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WEAK DOMINATION IN SEMI-TOTAL BLOCK GRAPHS

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Abstract

For any graph $G = (V, E)$, the semi-total block graph $T_b(G) = H$, whose set of vertices is the union of the set of vertices and blocks of G and in which two vertices are adjacent if and only if the corresponding vertices of G are adjacent or the corresponding members are incident in G. A dominating set S of G is said to be weak dominating set, if every vertex $u \in V - S$ is adjacent to a vertex $v \in S$, such that $\deg(v) \leq \deg(u)$. A dominating set D of a graph H is a weak dominating set of H, if every vertex in $V[H] - D$ is weakly dominated by at least one vertex in D. A weak semi-total block domination number $\gamma_{wtb}(G)$ of G is the minimum cardinality of a weak semi-total block dominating set of G. In this paper, we obtained some sharp bounds for $\gamma_{wtb}(G)$. Also some upper and lower bounds on $\gamma_{wtb}(G)$, in terms of elements of G and other dominating parameters of G were obtained.

1. Introduction

All the graphs considered here are finite, without loops and multiple edges, undirected and connected. Graph terminology not found here can be

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found in [6]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E, such that $|V| = p$ and $|E| = q$. Degree of an edge $= uv$ of a graph G is defined as $\deg(u) + \deg(v) - 2$ and is denoted as $\delta'(G)$. In general we use $\langle X \rangle$ to denote the sub graph induced by the set of vertex $X.N(v)$ and $N[v]$ denote the open and closed neighborhood of a vertex v, respectively. The minimum (maximum) degree among the vertices of G is denoted by $\delta(G)[\Delta(G)]$. Also $\beta_0(G)$, $[\beta_1(G)]$ is the minimum number of vertices (edges) in a maximal independent set of vertices (edge) of G. A vertex cover in a graph G is a set of vertices that covers all the edges of G. The vertex covering number $\alpha_0(G)$ is the minimum cardinality of a vertex cover in G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G. An edge covering number $\alpha_1(G)$ is the minimum cardinality of an edge cover in G.

The minimum number of colors in any coloring of a graph G such that no two adjacent vertices have the same color is called the chromatic number of G and is denoted by $\chi(G)$.

A set $D \subseteq V(G)$ is a dominating set, if for every vertex $v \in V(G) - D$ is adjacent to at least one vertex $u \in D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A thorough study of domination appears in [8, 9].

We begin by recalling some standard definitions from domination theory.

A set $S \subseteq V(G^2)$ is a dominating set, if each vertex in $V(G^2) - S$ has one neighbor in S. The square domination number of G is denoted by $\gamma(G^2)$ is the minimum cardinality of square dominating set of G^2. This concept was introduced by [18].

A dominating set $D \subseteq V(G)$ is a connected dominating set, if the induced sub graph $\langle D \rangle$ has one component. The connected domination number $\gamma_c(G)$ is the minimum cardinality of a connected dominating set of G.

Further, a set $F \subseteq E(G)$ is called an edge dominating set, if for every edge $e \in E(G) - F$ is adjacent to at least one edge $f \in F$. An edge domination number $\gamma'(G)$ is the minimum cardinality of an edge dominating set of G, see [1].

An edge dominating set $F \subseteq E(G)$ is said to be connected edge dominating set, if the induced sub graph $\langle F \rangle$ is connected. An edge connected domination number $\gamma'_c(G)$ is the minimum cardinality of an edge connected dominating set of G, for details see [2].

A dominating set $S \subseteq V(G)$ is called an end dominating set, if S contains all the end vertices of G. An end domination number $\gamma_e(G)$ is the minimum cardinality of an end dominating set of G. Domination related parameters were now studied in graph theory [8, 9, 15].

The concept of perfect domination was introduced and studied in [4]. A dominating set $D \subseteq V(G)$ is said to be perfect dominating set, if for every elements $v \in V(G) - D$ is dominated by exactly one element $v \in D$. The perfect domination number $\gamma_p(G)$ is the minimum cardinality of a perfect dominating set of G.

A dominating set $S \subseteq V(G)$ is called a double dominating set, if every vertex of $V(G)$ is dominated by at least two vertices in S. The double domination number $\gamma_{dd}(G)$ is the minimum cardinality of double dominating set of G. The concept of double domination was introduced in [7].

In [3], the author showed that a roman domination function of graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of Roman domination function in G is the value $f(v) = \Sigma_{u \in v} f(u)$. The Roman domination number of G is denoted by $\gamma_R(G)$, equals the smallest weight of a Roman dominating function on G.

