and } 0 < \alpha < \min_{x \in [0, 2]} \left(\sum_{j=1}^{n} a_{ij}(x) + b_i(x)\right) \quad (1.4)$$

Further, the functions $f_i(x), a_{ij}(x), b_i(x), 1 \le i, j \le n$ are assumed to be in $C^2([0, 2])$. Based on the foregoing assumptions, $\vec{u} \in C^2$ where $C = C^0([-1, 2]) \cap C^1([0, 2]) \cap C^2((0, 1) \cup (1, 2])$. The problems (1) and (2) can be rephrased as follows:

$$\vec{L}\vec{u}(x) = \begin{cases} \vec{L}_1\vec{u}(x) = E\vec{u}'(x) + A(x)\vec{u}(x) = \vec{g}(x) \text{ on } (0, 1] \\ \vec{L}_2\vec{u}(x) = E\vec{u}'(x) + A(x)\vec{u}(x) + B(x)\vec{u}(x-1) = \vec{f}(x) \text{ on } (0, 2] \end{cases} \quad (1.5)$$

where $g(x) = f(x) - B(x)\vec{\phi}(x-1)$ and $\vec{u}(0) = \vec{\phi}(0)$. The reduced problem corresponding to (1.5) is given by

$$\begin{cases} A(x)\vec{u}_0(x) = \vec{f}(x) - B(x)\vec{\phi}(x-1) \text{ on } (0, 1] \\ A(x)\vec{u}_0(x) + B(x)\vec{u}_0(x)(x-1) = \vec{f}(x) \text{ on } (0, 2] \end{cases}$$

2. Analytical Results

Lemma 2.1 Maximum Principle. *Let* $\vec{\zeta}$ *be any function in the domain of* $\vec{L}$ *such that* $\vec{\beta}\vec{\zeta}(0) \ge \vec{0}$, *then* $\vec{L}\vec{\zeta}(x) \ge \vec{0}$ *on* $(0, 2]$ *implies that* $\vec{\zeta}(x) \ge \widetilde{0}$ *on* $[0, 2]$.

Lemma 2.2. *Let* $\vec{\zeta}$ *is any function in the domain of* $\vec{L}$, *such that for each* $x \in [0, 2]$, *then*

$$| \vec{\zeta}(x) | \le C \max\left\{\| \vec{\beta}\vec{\zeta}(0) \|, \frac{1}{\alpha} \| \vec{L}\vec{\zeta} \|\right\}.$$

Lemma 2.3. *Let* $\vec{u}$ *be the solution of* (1.1), (1.2). *Then, there exists a* *constant C such that, for* $i = 1, 2, \ldots, n,\ x \in (0, 2]$, *we have*

$$|u_i(x)| \le C\{\|\vec{l}\| + \|\vec{f}\|\},\ |u_i'(x)| \le C\varepsilon_i^{-1}\{\|\vec{l}\| + \|\vec{f}\|\},\ |u_i''(x)|$$

$$\le C\varepsilon_i^{-2}\{\|\vec{l}\| + \|\vec{f}\| + \|\vec{f}'\|\}.$$

Proof. The proof is similar to [8].

3. Bounds on the Solution and Its Derivatives

A Shishkin decomposition of $\vec{u}$ is given by $\vec{u} = \vec{v} + \vec{w}$, where $\vec{v} = (v_1, v_2, \ldots, v_n)^T$ is the solution of

$$\vec{L}_1\vec{v}(x) = E\vec{v}'(x) + A(x)\vec{v}(x) = \vec{g}(x) \text{ on } (0, 1] \tag{1.6}$$

$$\vec{L}_2\vec{v}(x) = E\vec{v}'(x) + A(x)\vec{v}(x) + B(x)\vec{v}(x-1) = \vec{f}(x) \text{ on } (0, 2] \tag{1.7}$$

with $\vec{\beta}\vec{v}(0) = \vec{u}_0(0) - E\vec{u}_0'(0)$ and $\vec{w} = (w_1, w_2, \ldots, w_n)^T$ is the solution of

$$\vec{L}_1\vec{w}(x) = E\vec{w}'(x) + A(x)\vec{w}(x) = \vec{0} \text{ for } x \in (0, 1]$$

$$\vec{L}_2\vec{w}(x) = E\vec{w}'(x) + A(x)\vec{w}(x) + B(x)\vec{w}(x-1) = \vec{0} \text{ for } x \in (0, 2]$$

with $\vec{\beta}\vec{w}(0) = \vec{\beta}(\vec{u}(0) - \vec{v}(0))$. Here, $\vec{v}$ is called the smooth component of $\vec{u}$ and $\vec{w}$, the singular component.

