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SPLIT GEODETIC DOMINATION NUMBER OF A GRAPH

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Abstract

In this paper, we define a new parameter split geodetic domination number of a graph. A set $S \subseteq V(G)$ is said to be a split geodetic dominating set of G if S is both a split geodetic set and a dominating set of G the induced subgraph $\langle V - S \rangle$ is disconnected. The minimum cardinality of the split geodetic dominating set of G is called split geodetic domination number of G and is denoted by. $\gamma_{gs}(G)$. The split geodetic domination number of some standard graphs are determined. For any positive integers a, b, c with $a < a + 1 = b < c$, then there exists a connected graph G such that $g(G) = a$, $g_s(G) = b$, $\gamma_{gs}(G) = c$. Also, for any three positive integers a, b, c with $a, b, c \geq 2$, there exist a connected graph G such that, $g(G) = a$, $\gamma_s(G) = b$, $\gamma_{gs}(G) = c$.

1. Introduction

We consider undirected finite graph without loops and multiple lines. The

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graph considered here have at least one component which is not complete and star graph. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [4]. Let v be a point in V. The open neighborhood of v is the set $N(v)$ consisting of all points u which are adjacent with v and $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of a point v in G. The distance $d(u, v)$ between two points u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. The closed interval $I[x, y]$ consists of all points lying on some $x - y$ geodesic of G and for a non empty subset S of $V(G)$, $I(S) = \cup_{x, y \in S} I[x, y]$. A set S of points is a geodetic set if $I[S] = V(G)$ and the minimum cardinality of a geodetic set is the geodetic number of G and is denoted by $g(G)$. A geodetic set of cardinality $g(G)$ is called a $g(G)$-set.

A point v is a simplicial point of a graph G if $\langle N(v) \rangle$ is complete. A simplex of a graph G is a sub graph of G which is a complete graph. A point in a graph G dominates itself and its neighbours. A set of points D in a graph G is a dominating set if each point of G is dominated by some point of D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G.

A subset S of $V(G)$ is a geodetic dominating set of G if S is a geodetic set and a dominating set of G. The minimum cardinality of a geodetic dominating set is called the geodetic domination number of G and is denoted by $\gamma_g(G)$. The study of geodetic domination number was initiated by Escuardo, Gera, Hansberg, Jafari Rad and Volkmann [2] in 2011. A dominating set D of a graph $G = (V, E)$ is a split dominating set if the induced subgraph $\langle V \setminus D \rangle$ is disconnected. The split domination number $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set of G. The study of split domination number was initiated by Veerabhadrappa R. Kulli and Bidarahalli Janakiram [6] in 1997.

A geodetic set S of a graph $G = (V, E)$ is a split geodetic set if the induced subgraph $\langle V - S \rangle$ is disconnected. The split geodetic number $g_s(G)$

of G is the minimum cardinality of a split geodetic set of G. The study of split geodetic number was initiated by Venkanagouda M. Goudar, Ashalatha K. S. and Venkatesha [8] in 2014. The following theorems are used in the sequel.

Theorem 1.1 [4]. *Let be a vertex of a connected graph G. The following statements are equivalent*:

(i) *v is a cut-vertex of G.*

(ii) *There exist vertices u and w distinct from v such that v is on every $u - w$ path.*

(iii) *There exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every $u - w$ path.*

Theorem 1.2 [1]. *Every geodetic set of a graph contains its extreme vertices.*

2. Split Geodetic Domination Number of a Graph

Definition 2.1. A set $S \subseteq V(G)$ is said to be *a* split geodetic dominating set of G if S is both *a* split geodetic set and a dominating set of G (the induced sub graph $\langle V - S \rangle$ is disconnected). The minimum cardinality of the split geodetic dominating set of G is called split geodetic domination number of G and is denoted by $\gamma_{gs}(G)$.

Example 2.2.

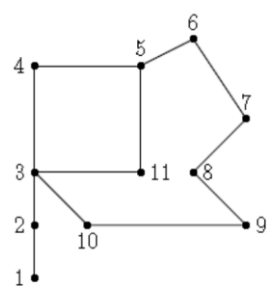

Figure 1. Graph G.