The concept of restrained domination in graph was introduced by Domke et al. (1999) see [5]. A set $S \subseteq V(G)$ is a restrained dominating set of G, if every vertex $V(G) - S$ is adjacent to a vertex in S and another vertex in $V(G) - S$. The restrained domination number of a graph G is denoted by $\gamma_{\mathrm{Re}s}(G)$ is the minimum cardinality of a restrained dominating set of G.

Analogously, a set D_ε of elements of G is an entire dominating set, if every element not in D_ε is either adjacent or incident to at least one element in D_ε. The entire domination number of G is denoted by $\gamma_\varepsilon(G)$ of G is the minimum cardinality of an entire dominating set of G, see [11].

The concept of split domination number introduced by [10]. A dominating set $D \subseteq V(G)$ is a split dominating set, if the induced sub graph $\langle V - D \rangle$ has more than one component. The split domination number of G is denoted by $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set of G.

A dominating set $S \subseteq V(G)$ is called non split dominating set, if the induced sub graph $\langle V - S \rangle$ is connected. The non split domination number $\gamma_{ns}(G)$ is the minimum cardinality of non split dominating set of G. For details see, [10].

A dominating set $D \subseteq V(G)$ is a strong non split dominating set, if the induced sub graph $\langle V - D \rangle$ is complete. The strong domination number of a graph G is denoted by $\gamma_{sns}(G)$ is the minimum cardinality of a strong non-split dominating set of G, see [10].

In [18], Sampathkumar and L. Pushpa Latha have shown weak domination number. A dominating set S is a weak dominating set of G, if for every vertex $u \in V(G) - S$ there is a vertex $v \in S$ with $\deg(v) \leq \deg(u)$ and u is adjacent to v. A weak domination number $\gamma_w(G)$ is the minimum cardinality of a weak dominating set of G.

A weak dominating set D is a weak dominating set of $L(G)$, if for every vertex $u \in V[L(G)] - D$ there is a vertex $v \in D$ with $\deg(v) \leq \deg(u)$ and u is adjacent to v. A weak domination number $\gamma_{wl}(G)$ is the minimum cardinality of a weak line dominating set of G, see [12].

A weak dominating set S is a weak dominating set of $B(G)$, if for every vertex $u \in V[B(G)] - S$ there is a vertex $v \in S$ with $\deg(v) \leq \deg(u)$ and u is adjacent to v. The weak block domination number $\gamma_{wb}(G)$ of G is the minimum cardinality of a weak block dominating set of $B(G)$. This concept is discussed in [13].

The purpose of this paper is to introduce the concept of weak domination in semi-total block graph and study its properties.

A dominating set D of a graph $T_b(G)$ is a weak semi-total block dominating set of G. If every vertex $u \in V[T_b(G)] - D$ is weakly dominated by at least one vertex $v \in D$ with $\deg(v) \leq \deg(u)$ and u is adjacent to v. The weak semi-total block domination number of G is denoted by $\gamma_{wtb}(G)$ is the minimum cardinality of a weak semi-total block dominating of G.

Further domination related graph valued functions has been studied in [14, 16].

2. Main Results

We develop the following results for some standard graphs.

Proposition 2.1. *For any cycle C_p with $p \geq 3$ vertices, then*

$$\gamma_{wtb}[C_p] = \left\{\begin{array}{ll} \frac{p}{3} & \text{if } p \equiv 0 (\operatorname{mod} 3) \\ \left\lceil \frac{p}{3} \right\rceil & \text{otherwise} \end{array}\right\}$$

Proposition 2.2. *For any star $K_{1,n}$, $n \geq 1$, then $\gamma_{wtb}[K_{1,n}] = \Delta(G)$.*

Proposition 2.3. *For any complete graph K_p with $p \geq 2$ vertices, then $\gamma_{wtb}[K_p] = 1$.*

Proposition 2.4. *For any path P_p with $p \geq 2$ vertices, then $\gamma_{wtb}[P_p] = diam(G)$.*

The following results give bounds and equalities on γ_{wtb} of tress.

Theorem 2.5. *For any nontrivial tree T, then $\gamma_{wtb}(T) = q$.*

Proof. For any nontrivial tree T, since each edge is a block, then in $T_b(T)$ these are the block vertices. Also these block vertices of $T_b(T)$ are the elements of a weak dominating set of $T_b(T)$. Hence $\gamma_{wtb}(T) = q$.