Lemma 3.1. *For all* $x \in [0, 2]$ *and for* $i = 1, 2, \ldots, n$, *there exists a constant C such that* $\|v_i^{(k)}\| \le C$ *for* $k = 0, 1$ *and* $\|v_i''\| \le C\varepsilon_i^{-1}$.

Proof of Lemma 3.1. The proof is analogous to [7].

The layer function associated with the solution $\vec{u}$ are denoted by

$$\mathfrak{B}_{p,i}(x) = e^{-(x-p)\alpha/\varepsilon_i},\ p = 0, 1; i = 1, 2, \ldots, n\, x \in [0, 2].$$

The following elementary properties of these layer functions for all $1 \le i \le j \le n,\ p = 0, 1$ should be noted.

$$\mathfrak{B}_{p,i}(x) < \mathfrak{B}_{p,j}(x) \text{ for all } x > 0,\ \mathfrak{B}_{p,i}(s) < \mathfrak{B}_{p,j}(t) \text{ for all } 0 \le s < t \le \infty,$$

$$\mathfrak{B}_{p,i}(p) = 1 \text{ and } 0 < \mathfrak{B}_{p,i}(x) < 1 \text{ for all } x > p.$$

The bounds of the singular component $\vec{w}$ derived in terms of these layer functions, are contained in the following lemma.

Lemma 3.2. *Let* $A(x), B(x)$ *satisfies* (1.3) *and* (1.4). *Then there exist a constant* C, *such that for each* $x \in [0, 1)$, *and* $i = 1, 2, \ldots, n$,

$$|w_i(x)| \leq C\mathfrak{B}_{0,n}(x), \; |w_i'(x)| \leq C\sum_{q=1}^{n} \frac{\mathfrak{B}_{0,q}(x)}{\varepsilon_q}, \; |\varepsilon_i w_i''(x)| \leq C\sum_{q=1}^{n} \frac{\mathfrak{B}_{0,q}(x)}{\varepsilon_q}$$

and for $x \in [1, 2]$

$$|w_i(x)| \leq C\mathfrak{B}_{1,n}(x), \; |w_i'(x)| \leq C\sum_{q=1}^{n} \frac{\mathfrak{B}_{1,q}(x)}{\varepsilon_q}, \; |\varepsilon_i w_i''(x)| \leq C\sum_{q=1}^{n} \frac{\mathfrak{B}_{1,q}(x)}{\varepsilon_q}.$$

4. The Shishkin Mesh

A piecewise uniform Shishkin mesh $\overline{\Omega}^N = \overline{\Omega}^{-N} \cup \overline{\Omega}^{+N}$ where $\overline{\Omega}^{-N} = \{x_j\}_0^{N/2}$ and $\overline{\Omega}^{+N} = \{x_j\}_{\frac{N}{2}+1}^{N/2}$ with N mesh intervals is now constructed by dividing the interval $[0, 2]$ into $2n+2$ subintervals as follows.

$$[0, 2] = [0, \tau_1] \cup (\tau_1, \tau_2] \ldots (\tau_{n-1}, \tau_n] \cup (\tau_n, 1] \cup (1, 1+\tau_1] \ldots (1+\tau_n, 2].$$

On each of the sub-intervals $[0, \tau_1], (\tau_i, \tau_i + 1], (1, 1+\tau_1], (1+\tau_i, 1+\tau_i+1), 1 \leq i \leq n-1$, a uniform mesh with $\frac{N}{4n}$ mesh-intervals is placed and on each of the sub-intervals $(\tau_n, 1]$ and $(1+\tau_n, 2]$, a uniform mesh with $\frac{N}{4}$ mesh-intervals is placed. The n transition points between the uniform meshes are defined by

$$\tau_n = \min\left\{\frac{1}{2}, \frac{\varepsilon_n}{\alpha} \ln N\right\} \text{ and} \tag{4.1}$$

$$\tau_i = \min\left\{\frac{\tau_{i+1}}{2}, \frac{\varepsilon_i}{\alpha} \ln N\right\}, \text{ for } i = 1, 2, \ldots, n-1. \tag{4.2}$$