For the graph G given in Figure 1, $S_1 = \{1, 7\}$ is a geodetic set,

$S_2 = \{1, 5, 9\}$ is a split geodetic set, $S_3 = \{1, 3, 5, 8\}$ is a split geodetic dominating set. Thus geodetic number, geodetic domination number and split geodetic domination number are different.

Theorem 2.3. *Every split geodetic dominating set of G contains its extreme points.*

Proof. Let v be an extreme point of G and let S be a split geodetic dominating set of G. Suppose $v \notin S$, then S is not a geodetic set. Hence S is not a split geodetic dominating set of G, which is a contradiction. Because every split geodetic dominating set is a geodetic set. Hence every split geodetic dominating set contains its extreme points.

Theorem 2.4. *Every split geodetic dominating set of G contains its end points.*

Proof. Since end points are extreme points. The result follows from Theorem 2.3.

Theorem 2.5. *Let G be a connected graph of order p, then* $2 \leq \gamma_{gs}(G) \leq p - 2$.

Proof. A split geodetic dominating set needs at least two points and hence $\gamma_{gs}(G) \geq 2$. Suppose that $\gamma_{gs}(G) \geq p - 1$. Consider $\gamma_{gs}(G) = p - 1$. Let v be a point of G and let $S = V(G) - \{v\}$ be a split geodetic dominating set of G. Then $\langle V - S \rangle$ is connected, which is a contradiction to our Definition 2.1. Therefore, $\gamma_{gs}(G) \leq p - 2$.

Theorem 2.6. *If a connected graph G has n extreme points, then* $\gamma_{gs}(G) \geq n$.

Proof. Let G be a connected graph with n extreme points. Then by Theorem 2.3, $\gamma_{gs}(G) \geq n$.

Remark 2.7. Consider the graph given in Figure 2.

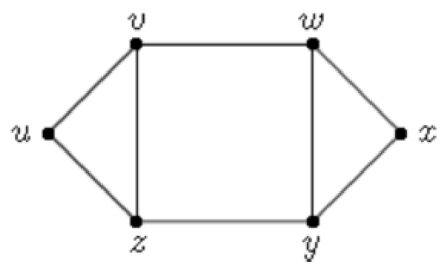

Figure 2. G.

The extreme points of graph G are u, x. The split geodetic dominating set of G is $\{u, v, x, y\}$. It is clear that $\gamma_{gs}(G) \geq 2$.

Theorem 2.8. *Let G be any connected graph of order $p \geq 4$. Then $\gamma_{gs}(G) = 2$ if and only if there exists a split geodetic dominating set $S = \{u, v\}$ of G such that $d(u, v) \leq 3$.*

Proof. Assume that $\gamma_{gs}(G) = 2$. Let S be a minimum split geodetic dominating set of G. Suppose that $d(u, v) \geq 4$, then it has at least 3 internal points such that $\gamma_{gs}(G) > 2$, which is a contradiction to our assumption. Hence $d(u, v) \leq 3$.

Conversely, Assume that there exists a split geodetic dominating set $S = \{u, v\}$ of G such that $d(u, v) \leq 3$. Then by Theorem 2.5, S is a minimum split geodetic dominating set of G. Hence $\gamma_{gs}(G) = 2$.

Theorem 2.9. *Let G be a connected graph with a cut point v. Then every split geodetic dominating set S of G contains at least one element from each component of $G - \{v\}$.*

Proof. Suppose there exists a component G_1 of $G - \{v\}$ such that G_1 contains no point of S. By Theorem 2.3, contains extreme points of G. Hence G_1 does not contain any extreme point of G. Let $u \in G_1$. Since S is a split geodetic dominating set of G, there exists a pair of points $x, y \in S$ such that $u \in I[x, y] \subseteq I[S]$ and $u \in N[S]$. Let the $x - y$ geodesic in G be $P : x = u_0$, $u_1, \ldots, u, \ldots, u_n = y$ in G with $u \neq x, y$. Since v is a cut point of G, by

Theorem 1.1, the $x - u$ sub path P_1 of P and the $u - y$ sub path P_2 of P both contains v, it follows that P is not a path, which is a contradiction. Hence S contains at least one element from each component of $G - \{v\}$.