Theorem 2.6. *For any nontrivial tree T, with l end vertices and s support vertices, then $\gamma_{wtb}(T) = s + l - 1$ if and only if all the non end vertices of a tree T is adjacent to at least one end vertex of T.*

Proof. For the necessity, suppose T has at least one nonendvertex v which is not adjacent to an end vertex. Then by Theorem 2, $\gamma_{wtb}(T) \neq s + l - 1$.

For the sufficiency, for any tree T with $p \geq 2$ vertices with s number of support vertices and l number of end vertices the total number of block vertices in $T_b(T) iss + l - 1$. Since for any tree T with $p \geq 2$ vertices each edge is a block and the weak domination in $T_b(T)$ is $s + l - 1$. Hence $\gamma_{wtb}(T) = s + l - 1$.

Theorem 2.7. *For any tree T, with $p \geq 4$ vertices, then $\gamma_{\mathrm{Re}s}(T) + \gamma(T)$ $-1 \leq \gamma_{wtb}(T)$ and $T \neq K_{1, n}$.*

Proof. Suppose $T = K_1$. Then $\gamma_{wtb}(T) \ngeq \gamma_{\mathrm{Re}s}(T) + \gamma(T) - 1$. Let $C = \{v_1, v_2, \ldots, v_n\} \subseteq V(T)$ be the set of all cut vertices in T. Suppose $C_1 \subseteq C$, such that $N[C_1] = V(T)$. Then C_1 forms a minimal dominating set of T. Let $D = \{v_1, \ldots, v_m\}$ be the set of all end vertices in T. Further, if $\forall v_j \in (T) - \{D \cup C_1\}$ is adjacent to least one vertex of $v_i\{D \cup C_1\}$ and at least one vertex of $V(T) - \{D \cup C_1\}$. Then $S = D \cup C_1$ forms a γ_{res}-set of T. Let $R = \{u_1, u_2, u_3, \ldots, u_p\} \subseteq V[T_b(T)]$ be the set of vertices with $\deg(u_i) = 2, \forall \in R$. Further, let $R_1 = \{u_1, u_2, u_3, \ldots, u_q\} \subseteq R$, such that $N[R_1] = [T_b(T)]$. Then R_1 forms a minimal dominating set of $T_b(T)$.

Suppose, $\forall\, u_j \in [T_b(T)] - R_1$ is adjacent to at least one vertex of $u_i \in R_1$ with $\deg(u_i) \leq \deg(u_j)$ and u_j is adjacent to u_i. Hence R_1 forms a minimal γ_{wtb}-set of T. Thus $|R_1| \geq |S| + |C_1| - 1$, gives $\gamma_{wtb}(T) \geq \gamma_{\mathrm{Re}s}(T) + \gamma(T) - 1$.

Theorem 2.8. *For any nontrivial tree T, $\gamma_{wb}(T) + \alpha_1(T) - 3 \leq \gamma_{wtb}(T)$.*

Proof. Let $E = \{e_1, \ldots, e_n\} \subseteq E(T)$ be the set of all end edges in T and

$E_1 = E(T) - E$ be the set of all non end edges which are not adjacent to the end edges of E, further, consider the subset $E'_1 \subseteq E_1$. Then $\{E \cup E'_1\}$ be the minimal set of edges which covers all the vertices of T, such that $|E \cup E'_1| = \alpha_1(T)$ Since the chromatic number of a tree T, $x(T) = 2$, then we now show that $\gamma_{wb}(T) + \alpha_1(T) - x(T) - 1 \leq \gamma_{wtb}(T)$. For any tree T, let $A = \{e_1, e_2, e_3, \ldots, e_m\}$ be the set of edges in T. Let $K = \{v_1, v_2, v_3, \ldots, v_k\}$ $= V[B(T)]$ be the set of vertices corresponding to the edges of A. Let $j \subseteq K$ be the set of vertices, such that $N[j] = V[B(T)]$ and if, $\forall v_i \in j$ has degree at least two and $\forall v_j \in V[B(T)] - J$ with $\deg(v_i) \leq \deg(v_j)$ and $N(v_i) \cap J.$ $= \{v_j\}$ Then J forms a $\gamma_{wb} - set$ of T. Let $H = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of all non end block vertices in $Tb(T)$ with $\deg(v_i) = 2, \forall v_i \in H$ and let $H_1 = \{v_1, v_2, \ldots, v_m\}$ be the set of all end block vertices in $b(T)$, such that $N[H \cup H_1] = V[T_b(T)]$. Then $D = \{H \cup H_1\}$ forms a minimal γ-set of T.

