Advances and Applications in Mathematical Sciences
Volume 21, Issue 11, September 2022, Pages 6529-6543

ON THE SEMI-LOCAL CONVERGENCE OF A THIRD ORDER METHOD FOR SOLVING NONLINEAR EQUATIONS

IOANNIS K. ARGYROS[1], SANTHOSH GEORGE[2] and CHRISTOPHER ARGYROS[3]

[1]Department of Mathematical Sciences
Cameron University
Lawton, OK 73505, USA
E-mail: iargyros@cameron.edu

[2]Department of Mathematical
and Computational Sciences
National Institute of Technology
Karnataka, India-575 025
E-mail: sgeorge@nitk.edu.in

[3]Department of Computing and Technology
Cameron University
Lawton, OK 73505, USA
E-mail: christopher.argyros@cameron.edu

Abstract

Local convergence of high order methods for solving nonlinear equations defined on abstract spaces has been studied extensively by a plethora of authors. But this is not the case for semi-local convergence of these methods which is certainly a more interesting case. A technique is developed based on majorizing sequences and the notion of restricted Lipschitz condition to provide a semi-local convergence analysis for the third convergent order Noor-Waseem method. Due to the generality of our technique it can be used on other high order methods. Numerical applications complete this article.

1. Introduction

In this article we are concerned with the task of finding a solution x_* for

2020 Mathematics Subject Classification: 49M15, 47H17, 65J15, 65G99, 41A25.
Keywords: Third order method, nonlinear equation, majorizing sequence, semi-local convergence.
Received August 15, 2021; Accepted March 31, 2022

the nonlinear equation

$$\mathcal{L}(x) = 0. \tag{1.1}$$

where, $\mathcal{L} : D \subset \mathcal{T}_1 \to \mathcal{T}_2$ is a differentiable in the sense of Fréchet, $\mathcal{T}_1$ and $\mathcal{T}_2$ stand for Banach spaces and $D \neq \emptyset$ is an open set. Throughout the article $B(x_0, \rho) = \{x \in \mathcal{T}_1 : \| x - x_0 \| < \rho\}$ and $B[x_0, \rho] = \{x \in \mathcal{T}_1 : \| x - x_0 \| < \rho\}$ for some $\rho > 0$.

A plethora of applications from applied as well as the theoretical disciplines can be reduced to determining the point $x_* \in D$. But this task is very difficult in general. Moreover, the closed form of x_* is hard to find unless in special cases. This forces researchers and practitioners to resort to iterative approximations to x_*. A plethora of such approximations can be found in the literature [1-13]. Among those the most use full are the high convergence order ones. We notice that many local convergence results exist for these methods relying on Taylor expansions and derivatives of order at least one higher than the order of the method. As an example consider the third order Noor-Waseen method [8] defined by

$$\begin{aligned} x_* \in D,\ y_k &= x_k - \mathcal{L}'(x_k)^{-1}\mathcal{L}(x_k) \\ x_{k+1} &= x_k - 4A_k^{-1}\mathcal{L}(x_k), \end{aligned} \tag{1.2}$$

where $A_k = 3\mathcal{L}'\left(\frac{2x_k + y_k}{3}\right) + \mathcal{L}'(y_k)$.

The existence of derivatives up to fourth order has been assumed although derivatives of order two and above do not appear on method (1.2). Moreover, method (1.2) may converge even if derivatives other than the first do not exist. Consider the academic and motivational example: in the scalar case for $D = \left[-\frac{1}{2}, \frac{3}{2}\right]$.

$$f(t) = 0, \tag{1.3}$$

where

$$f(t) = \begin{cases} t^5 - t^4 + t^3 \log t^2 & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Then, notice that $x_* = 1 \in D$ and

$$f'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

But then, the third derivative of f is unbounded on D. Therefore, convergence is not assured by the results in [8]. There are no uniqueness of x_* results or error bounds on $\| x_n - x_* \|, \| y_n - x_n \|, \| x_{n+1} - x_n \|$ that can be computed. The same observations can be made for the local results of other methods [1-7, 9-13]. Hence, there is a need to develop results using conditions only on the first derivative that appears on these methods. These results should also provide the uniqueness of x_* and the error bounds in advance. Moreover, they should be given for the more interesting semi-local case. It turns out that these objectives can be achieved not only for (1.2) but for other methods too in a similar way. This is the novelty and motivation of our article. That is to expand the applicability under weaker conditions for these methods.