Theorem 2.10. *For any two integers a and b with* $a \geq 2$ *and* $0 \leq b \leq 2$, *there exists a connected graph* G *of order* $p \geq 4$, *such that* $\gamma_{gs}(G) \leq p - \left\lfloor \frac{2diam(G)}{3} \right\rfloor$.

Proof. Let G be a connected graph with diameter $d = 3a + b$. Consider a shortest path $P : u_0, u_1, \ldots u_d$ of length d and let $A = \{u_0, u_3, \ldots u_{3a}, u_{3a+b}\}$ where, $|A| = a + 2$ when $b = 0$ and $|A| = a + 2$ when $b = 1, 2$. Let $S = (V(G)/(V(P)/A))$ where S gives split geodetic dominating set of G. Then

$$S = (V(G) \setminus (V(P) \setminus A)) = p - \left\lfloor \frac{2(3a + b)}{3} \right\rfloor$$

$$\gamma_{gs}(G) \leq p - \left\lfloor \frac{2diam(G)}{3} \right\rfloor.$$

Theorem 2.11. *For any two integers* $m, n \geq 2$, $\gamma_{gs}(K_{m,n}) = \min\{m, n\}$.

Proof. Let $= K_{m,n}$, such that $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the partitions of G with $m \leq n$. Now, let $S = \{u_1, u_2, \ldots u_m\}$ for all $v_i \in I[u_i, u_j] \subseteq I[S]$ for $1 \leq i \neq j \leq m$. Since, $\langle V - S \rangle$ is disconnected. Therefore S is a split geodetic dominating set of G and is minimum. Because if we remove one point from S, then the set will not be a geodetic set and not a split geodetic dominating set. Thus $\gamma_{gs}(K_{m,n}) = \min\{m, n\} = m$.

Theorem 2.12. *If T is a caterpillar graph, then* $\gamma_{gs}(T) = p - d + 2$.

Proof. Let T be caterpillar graph. Let $d = d(u, v)$ and let $P : u = v_0, v_1, \ldots v_d = v$ be a $u - v$ path. Let k be the number of end points of T. Then $p = d - 1 + k$. By Theorem 2.4, end points belongs to S. So that $S = V(G) - \{v_1, \ldots, v_{d-1}\}$ is connected. Then $I[S] = V[G]$, $V - (S \cup v_2)$ is disconnected and so, $\gamma_{gs}(T) = |S| + 1 = k + 1 = p - d + 2$.

Theorem 2.13. *For the wheel* $W_p = K_1 + C_{p-1}$, $(p \geq 6)$,

$$\gamma_{gs}(W_p) = \frac{p+2}{2} \text{ if } p \text{ is evevn and } \frac{p+1}{2} \text{ if } p \text{ is odd.}$$

Proof. We prove this theorem by consider the following cases.

Case (i)**.** Let p be an even.

The set $S = \{y_1, y_3, y_5, \ldots, y_{p-1}, x\}$ is a minimum split geodetic dominating set of W_p. The points $y_2, y_4, \ldots y_p$ belongs to the geodesic path such that $I[S] = V(W_p)$ and the set S has $\frac{p}{2} + 1$ points. Hence $\gamma_{gs}(W_p) = \frac{p+2}{2}$.

Case (ii)**.** Let p be an odd.

The set $S = \{y_1, y_3, y_5, \ldots, y_{p-2}, x\}$ is a minimum split geodetic dominating set of W_p. The induced sub graph $\langle V - S \rangle$ is disconnected and the set S has $\frac{p-1}{2} + 1$ points. Hence $\gamma_{gs}(W_p) = \frac{p+1}{2}$.

Theorem 2.14. *For cycle* C_p *of order* $p \geq 6$, *then* $\gamma_{gs}(C_p) = \left\lceil \frac{p}{3} \right\rceil$.

Proof. Case (i)**.** Let $C_{3n-1} : v_1, v_2, \ldots, v_{3n-1}, v_1$ be a cycle with $3n - 1 = p$ for $n \geq 3$ points. The set $S = \{v_1, v_4, \ldots, v_{3n-2}\}$ is a minimum split geodetic dominating set and the set S has $\left\lceil \frac{3n-1}{3} \right\rceil$ points. Hence $\gamma_{gs}(C_{3n-1}) = \left\lceil \frac{p}{3} \right\rceil$.