Further, let $H_2 = \{v_1, v_2, v_3, \ldots, v_p\}$ be the set of all end vertices of T adjacent to end block vertices of H_1 in $T_b(T)$, then $\forall v_j \in V[T_b(T)]$ $-\{H \cup H_2\}$ is adjacent to at least one vertex of $v_i \in \{H \cup H_2\}$ with $\deg(v_i) \leq \deg(v_j)$ and v_j is adjacent to v_i. Then $\{H \cup H_2\}$ forms a minimal weak semi-total block dominating set of T. Therefore $|H \cup H_2| \leq |J|$ $+|E \cup E'_1| - x(T) - 1$, gives $\gamma_{wtb}(T) \geq \gamma_{wb}(T) + \alpha_1(T) - 3$.

Theorem 2.9. *For any nontrivial tree* T, $\gamma_w(T) + \gamma_R(T) - 4 \leq \gamma_{wtb}(T)$ $+ \gamma'(T)$.

Proof. Let $D = \{v_1, v_2, v_3, \ldots, v_n\} \subseteq V(T)$ be the set of end vertices in T. Let $D_1 = V(T) - D$ be the set of minimum degree vertices in T and $D'_1 \subset D_1$, such that $N[D \cup D'_1] = V(T)$. Furthermore, $\forall v_p \in V[T] - \{D \cup D'_1\}$ is adjacent to at least one vertex $v_q \in \{D \cup D'_1\}$ such that $\deg(v_q) \leq \deg(v_p)$ and $N(v_p) \cap \{D \cup D'_1\} = \{v_q\}$. Hence $D \cup D'_1$ forms a minimal γ_w-set of T. Let $E = \{e_1, e_2, \ldots, e_n\} \subseteq E(T)$ be the set of edges and $H \subseteq E$ be the minimal set of edges which covers all the edges of (T). Then H forms a γ'-set of T. Further, let a function $f : V(T) \to \{0, 1, 2\}$ and partition the vertex set $V(T)$ into $\{V_0, V_1, V_2\}$ induced by f with $Vi = ni$, for $0, 1, 2$. Suppose the set V_2 dominates V_0. Then $R = V_1 \cup V_2$ forms a minimal Roman dominating set of T. For any tree T, each edge is a block. Let $A = \{B_1, B_2, \ldots, B_n\}$ be the blocks of T. Let $S = \{v_1, v_2, \ldots, v_m\}$ be the block vertices in $T_b(T)$ corresponding to the blocks of A, such that $|S| = \gamma_{wtb}$-set of T. Further, suppose $S_1 \subseteq S$ be the end bock vertices adjacent to D, in $T_b(T)$. Then $\forall v_j \in V[T_b(T)] - \{S \cup S_1\} \cup D$ is adjacent to at least one vertex of $v_i \in \{S - S_1\} \cup D$, such that $\deg(v_i) \leq \deg(v_j)$ and v_j is adjacent to v_i. Thus $\{S - S_1\} \cup D$ forms a γ_{wtb}-set of T. Hence $|\{S - S_1\} \cup D| + |H| \geq |D \cup D_1'| + |R| - 4$, gives $\gamma_{wtb}(T) + \gamma'(T) \geq \gamma_w(T) + \gamma_R(T) - 4$.

Theorem 2.10. *For any nontrivial tree T, then $\gamma_\varepsilon(T) + C \leq \gamma_{wtb}(T) + 3$, where C is the cut vertices of T.*

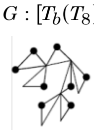
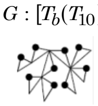
Proof. Let $C = \{c_1, c_2, \ldots, c_p\} \subseteq V(T)$ be the set of all cut vertices of. Then $C_1 \subseteq C$ forms a γ-set of T. Let $F \subseteq E(T)$ be the edge dominating set of T. Further, we consider $C_1' \subseteq C_1$ and $F' \subset F$, such that each element of T is adjacent or incident to at least one element of $\{C_1' \cup F'\}$. Then $\{C_1' \cup F'\}$ forms an entire dominating set of T. In $T_b(T)$, let $B = \{v_1, v_2, v_3, \ldots, v_m\}$ be the set of block vertices which are at a distance 2, such that $N[B] = V[T_b(T)]$. Then $|B| = \gamma_{tb}(T)$. Suppose, let $A = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of end block vertices in $T_b(T)$ and $A_1 = \{v_1, v_2, v_3, \ldots, v_p\}$ be the set of end vertices adjacent to A, with $\deg(v_i) = 2, \forall v_i \in A_1$. Then $\forall v_j \in V[T_b(T)] - \{B - A\} \cup A_1$ is adjacent to at least one vertex of $v_k, \forall v_k \in \{B - A\} \cup A_1$, such that $\deg(v_k) \leq \deg(v_j)$ and v_j is adjacent to v_k. Then $\{B - A\} \cup A_1$ forms a γ_{wtb} set of T. Clearly $|\{B - A\} \cup A_1| + 3 \geq |\{C_1' \cup F'\}| + |C|$, gives $\gamma_{wtb}(T) + 3 \geq \gamma_\varepsilon(T) + C$.