Majorizing sequences for method (1.2) are introduced and studied in Section 2. The semi-local convergence is given in Section 3 for method (1.2). Numerical applications appear in Section 4. Concluding remarks in Section 5 complete this article.

2. Majorizing Sequences

Scalar sequences are developed that majorize method (1.2). Let $L_0 > 0, L > 0, L_1 > 0$ and $\eta \geq 0$ be given constants. Define sequence $\{t_n\}$ by

$$t_0 = 0, s_0 = \eta$$

$$t_{n+1} = s_n + \frac{2L(s_n - t_n)^2}{1 - \frac{L_0}{6}(t_n - 2s_n)}$$

$$s_{n+1} = t_{n+1} + \frac{L(2(s_n - t_n) + t_{n+1} - t_n)(t_{n+1} - t_n)}{2(1 - L_0 t_{n+1})}. \tag{2.1}$$

Next, we present convergence criteria for sequence $\{t_n\}$.

Lemma 2.1. *Suppose that for all* $n = 0, 1, 2, \ldots$

$$L_0(t_n + 2s_n) < 6 \text{ and } L_0 t_{n+1} < 1. \tag{2.2}$$

Then, sequence $\{t_n\}$ *is nondecreasing bounded from above by* $\frac{1}{L_0}$ *and converges to its unique least upper bound* $t^* \in \left[0, \frac{1}{L_0}\right]$.

Proof. It follows by (2.1) and (2.2) that $0 \leq t_n \leq s_n \leq t_{n+1} < \frac{1}{L_0}$. Hence, we conclude $\lim_{n\to\infty} t_n = t^*$. □

The second convergence result contains stronger criteria than (2.1) but which are easier to verify. Consider recurrent polynomials on the interval $[0, 1)$ by

$$f_n^{(1)} = 2Lt^{n-1}\eta + \frac{L_0}{6}(3(1 + t + \ldots + t^n)\eta + 2t^n\eta) - 1,$$

$$f_n^{(2)}(t) = L(3+t)(1+t)t^{n-1}\eta + 2L_0(1 + t + \ldots + t^{n+1})\eta - 2,$$

$$g_1(t) = 2Lt - 2L + \frac{L_0}{6}(5t^2 - 2t)$$

and

$$g_2(t) = L(3+t)(1+t)t - L(3+t)(1+t) + 2L_0t^2.$$

It follows by these definitions that

$$g_1(0) = -2L,\ g_1(1) = \frac{L_0}{2},\ g_2(0) = -3L,\ g_2(t) = 2L_0.$$

The intermediate value theorem asserts that polynomials g_1 and g_2 have zeros in the interval $(0, 1)$. Denote by δ_1 and δ_2 the smallest such zeros. respectively.

It turns out that these polynomials are related.

Lemma 2.2. *The following assertions hold*

(1) $f_{n+1}^{(1)}(t) = f_n^{(1)}(t) + g_1(t)t^{n-1}\eta$, $f_{n+1}^{(1)}(t) = f_n^{(1)}(t)$ *at* $t = \delta_1$.