The construction leads to a class of $2n$ possible Shishkin piecewise uniform meshes $M_{\vec{b}}$ where $\vec{b}$ denotes an n-vector with $b_i = 0$ if $\tau_i = \frac{\tau_{i+1}}{2}$ and $b_i = 1$, otherwise. It is to be noted that on any such mesh

$$h_j = x_j - x_{j-1} \leq C_{N-1}, 1 \leq j \leq N$$

$$\tau_i \leq C\varepsilon_i \ln N, \mathfrak{B}_{p,i}(p + \tau_i) = N - 1; \text{ if } b_i = 1$$

$$\tau_i = 2^{-(j-i+1)}\tau_j + 1, \text{ for } i \leq j, \text{ if } b_k = 0 \ \forall k, i \leq k \leq j$$

5. The Discrete Problem

On the mesh constructed above, the problem (1.1)-(1.2) is discretized and the corresponding discrete problems is given by

$$\vec{L}^N\vec{U}(x_j)ED^-\vec{U}(x_j) + A(x_j)\vec{U}(x_j) + B(x_j)\vec{U}(x_j - 1) = \vec{f}(x_j), \ x_j \in \Omega^N \tag{5.1}$$

$$\vec{B}^N\vec{U}(x_j) = \vec{U}(x_j) - ED^+\vec{U}(x_j) = \vec{l}(x_j), \ x_j \in \Omega^N \tag{5.2}$$

where $D^-\vec{U}(x_j) = \dfrac{\vec{U}(x_j) - \vec{U}(x_j - 1)}{x_j - x_j - 1}$, $D^+\vec{U}(x_j) = \dfrac{\vec{U}(x_{j+1}) - \vec{U}(x_j)}{x_{j+1} - x_j}$, $j = 1, 2, \ldots, N$

The problem (5.1)-(5.2) can be rewritten as follows.

$$\begin{cases} (\vec{L}_1^N\vec{U})_i(x_j) = ED^-\vec{U}(x_j) + A(x_j)\vec{U}(x_j) = \vec{f}(x_j) - B(x_j)\vec{\varphi}(x_j - 1), \ x_j \in \Omega^{-N} \\ (\vec{L}_2^N\vec{U})_i(x_j) = ED^-\vec{U}(x_j) + A(x_j)\vec{U}(x_j) + B(x_j)\vec{U}(x_j - 1) = \vec{f}(x_j), \ x_j \in \Omega^{+N} \end{cases} \tag{5.3}$$

with $\vec{\beta}^N\vec{U}(x_j) = \vec{U}(x_j) - ED^+\vec{U}(x_j) = \vec{l}(x_j), \ x_j \in \Omega^{*N}$.

Here D^- and D^+ are called the backward and forward difference operators. The operators satisfies the following discrete maximum principle.

Lemma 5.1. *If* $\vec{\Upsilon}(x_j)$ *is any function in the domain of* $\vec{L}^N$. *The inequalities* $\beta\vec{\Upsilon}(0) \geq \vec{0}$ *and* $\vec{L}^N\vec{\Upsilon}(x_j) \geq \vec{0}$ *for* $1 \leq j \leq N$, *then* $\vec{\Upsilon}(x_j) \geq \vec{0}$ *for all* $0 \leq j \leq N$.

Lemma 5.2 (Stability result)**.** *Let* $\vec{Q}$ *be any vector-valued function in the domain of* $\vec{L}^N$. *Then*

$$\| \vec{Q}(x_j) \| \leq \max \left\{ \| \vec{\beta}^N \vec{Q}(0) \|, \frac{1}{\alpha} \| \vec{L}^N \vec{Q}(x_j) \|_{x_j \in \Omega^N} \right\}.$$

6. The Local Truncation Error

From the discrete stability result, it is seen that in order to bound the error $\vec{U} - \vec{u}$, it suffices to bound $\vec{L}^N (\vec{U} - \vec{u})$. Notice that, for $x_j \in \Omega^N$,

$$\vec{L}^N (\vec{u}(x_j) - \vec{U}(x_j)) = \vec{L}^N \vec{u}(x_j) - \vec{L}^N \vec{U}(x_j)$$

$$= (\vec{L} = \vec{L}^N) \vec{u}(x_j)$$

and $(\vec{L} = \vec{L}^N) \vec{u}(x_j) = \varepsilon_i (D^- - D) v_i(x_j) + \varepsilon_i (D^- - D) w_i(x_j)$

which is the local truncation of the first derivative. Then, by the triangle inequality,

$$| (\vec{L} - L^N) \vec{u}(x_j) | \leq | \varepsilon_i (D^- - D) v_i(x_j) | + | \varepsilon_i (D^- - D) w_i(x_j) |$$