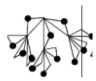
Theorem 2.11. *For any nontrivial tree T, then $\gamma_c'(T) + \beta_1(T) - 1 \leq \gamma_{wtb}(T)$ and $T \neq P_p$, $p \geq 8$.*

Proof. Suppose $T = P_p$ with $P \geq 8$. Then $\gamma_{wtb}(P_p) < \gamma'_c(P_p) + \beta_1(P_p) - 1$. Let $F = \{e_1, e_2, \ldots, e_n\} \subseteq E(T)$ be the set of all end edges and $F_1 = \{e_1, e_2, \ldots, e_m\}$ be the set of edges adjacent to the end edges of F, $F_2 = E(T) - \{F \cup F_1\}$. Suppose $F'_2 \subseteq F_2$. Then for every element $e_j \in E(T) - \{F_1 \cup F'_2\}$ is adjacent to at least one element $e_i \in \{F_1 \cup F'_2\}$. Hence $S = \{F_1 \cup F'_2\}$ forms a γ'-set of T.

Further, if the induced sub graph $\langle S \rangle$ has one component then S is a minimal γ'_c-set of T. Let $A \subseteq E(T)$ be the maximal independent set with $|A| = \beta_1(T)$. Suppose $H = \{v_1, v_2, \ldots, v_n\}$ be the block vertices in $T_b(T)$ corresponding to the blocks, $B = \{B_1, B_2, \ldots, B_n\}$ of T. Since in semi total block graph $T_b(T)$, each block is K_3, then $N[H] = V[T_b(T)]$. Then H is a γ_{tb}-set of. If H satisfies all the conditions of weak dominating set of T, then H is a γ_{wtb}-set of T. Hence $|H| \geq |S| + |A| - 1$, gives $\gamma_{wtb}(T) \geq \gamma'_c(T) + \beta_1(T) - 1$. It is also possible to satisfies the inequality of the theorem by considering. Suppose $H_1 = \{v_1, v_2, \ldots, v_m\} \subseteq H$ be the set of end block vertices and $C \subseteq V(T)$ be the end vertices of T. Then $\forall v_i \in V[T_b(T)] - \{H - H_1\} \cup C$ is adjacent to at least one vertex of $v_k \in \{H - H_1\} \cup C$ with $\deg(v_k) \leq (v_l)$. Hence $\{H - H_1\} \cup C$ is γ_{wtb}-set of T. Thus $|\{H - H_1\} \cup C| \geq |S| + |A| - 1$, which gives $\gamma_{wtb}(T) \geq \gamma'_c(T) + \beta_1(T) - 1$.

To envisage the results on trees, different graph parameters of some 3-ary trees tabulated below.

Graph: Tree	Δ (T)	Δ' (T)	γ (T)	γ_c (T)	γ_t (T)	i (T)	$Diam$ (T)	γ_s (T)	γ_{ss} (T)	γ_{ns} (T)	γ_{res} (T)	γ_w (T)	γ_{wtb} (T)
$G : [T_b(T_8)]$ 	3	4	3	4	6	3	4	2	3	5	5	5	8
$G : [T_b(T_{10})]$ 	4	5	3	4	6	3	5	3	4	7	7	7	9

$G : [T_b(T_{13})]$ 	4	5	3	4	4	3	5	3	4	10	2	10	12

$G : S_{m+1} \square P$	i (G)	γ (G)	γ_t (G)	γ_s (G)	γ_{ss} (G)	γ_{sns} (G)	γ_{en} (G)	γ_w (G)	γ_{dd} (G)	γ_{ns} (G)	γ_{res} (G)	γ_p (G)	γ_{wtb} (G)	γ_c (G)
$m = 3$ $T_b(G)$: 	4	2	2	2	5	6	5	3	4	4	2	4	3	2
$m = 4$ $T_b(G)$: 	5	2	2	2	5	8	6	4	4	5	2	5	4	2

The following results depict bounds on γ_{wtb} of general graph.

Theorem 2.12. *For any connected* (p, q) *graph* G, *with* $p \geq 2$ *vertices*, *then* $\gamma_{wtb}(G) \leq \gamma_{dd}(G) + \Delta(G) + 3$.