(2) $f_{n+1}^{(2)}(t) = f_n^{(2)}(t) + g_2(t)t^{n-1}\eta$,

$$f_{n+1}^{(2)}(t) = f_n^{(2)}(t) \ \text{at} \ t = \delta_2.$$

Proof. It follows by the definition of these polynomials in turn that:

(1) $f_{n+1}^{(1)}(t) = f_{n+1}^{(1)} - f_n^{(1)}(t) + f_n^{(1)}(t)$

$$= 2Lt^n\eta + \frac{L_0}{6}(3(1 + t + \ldots + t^{n+1})\eta + 2t^{n+1}\eta) - 1$$

$$- 2Lt^{n-1}\eta - \frac{L_0}{6}(3(1 + t + \ldots + t^n)\eta + 2t^n\eta) + 1 + f_n^{(1)}(t)$$

$$= f_n^{(1)}(t) + g_1(t)t^{n-1}\eta,$$

and

$$f_{n+1}^{(1)}(1) = f_n^{(1)}(t) \text{ at } t = \delta_1,$$

since $g_1(\delta_1) = 0$.

(2) $f_{n+1}^{(2)}(t) = f_{n+1}^{(2)} + L(3 + t)(1 + t)t^n\eta - L(3 + t)(1 + t)t^{n-1}\eta + 2L_0t^{n+1}\eta$

$$= f_n^{(2)}(t) + g_2(t)t^{n-1}\eta$$

and

$$f_{n+1}^{(2)}(t) = f_n^{(2)}(t) \text{ as } t = \delta_2,$$

since $g_2(\delta_2) = 0$. □

Let

$$a = \frac{2L\eta}{1 - \frac{L_0\eta}{3}}, \ b = \frac{L(2\eta + t_1)t_1}{2(1 - L_0t_1)}, \ \eta \neq 0, \ c = \max\{a, b\}$$

$$\delta_0 = \min\{\delta_1, \delta_2\}, \ \delta = \max\{\delta_1, \delta_2\}.$$

Lemma 2.3. *Suppose*

$$L_0 t_1 < 1,\ 2L_0\eta < 1 \tag{2.3}$$

and

$$0 \le c \le \delta_0 \le \delta \le 1 - L_0\eta. \tag{2.4}$$

Conditions (2.3) *and* (2.4) *determine the smallness of* η. *Then, sequence* $\{t_n\}$ *is nondecreasing, bounded from above by* $t^{**} = \frac{\eta}{1-\delta}$ *and converge to its unique least upper bound* $t^* \in \left[0, \frac{\eta}{1-\delta}\right]$. *The following error estimates also hold*

$$0 \le s_n - t_n \le \delta^n \eta \tag{2.5}$$

and

$$0 \le t_{n+1} - s_n \le \delta^{n+1}\eta. \tag{2.6}$$

Proof. The following items shall be shown using induction

$$0 \le \frac{2L(s_k - t_k)}{1 - \frac{L_0}{6}(t_k - 2s_k)} \le \alpha, \tag{2.7}$$

$$0 \le \frac{L(2(s_n - t_n) + (t_{n+1} - t_n))(t_{n+1} - t_n)}{2(1 - L_0 t_{n+1})} \le \gamma(s_n - t_n) \tag{2.8}$$

and

$$0 \le t_k \le s_k \le t_{k+1}. \tag{2.9}$$

These estimates hold for $k = 0$ by (2.1), the choice of a, b, conditions (2.3) and (2.4). It follows that $0 \le t_1 - s_0 \le \alpha(s_0 - t_0) = \alpha\eta$, $0 \le s_1 - t_1 \le \alpha(s_0 - t_0) = \alpha\eta$, so $t_1 \le \eta + \alpha\eta = \frac{1-\alpha^2}{1-\alpha}\eta < \frac{\eta}{1-\alpha} = t^{**}$. Suppose $0 \le s_k - t_k \le \alpha^k\eta, 0 \le t_{k+1} - s_k \le \alpha^{k+1}\eta$ and $t_{k+1} \le \frac{1-\alpha^{k+2}}{1-\alpha}\eta < t^{**}$. hold for all $k \le n$. Then, evidently (2.7) holds if

$$2L\alpha^k\eta + \frac{L_0}{6}\left[\frac{1-\alpha^{k+1}}{1-\alpha}\eta + 2\left(\frac{1-\alpha^{k+1}}{1-\alpha}\eta + \alpha^k\eta\right)\right] - \alpha \le 0.$$