The proof is similar to [8]

Analogous to the continuous case, the discrete solution $\vec{U}$ can be split into $\vec{V}$ and $\vec{W}$ which are defined to be the solutions of the following discrete problems.

$$(\vec{L}_1^N \vec{V})_i (x_j^*) = ED^- \vec{V}(x_j) + A(x_j) \vec{V}(x_j) = \vec{f}(x_j) - B(x_j) \vec{\varphi}(x_j - 1),\ x_j \in \Omega^{-N}$$

$$(\vec{L}_2^N \vec{V})_i (x_j^*) = ED^- \vec{V}(x_j) + A(x_j) \vec{V}(x_j) + B(x_j) \vec{V}(x_j - 1) = \vec{f}(x_j),\ x_j \in \Omega^{+N}$$

$$\vec{\beta}^N \vec{V}(x_j) = \vec{\beta}^N \vec{v}(x_j)$$

and

$$(\vec{L}_1^N \vec{W})(x_j) = 0,\ x_j \in \Omega^{-N},\ (\vec{L}_2^N \vec{W})(x_j) = 0,\ x_j \in \Omega^{+N}$$

The error at each point $x_j \in \overline{\Omega}^N$ is denoted by $\vec{U}(x_j) - \vec{u}(x_j)$. Then the local truncation error $\vec{L}^N \vec{e}(x_j)$ has the decomposition $\vec{L}^N \vec{e}(x_j) = \vec{L}^N (\vec{V} - \vec{v})(x_j) + \vec{L}^N (\vec{W} - \vec{w})(x_j)$. It is to be noted that for any smooth function ψ, the following two distinct estimates of the local truncation of its first derivative hold.

$$|(D^- - D)\vec{\sigma}(x_j)| \leq 2 \max_{s \in I_j} |\vec{\sigma}'(s)|$$

and $|(D^- - D)\vec{\sigma}(x_j)| \leq \frac{h_j}{2} \max_{s \in I_j} |\vec{\sigma}''(s)|$ (6.4)

where $I_j = x_j - x_{j-1}$.

7. Error Estimate

Theorem 6.1. *Let* $A(x)$ *and* $B(x)$ *satisfy* (1.3) *and* (1.4). *Let* v *denote the smooth component of the solution of the problem* (1.1), (1.2) *and* $\vec{V}$ *denote the smooth component of the solution of the problem* (5.1), (5.2). *Then*

$$|\vec{L}^N (\vec{V} - \vec{v})(x_j)| \leq CN^{-1}.$$

Theorem 6.2. *Let* $A(x)$ *and* $B(x)$ *satisfy* (1.3) *and* (1.4). *Let* $\vec{w}$ *denote the singular component of the solution of the problem* (1.1), (1.2) *and* $\vec{W}$ *denote the singular component of the solution of the problem* (5.1), (5.2). *Then*

$$|(\vec{L}^N (\vec{W} - \vec{w}))_i (x_j)| \leq CN^{-1} \ln N.$$

Theorem 6.3. *Let* $A(x)$ *and* $B(x)$ *satisfy* (1.3) *and* (1.4). *Let* $\vec{u}$ *denote the solution of the continuous problem* (1.1), (1.2) *and* $\vec{U}$ *be the solution of the discrete problem* (5.1), (5.2). *Then*

$$|\vec{U}(x_j) - \vec{u}(x_j)| \leq CN^{-1} \ln N.$$

Proof. It is clear that, in order to prove the above theorem it suffices to prove that $\| (\vec{L}^N (\vec{U} - \vec{u})) \| \leq CN^{-1} \ln N$. But, $\| (\vec{L}^N (\vec{U} - \vec{u})) \| \leq \| \vec{L}^N (\vec{V} - \vec{v}) \| + \| (\vec{L}^N (\vec{W} - \vec{w})) \|$. Hence using theorem (6.1) and (6.2), the above result is derived.

Numerical Experiments

The numerical method proposed above is illustrated through an example presented in this section.