Case (ii)**.** Let $C_{3n} : v_1, v_2, \ldots, v_{3n}, v_1$ be a cycle with $3n = p$ for $n \geq 2$ points. The set $S = \{v_1, v_4, \ldots v_{3n-2}\}$ is a minimum split geodetic dominating set and it is clear that $v_2, v_3, v_5, v_6, \ldots v_{3n-1}, v_{3n}$ lies on geodesic path. The set S has $\left\lceil \frac{p}{3} \right\rceil$ points. Hence $\gamma_{gs}(C_{3n}) = \left\lceil \frac{p}{3} \right\rceil$.

Case (iii)**.** Let $C_{3n+1} : v_1, v_2, \ldots, v_{3n+1}, v_1$ be a cycle with $3n + 1 = p$ for

$n \geq 2$ points. The set $S = \{v_1, v_4, \ldots, v_{3n+1}\}$ is a minimum split geodetic dominating set. The set S has $\left\lceil \frac{p}{3} \right\rceil$ points. Hence $\gamma_{gs}(C_{3n+1}) = \left\lceil \frac{p}{3} \right\rceil$.

Theorem 2.15. *Let G' be the graph obtained by adding an end line (v, u) to a cycle $C_p = G$ of order $p \geq 6$ where $v \in G$ and $u \notin G$. Then* $\gamma_{gs}(G') = \left\lfloor \frac{p+1}{3} \right\rfloor + 1$.

Proof. Let G' be the graph obtained by adding an end line (v, u) to a $C_p = G$ cycle of order $p \geq 6$ such that $v \in G$ and $u \notin G$. We discuss the following cases:-

Case (i). $G = C_{3n-1=p}$, $n \geq 3$.

Let $S = \{u, v_3, v_5, v_7, v_{10}, \ldots, v_{3n-2}\}$ be a split geodetic dominating set of G', where u is an end point of G'. Suppose $S' = (S - \{v_i\})$, $v_i \in S$, then S' is not a dominating set. Hence S is a minimum split geodetic dominating set of G. Thus, $\gamma_{gs}(G') = \left\lfloor \frac{p+1}{3} \right\rfloor + 1$.

Case (ii). $G = C_{3n=p}$, $n \geq 2$.

The set $S = \{u, v_3, v_5, v_8, \ldots, v_{3n-1}\}$ is a split geodetic dominating set of G' where u is an end point of G' and $I[S] = V[G']$. Thus $\langle V - S \rangle$ is disconnected. Hence $\gamma_{gs}(G') = \left\lfloor \frac{p+1}{3} \right\rfloor + 1$.

Case (iii). $G = C_{3n+1=p}$, $n \geq 2$.

The set $S = \{u, v_3, v_6, v_9, \ldots, v_{3n}\}$ is a split geodetic dominating set of G' and $I[S] = V[G']$. Thus $\langle V - S \rangle$ is disconnected. Hence $\gamma_{gs}(G')$ $= \left\lfloor \frac{p+1}{3} \right\rfloor + 1$.

Observation 2.16.

(i) For path P_p, $p \geq 5$, then $\gamma_{gs}(P_p) = \left\lfloor \frac{p+2}{3} \right\rfloor$.

(ii) Let G' be the graph obtained by adding an l end lines $(v, u_1), (v, u_2), (v, u_3), \ldots, (v, u_l)$ to a cycle $C_p = G$ of order $p \geq 6$ where $v \in G$ and $u_1, u_2, \ldots, u_l \notin G$. Then $\gamma_{gs}(G') = \left\lfloor \frac{p+1}{3} \right\rfloor + l$.

(iii) $\gamma_{gs}(K_2 \times P_p) = \left\lceil \frac{p}{2} \right\rceil + 1,\ p \geq 2$.

3. Realization Results

In this section, we give the realization results for the split geodetic dominating number of a graph G.