Proof. Let $D = \{v_1, v_2, v_3, \ldots, v_n\} \subseteq V(G)$ be the set of non end vertices with $\deg(v_i) \geq 2$, $\forall v_i \in D$, $1 \leq i \leq n$, such that $N[D] = V(G)$, then D forms a minimal dominating set of G. Let $V_1 = V(G) - D$ and $D_1 = \{v_1, v_2, v_3, \ldots, v_m\} \subseteq V_1$, then $D \cup D_1$ forms a double dominating set of G. For any graph G, there exists at least one vertex $v \in V(G)$ with $\deg(v) = \Delta(G)$. Let $A = \{v_1, \ldots, v_p\}$ be the block vertices in $T_b(G)$. Suppose $B = \{v_1, \ldots, v_q\}$ be the set of all end vertices in G and $B_1 = \{v_1, v_2, \ldots, v_l\}$ be the set of vertices adjacent to the end vertices of B in $T_b(G)$.

In $T_b(G)$, $\forall v_j \in V[T_b(G)] - \{A - B_1\} \cup B$ is adjacent at least one vertex of v_k, $v_k \in \{A - B_1\} \cup B$, such that $\deg(v_k) \leq \deg(v_j)$. Hence $S = \{A - B_1\} \cup$ forms a minimal γ- of G. Thus $|S| \geq |D \cup D_1| + |\deg(v)| + 3$, gives $\gamma_{wtb}(G) \leq \gamma_{dd}(G) + \Delta(G) + 3$.

In the following theorem we obtained lower bound for $\gamma_{wtb}(G)$ in terms of weak line domination number and minimum edge degree of G.

Theorem 2.13. *For any connected* (p, q) *graph* G, *with* $p \geq 3$ *vertices, then* $\gamma_{wl}(G) + 2 \leq \gamma_{wtb}(G) + \delta'(G)$.

Proof. Let $H = \{v_1, v_2, v_3, \ldots, v_n\} \subseteq V[L(G)]$ be the set of vertices with $\deg(v_i) \geq 1$, $\forall v_i \in H$, $1 \leq i \leq n$, such that $N[H] = V[L(G)]$. Then H forms a minimal dominating set of $L(G)$. Suppose there exists a set $H_1 \subseteq [L(G)] - H$, such that $\forall v_j \in H$, $\deg(v_i) \leq \deg(v_j)$, $\forall v_i \in H_1$ and v_i is adjacent to v_j. Then $\{H \cup H_1\}$ forms a minimal γ_{wl}-set of G. Let $e \in E(G)$ with $\deg(e) = \delta'(G)$.

Assume M be the set of end vertices. Now we consider the following cases:

Case 1. Suppose $M = \phi$. Let $\{B_1, \ldots, B_K\}$ be the number of blocks in G. In $T_b(G)$, let $A = \{v_1, \ldots, v_n\}$ be the set of block vertices corresponding to the blocks of G. Suppose $A_1 = \{v_1, v_2, \ldots, v_m\} \subset A$. Further let $B = v[T_b(G)] - A$ be the set of vertices with minimum degree, such that $\forall v_j \in v[T_b(G)] - \{A_1 \cup B\}$ is adjacent to at least one vertex of $v_i \in \{A_1 \cup B\}$ with $\deg(v_i) \leq \deg(v_j)$ and v_j is adjacent to v_i. Then $D = \{A_1 \cup B\}$ forms a γ_{wtb} set of G. Hence $|D| + |\deg(e)| \geq |\{H \cup H_1\} + 2$ gives the result.

Case 2. Suppose $M = \phi$. Let $C = \{v_1, \ldots, v_p\}$ be the set of all end vertices in G and $C_1 = \{v_1, \ldots, v_q\}$ be the set of vertices adjacent to the end vertices of C in $T_b(G)$. Then $\forall v_j \in v[T_b(G)] - \{D \cup C_1\}$ is adjacent to at least one vertex of $v_k, v_k \in \{D \cup C_1\}$ such that $\deg(v_k) \leq \deg(v_l)$ and v_l is adjacent to v_k. Then $D \cup C_1$ forms a minimal γ_{wtb}-set of G. Clearly $|D \cup C_1| + |\deg(e)| \geq |\{H \cup H_1\}| + 2$, gives $\gamma_{wtb}(G) + \delta'(G)\gamma_{wl}(G) + 2$.

Theorem 2.14. *For any connected graph* (p, q) *graph* G, *with* $p \geq 2$ *vertices, then* $\gamma_{wtb}(G) \leq \gamma_{sns}(G) + 1$.