or

$$f_k^{(1)}(t) \leq 0 \text{ at } t = \alpha. \tag{2.10}$$

Define function $f_\infty^{(1)}$ on the interval $[0, 1)$ by

$$f_k^{(1)}(t) = \lim_{k \to \infty} f_k^{(1)}(t).$$

It follows by the definition of $f_\infty^{(1)}$ and $f_k^{(1)}$ that

$$f_\infty^{(1)}(t) = \frac{L_0 \eta}{2(1-t)} - 1. \tag{2.11}$$

Then, by Lemma 2.2 (1) and (2.11), estimate (2.10) holds if

$$\frac{L_0 \eta}{2(1-t)} - 1 \leq 0 \text{ at } t = \alpha.$$

But this is true by the right hand side of (2.4). Similarly, (2.8) holds if

$$\frac{L[2(s_k - t_k) + (1+\gamma)(s_k - t_k)](1+\gamma)(s_k - t_k)}{2(1 - L_0 t_{k+1})} \leq \gamma(s_k - t_k),$$

or

$$\frac{L(3+\gamma)(1+\gamma)(s_k - t_k)}{2(1 - L_0 t_{k+1})} \leq \gamma,$$

or

$$L(3+\gamma)(1+\gamma)t^k \eta + 2\gamma L_0 \frac{1 - \gamma^{k+2}}{1-\gamma} \eta - 2\gamma \leq 0,$$

or

$$f_k^{(2)}(t) \leq 0 \text{ at } t = \gamma_2. \tag{2.12}$$

Define function $f_\infty^{(2)}$ on the interval $[0, 1)$ by

$$f_\infty^{(2)}(t) = \lim_{k \to \infty} f_k^{(2)}(t).$$

By the definition of $f_k^{(2)}$ and $f_\infty^{(2)}$, we get

$$f_\infty^{(1)}(t) = 2\left(\frac{L_0 \eta}{1-t} - 1\right).$$

Bence, (2.12) holds, if

$$\frac{L_0\eta}{1-t} - 1 \leq 0.$$

But this holds by the right hand side of (2.4). It then also follows from (2.1) that (2.9) holds. The induction for assertions (2.7) and (2.8) is completed. Hence, we deduce $\lim_{k\to\infty} t_k = t^* \leq t^{**}$. □

3. Semi-local Convergence

The conditions (H) are needed:

Assume:

(H1) There exists $x_0 \in D, \eta \geq 0$ such that $\mathcal{L}'(x_0)^{-1} \in L(T_2, T_1)$ and $\| \mathcal{L}'(x_0)^{-1}\mathcal{L}(x_0) \| \leq \eta$.

(H2) There exists $L_0 > 0$ such that for each $x \in D$

$$\| \mathcal{L}'(x_0)^{-1}(\mathcal{L}'(z) - \mathcal{L}'(x_0)) \| \leq L_0 \| z - x_0 \|.$$

Define $D_1 = U\left(x_0, \frac{1}{L_0}\right) \cap D$.

(H3) There exist $L > 0, L_1 > 0$ such that for each $u, w \in D_1$

$$\| \mathcal{L}'(x_0)^{-1}(\mathcal{L}'(u) - \mathcal{L}'(w)) \| \leq L_0 \| u - w \|,$$

and

$$\| \mathcal{L}'(x_0)^{-1}\mathcal{L}'(u) \| \leq L_1.$$

(H4) Conditions of Lemma 2.1 or Lemma 2.3 hold

and

(H5) $U[x_0, t^*] \subset D$.

Then, the following semi-local result for method (1.2) can be shown under conditions H.