Theorem 3.1. *For any positive integers* a, b, c *with* $a < a + 1 = b < c$, *then there exists a connected graph* G *such that* $g(G) = a$, $g_s(G) = b$, $\gamma_{gs}(G) = c$.

Proof. Let $K_{2,2}$ be a complete graph with partite sets $X = \{y, z\}$ and $Y = \{m, n\}$. Let H_1 be a graph obtained from $K_{2,2}$ by adding new points $z_1, z_2, \ldots, z_{a-1}$ and joining $z_i (1 \leq i \leq a-1)$ with z. Let H_2 be $(c-b)$ copies of path $P : g_i, h_i, u_i, v_i, (1 \leq i \leq c-b)$. Let G be a graph obtained from H_1 and H_2 by adding two points w and x and joining $v_i (1 \leq i \leq c-b)$. to the point x and joining $g_i (1 \leq i \leq c-b)$ to w and also join x and y. The resulting graph G is given in Figure 3.

Let $S = \{z_1, \ldots z_{a-1}\}$ be the set of all extreme points of G. By Theorem 1.2, S belongs to every geodetic set of G. Here S is not a geodetic set of G. However the set $S_1 = \{w, z_1, \ldots, z_{a-1}\}$ is the minimum geodetic set of G. Thus $g(G) = a$. The set $S_2 = S_1 \cup \{y\}$ is the minimum split geodetic set of G. Thus $a + 1 = b$ gives $g_s(G) = b$. The set $S_3 = S_2 \cup \{u_1, u_2, \ldots, u_{c-b}\}$ is the minimum split geodetic dominating set of G. Thus $\gamma_{gs}(G) = b + [c-b] = c$.

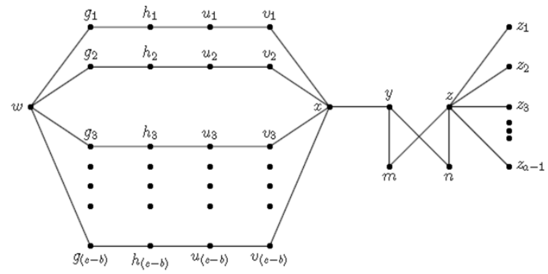

Figure 3. G

Theorem 3.2. *For any three integers a, b, c with $a, b, c \geq 2$, there exists a connected graph G such that $g(G) = a, \gamma_s(G) = b, \gamma_{gs}(G) = c$.*

Proof. We shall prove this theorem by considering 3 cases.

Case (i)**.** $a = b = c$

Consider $G = K_{a,p}, p > 2$, the complete bipartite graph on $a + p$ points. Then $g(G) = \gamma_s(G) = \gamma_{gs}(G) = a$.

Case (ii)**.** $a < b = c$

Let $K_{a-1,a-1}$ be the complete bipartite graph with bipartite sets $U = \{u_1, u_2, \ldots, u_{a-1}\}$ and $V = \{h_1, h_2, \ldots, h_{a-1}\}$. Let $P : l_1, l_2, \ldots, l_{3(b-a+1)+1}$ be a path of order $3(b - a + 1) + 1$. Let G be the graph obtained from $K_{a-1,a-1}$ and P by identifying the points u_1 and l_1. The resulting graph G is given in Figure 4.

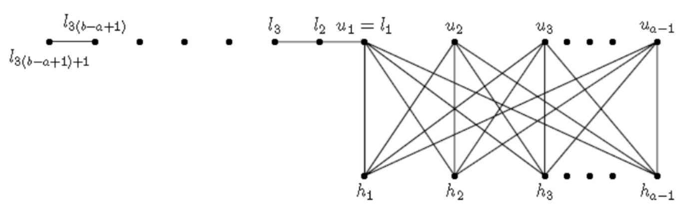

Figure 4. G

The set $\{l_{3(b-a+1)+1}, u_1, u_2, \ldots, u_{a-1}\}$ is the minimum geodetic set of G. The set $\{l_{3(b-a+1)+1}, l_{3(b-a+1)-2}, \ldots, l_4, u_1, \ldots, u_{a-1}\}$ is both a minimum split dominating set and a minimum split geodetic dominating set of G. Thus $g(G) = a$ and $\gamma_s(G) = b = c = \gamma_{gs}(G)$.