Proof. Let $R = \{v_1, \ldots, v_n\} \subseteq V(G)$ be the set of all end vertices in G and

$S = \{V_1, \ldots, v_m\} \subseteq V(G)$ be the set of all non end vertices which are adjacent to the end vertices of G. Further, let $S_1 \subseteq (G)$ be the set of non end vertices which are not adjacent to end vertices of R and $S_1' \subseteq S_1$ be the set of vertices of G, such that $N[R \cup S_1'] = V(G)$. Hence $\{R \cup S_1'\}$ form a minimal γ-set of G. Further if the induced sub graph $\langle V(G) - \{R \cup S_1'\}\rangle$ is complete, then $\{R \cup S_1'\}$ itself is a γ_{sns}-set of G. Otherwise, if $R = \phi$, then $\{S_1'\}$ is a γ-set of G and if $\langle V(G) - S_1'\rangle$ is complete sub graph, then $\{S_1'\}$ is also a γ_{sns}-set of G. If not then select the $|B| \leq |S' \cup \{v_j\}| + 1$, gives $\gamma_{wtb}(G) \leq \gamma_{sns}(G) + 1$.

Theorem 2.15. *For any connected* (p, q) *graph* G, *with* $p \geq 2$ *vertices, then* $\gamma_{wtb}(G) \leq \gamma_{sn}(G) + \alpha_0(G) - 1$.

Proof. Let $B = \{v_1, \ldots, v_p\}$ be the vertex set of G. Suppose $D = \{v_1, v_2, v_3, \ldots, v_q\} \subseteq B$ be a minimal dominating set of G. Further, if the induced sub graph $\langle V - D\rangle$ is connected then D itself is a γ_{ns}-set of G. Otherwise, there exists a vertex $w \in V(G) - D$ with $\deg(w) = 0$, then $D \cup (w)$ forms a non split dominating set of G.

Let $C \subseteq B$ be the minimal set of vertices which covers all the edges of G with $|C| = \alpha_0(G)$. Let $S = \{v_1, v_2, \ldots, v_n\} \subseteq V[T_b(G)]$ be the set of vertices with minimum degree and $\forall v_i \in S, 1 \leq i \leq n$ is adjacent to at least one vertex of $v_j \in V[T_b(G)] - S$, such that $N[S] = V[T_b(G)]$. Furthermore, if $\deg(v_i) \leq \deg(v_j)$ with $N(v_j) \cap S = \{v_i\}$. Hence S forms minimal $\gamma_{wtb}(G) \leq \gamma_{ns}(G) + \alpha_0(G) - 1$.

Theorem 2.16. *For any connected* (p, q) *graph, with* $p \geq 2$ *vertices, then* $\gamma_c(G) \leq \gamma(G^2) \leq \gamma_{wtb}(G) + 1$ *and* $G \neq C_p$, $p \geq 5$.

Proof. Suppose $G = C_p$, $p \geq 5$. Then by Theorem 1, $\gamma_{wtb}(C_p) + 1 \ngeq \gamma_c(C_p) + \gamma(C_p^2)$, $p \geq 5$. Let $S = \{v_1, v_2, \ldots v_n) \subseteq V(G^2)$, such that $[s] = (G^2)$. Then S forms a minimal γ-set of G^2. Let $D \subseteq (G)$ be a minimal dominating set of G. Further, if the induced sub graph $\langle D\rangle$ has one component then D itself is a connected dominating set of G. Otherwise, there exists a vertex set $\{v_i\} \in V(G) - D$, such that $D \cup \{v_i\}$ is connected. Hence $D \cup \{v_i\}$ forms a connected dominating set of G. Let $B = \{B_1, B_2, \ldots, B_n\}$ be the blocks in G. Let $F = \{v_1, v_2, \ldots, v_m\}$ be the set of block vertices in $T_b(G)$ corresponding to the blocks of B, such that $N[S] = [T_b(G)]$. Then be the minimal dominating set of $T_b(G)$ and if, $\forall v_p \in F$ with $\deg(v_p) \leq \deg(v_q)$, $\forall \in v_q$, $V[T_b(G)] - F$ and v_q is adjacent to v_p. Then F forms γ_{wtb}-set of G. Thus $|F| + 1 \geq |D \cup \{v_i\}| + |S|$. Hence $\gamma_{wtb}(G) + 1 \geq \gamma_c(G) + \gamma(G^2)$. Otherwise, let $F' = [T_p(G)] - F$ be the set of vertices with minimum degree and satisfies the definition of weak dominating set. Then $F' = \gamma_{wtb}$-set, gives $|F^1| + 1 \geq |D \cup \{v_j\}| + |S|$ so that $\gamma_{wtb}(G) + 1 \geq \gamma_c(G) + \gamma(G^2)$.