Example 1. Consider the initial value problem

$$E\vec{u}'(x) + A(x)\vec{u}(x) = \vec{g}(x)\ \forall\, x \in (0, 1]$$

$$E\vec{u}'(x) + A(x)\vec{u}(x) + B(x)\vec{u}(x-1) = \vec{f}(x)\ \forall\, x \in (1, 2]$$

with $u_1(0) - \varepsilon u_1'(0) = 1,\ u_2(0) - \varepsilon u_2'(0) = 1,\ u_3(0) - \varepsilon u_3'(0) = 1.$

Where $A(x) = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$, $B(x) = diag(-1, -1, -1)$, $\vec{f}(x) = (3, 0, 1)^T$,

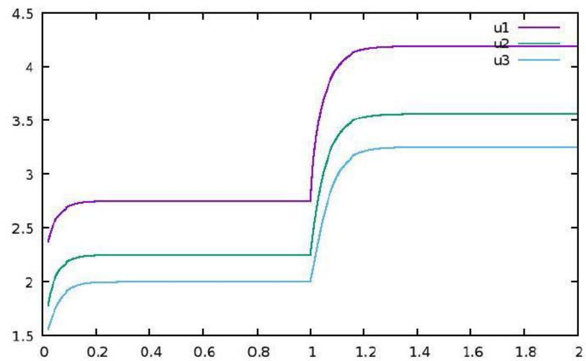
$\vec{g}(x) = (1, 1, 1)^T$, $E = diag(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

The numerical solution obtained by applying the fitted mesh method (6.1) and (6.2) to the Example is shown in Figure 1. The order of convergence and the error constant are calculated and are presented in Table 1.

Table I. Values of D_ε^N, D^N, p^N, p^* and $C_{p^*}^N$ generated for the example.

η	Number of mesh points			
	48	96	192	384
0.100E+01	0.425E-01	0.244E-01	0.132E-01	0.690E-02
0.125E+00	0.836E-01	0.660E-01	0.479E-01	0.322E-01
0.156E-01	0.836E-01	0.660E-01	0.479E-01	0.322E-01
0.195E-02	0.836E-01	0.660E-01	0.479E-01	0.322E-01
0.244E-03	0.836E-01	0.660E-01	0.479E-01	0.322E-01
D^N	0.836E-01	0.660E-01	0.479E-01	0.322E-01
p^N	0.340E+00	0.462E+00	0.572E+00	

C_p^N	0.149E+01	0.149E+01	0.137E+01	0.116E+01
The Order of Convergence = 0.3404967E+00				
The Error Constant = 0.1485278E+01				



References

[1] E. P. Doolan, J. J. H. Miller and W. H. A. Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, 1980.

[2] M. K. Kadalbajoo and K. K. Sharma, Numerical analysis of singularly perturbed delay differential equations with layer behavior, Applied Mathematics and Computations 157(1) (2004).

[3] S. Valarmathi and J. J. H. Miller, A parameter uniform finite difference method for a singularly perturbed linear dynamical systems, Int. J. Numer. Anal. Model 43(2) (2013), 67-80.

[4] Valarmathi Sigamani, John J. H. Miller, Ramanujam Narasimhan, Paramasivam Mathiazhagan and Franklin Victor, Editors, Differential equations and numerical analysis, Springer Science and Business Media LLC 172 (2016).

[5] M. K. Kadalbajoo and K. K. Sharma, A numerical method based on finite difference for boundary value problems for singularly perturbed delay differential equations, Applied Mathematics and Computation 197(2) (2008), 692-707.

[6] M. K. Kadalbajoo and K. K. Sharma, Numerical analysis of singularly perturbed delay differential equations with layer behavior, Applied Mathematics and Computation 157(1) (2004), 11-28.

[7] N. Shivaranjani, N. Ramanujam, J. J. H. Miller and S. Valarmathi, A parameter-uniform Numerical method for an initial value problem for a system of singularly perturbed delay differential equations, A. R. Ansari (ed.), Advances in Applied Mathematics, Springer Proceedings in Mathematics and Statistics 87 (2014), 127-138.

[8] Selvaraj Dinesh and Mathiyazhagan Joseph Paramasivam, A Parameter-Uniform numerical method for a singularly perturbed initial value problem with Robin initial condition, Malaya Journal of Matematik 8(2) (2020), 414-420.

[9] Dinesh Selvaraj and Joseph Paramasivam Mathiyazhagan, Parameter uniform convergence for a system of two singularly perturbed initial value problems with different perturbation parameters and Robin initial conditions, Malaya Journal of Matematik, 2021.