Case (iii). $b < a = c$

Consider a complete bipartite graph $K_{b-1, b-1}$ with bipartite sets $U = \{u_1, u_2, \ldots, u_{b-1}\}$ and $V = \{h_1, h_{2,} \ldots h_{b-1}\}$. Let G be the graph obtained from $K_{b-1, b-1}$ by adding new points $h_1, h_2, \ldots, h_{a-b+1}, x$ and joining each $h_i(1 \le i \le h_{a-b+1})$ to x and x to each $v_i(1 \le i \le b-1)$. The resulting graph is given in Figure 5.

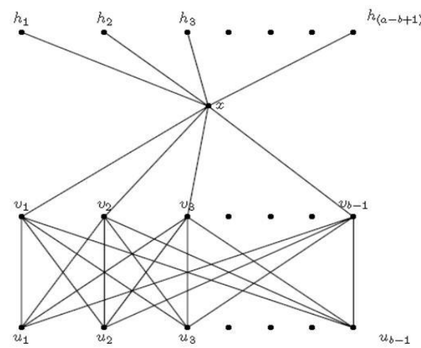

Figure 5.G

The set $\{v_1, v_2, \ldots, v_{b-1}, h_1, h_2, \ldots, h_{a-b+1}\}$ is both a minimum geodetic set and a minimum split geodetic dominating set of G. The set $\{u_1, u_2, \ldots, u_{b-1}, x\}$ is the minimum split dominating set of G. Thus $\gamma_s(G) = b$ and $g(G) = a = c = \gamma_{gs}(G)$.

Theorem 3.3. *For any two integers a and b with* $2 \le a \le b$, *there exists a connected graph G such that* $g_s(G) = a, \gamma(G) = b$ *and* $\gamma_{gs}(G) = a + b$.

Proof. Let $C_{3p} : u_1, u_2, \ldots, u_{3p}, u_1$ with $p \ge 2$ be a cycle of order $3p$. Let $u = u_{\lceil 3p/2 \rceil + 1}$ be the opposite point of u_1 in C_{3p}. Let $P : y_0, y_1, \ldots y_{3(b-p)}$ be a path of order $3(b-p)$. Let G be a graph obtained from P and C_{3p} by adding $a-2$ new points $z_1, z_2, \ldots, z_{a-2}$ and joining each to $z_i(1 \le i \le a-2)$ to u_1 and also join u and y_0. The resulting graph G

is given in Figure 6. Let $S = \{z_1, z_2, \ldots, z_{a-2}, y_{3(b-p)}\}$ be the set of all extreme points of G. By Theorem 1.2, S belongs to every geodetic set of G. Here, S is not a split geodetic set of G. However, the set $S_1 = \{z_1, z_2, \ldots, z_{a-2}, u, y_{3(b-p)}\}$ is the split geodetic set of G, so that $g_s(G) = a$.

The set $S_2 = \{u_1, u_4, u_7, \ldots, u, \ldots, u_{3p-2}, y_2, \ldots, y_{3(b-p)-1}\}$ is the minimum dominating set of G. Thus $\gamma(G) = b$. Also, the union of S_1 and S_2 gives minimum split geodetic dominating set of G, so that $\gamma_{gs}(G) = |S_1 \cup S_2| = a + b$.

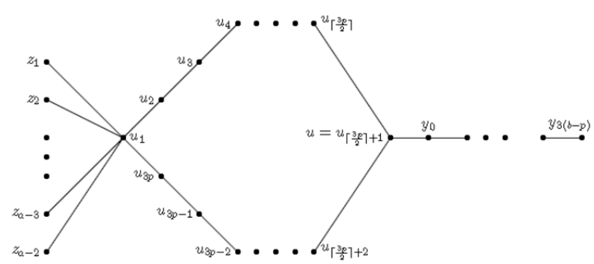

Figure 6. G

4. Conclusion

In this paper, we define the split geodetic domination number of a graph. This work can be extended to find split edge geodetic domination number, upper split geodetic domination number, etc. The findings united in this paper would support to the readers to develop various useful applications to Science and Technology.

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