Theorem 2.17. *For any connected (p, q) graph G, with $p \geq 2$ vertices, then $\gamma_{wtb}(G) \leq \gamma_e(G) + \gamma_p(G) + 2$.*

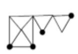
Proof. Suppose A be the vertex set of G and let $A_1 \subseteq A$ be the set of all end vertices in G. Further, if $V_1 = V(G) - A_1$ and $A_1' \subseteq V_1$, such that $N[A_1 \cup A_1'] = V(G)$. Then $\{A_1 \cup A_1'\}$ forms a minimal γ_e-set of G. Suppose $S \subseteq V(G)$ is a minimal dominating set of G. If each vertex $v \in V(G) - S$ is dominated by exactly one vertex of S. Then S forms a perfect dominating set of G. Let $D = \{v_1, v_2, \ldots, v_n\} \subseteq V[T_b(G)]$ be the set of vertices with minimum degree. Then D forms a γ_{tb}-set of G, such that $\forall v_j \in [T_b(G)] - D$, $\deg v_i \leq v_j$, $\forall v_i \in D$ with $N(v_j) \cap D = \{v_i\}$. Hence forms minimal weak semi-total block dominating set of G. Hence $|D| \leq |\{A_1 \cup A_1'\}| + |S| + 2$, gives $\gamma_{wtb}(G) \leq \gamma_e(G) + \gamma_p(G) + 2$.

Theorem 2.18. *For any connected (p, q) graph G, with $p \geq 3$ vertices, then $\gamma_s(G) + \beta_0(G) - 3 \leq \gamma_{wtb}(G)$ and $G \neq W_p, C_p$ with $p \geq 8$.*

Proof. For the graph $G = W_p, C_p$ with $p \geq 8$, $\gamma_{wtb}(G) < \gamma_s(G) + \beta_0(G) - 3$ and hence the result does not holds. Let $B = \{v_1, v_2, \ldots, v_p\} \subseteq V(G)$ be the set of all end vertices and

$C = \{v_1, v_2, \ldots, v_q\} \subseteq V(G) - B$. Further, consider the subset $C_1 \subseteq C$, such that $N(B) \cap N(C_1) = u \in V(G) - \{B \cup C_1\}$, then $\{B \cup C_1\}$ is an independent set with $|B \cup C_1| = \beta_0$. Suppose $D \subseteq V(G)$ be the minimal dominating set of G. Assume the induced sub graph $\langle V(G) - D \rangle$ has more than one component. Then D is a minimal split dominating set of G. Suppose $R = \{B_1, B_2, \ldots, B_n\}$ be the blocks of G. Let $S = \{v_1, v_2, \ldots, v_n\}$ be the block vertices in $T_b(G)$ corresponding to the blocks of R. Further, let S_1 be the set of vertices with minimum degree and $S_1' \subseteq S_1$, such that $N[S \cup S_1']$. $= V[T_b(G)]$ Then $S \cup S_1'$ forms a γ-set of $T_b(G)$ and if, $\forall v_i \in V[T_b(G)]$ $- S \cup S_1'$ is adjacent to at least one vertex of $v_i \in S \cup S_1'$, such that $\deg(v_i) \leq \deg(v_j)$. Hence $S \cup S_1'$ forms a minimal weak semi-total block dominating set of G. Thus $|S \cup S_1'| \geq |D| + |B \cup C_1| - 3$, gives $\gamma_{wt}(G) \geq \gamma_s(G) + \beta_0(G) - 3$.

In the following table, we have tabulated all domination parameters and other related parameters of some graphs. All the inequalities discussed in theorem 2.12 to 2.18 are evident from these values of parameters.

Graph G	$i(G)$	$\gamma(G)$	$\gamma_t(G)$	$\gamma_c(G)$	$\gamma_s(G)$	$\gamma_{ss}(G)$	$\gamma_{sns}(G)$	$\gamma_{en}(G)$	$\gamma_w(G)$	$\gamma_{dd}(G)$	$\gamma_{ns}(G)$	$\gamma_{res}(G)$	$\gamma_p(G)$	$\gamma_{wtb}(G)$
$G: T_{m,n}$ $m = 4$, $n = 2$ 	3	3	5	5	2	3	4	3	3	3	3	3	3	4
$G: T_{m,n}$ $m = 4$, $n = 2$ 	3	3	5	4	3	5	6	5	4	5	5	4	4	5

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