Theorem 3.1. *Assume conditions H. Then, iteration* $\{x_n\}$ *given by* *method* (1.2) *is well defined in* $U[x_0, t^*]$, *remains in* $U[x_0, t^*]$ *for each* $n = 0, 1, 2, \ldots$ *and converges to a solution* x^* *of equation* $F(x) = 0$ *in* $U[x_0, t^*]$. *Moreover, the following assertions hold*

$$\| y_n - x_n \| \le s_n - t_n, \tag{3.1}$$

$$\| x_{n+1} - y_n \| \le t_{n+1} - s_n \tag{3.2}$$

and

$$\| x^* - x_n \| \le t^* - t_n. \tag{3.3}$$

Proof. By condition (H1) and (2.1),

$$\| y_0 - x_0 \| = \| \mathcal{L}'(x_0)^{-1}\mathcal{L}(x_0) \| \le \eta = s_0 - t_0.$$

Hence, (3.1) holds for $n = 0$ and $y_0 \in U[x_0, t^*]$. Let $u \in U[x_0, t^*]$. Using condition (h1) we get

$$\| \mathcal{L}'(u)^{-1}(\mathcal{L}'(u) - \mathcal{L}'(x_0)) \| \le L_0 \| u - x_0 \| \le L_0 t^* < 1,$$

so the Banach lemma on linear invertible operators [10] assures that $\mathcal{L}'(u)^{-1}$ exists and

$$\| \mathcal{L}'(u)^{-1}\mathcal{L}'(x_0) \| \le \frac{1}{1 - L_0 \| z - x_0 \|}. \tag{3.4}$$

We can write by method (1.2)

$$\begin{aligned} x_{k+1} &= y_k + \mathcal{L}'(x_k)^{-1}\mathcal{L}(x_k) \\ &\quad - 4A_k^{-1}\mathcal{L}(x_k) \\ &= y_k + (\mathcal{L}'(x_k)^{-1} - 4A_k^{-1})\mathcal{L}(x_k) \\ &= y_k + (4A_k^{-1} - \mathcal{L}'(x_k)^{-1})\mathcal{L}(x_k) \\ &= y_k - A_k^{-1}(4\mathcal{L}'(x_k) - A_k)\mathcal{L}'(x_k)^{-1}\mathcal{L}(x_k). \end{aligned} \tag{3.5}$$

Some estimates are needed assuming (3.1) and (3.2) for all $k \leq n$

$$4\mathcal{L}'(x_k) - 3\mathcal{L}'\left(\frac{2x_k + y_k}{3}\right) - \mathcal{L}'(y_k)$$

$$= 3\left[\mathcal{L}'(x_k) - \mathcal{L}'\left(\frac{2x_k + y_k}{3}\right)\right] - [\mathcal{L}'(x_k) - \mathcal{L}'(y_k)]$$

so by (H2) and (H3)

$$\left\| \mathcal{L}'(x_k)^{-1}\left[4\mathcal{L}'(x_k) - 3\mathcal{L}'\left(\frac{2x_k + y_k}{3}\right) - \mathcal{L}'(y_k)\right] \right\|$$

$$\leq 3L\left\| x_k - \frac{2x_k + y_k}{3} \right\| + L\| y_k - x_k \|$$

$$\leq 2L\| y_k - x_k \| \leq 2L(s_k - t_k); \tag{3.6}$$

$$\| (4\mathcal{L}'(x_0))^{-1}(A_k - 4\mathcal{L}'(x_0)) \|$$

$$\leq \frac{1}{4}\left[3\left\| \mathcal{L}'(x_0)^{-1}\left(\mathcal{L}'\left(\frac{2x_k + y_k}{3}\right) - \mathcal{L}'(x_0)\right) \right\|\right.$$

$$\left. + \| \mathcal{L}'(x_0)^{-1}(\mathcal{L}'(x_k) - \mathcal{L}'(x_0)) \|\right]$$

$$\leq \frac{1}{4} L_0\left[\left\| \frac{2x_k + y_k}{3} - x_0 \right\| + \| y_k - x_0 \|\right]$$

$$\leq \frac{1}{4} L_0\left[\frac{2\| x_k + x_0 \| + \| y_k - x_0 \|}{3} + \| y_k - x_0 \|\right]$$

$$\leq \frac{L_0}{4}\left[\frac{1}{3}(2t_k + s_k) + s_k\right]$$

$$= \frac{L_0}{6}(t_k + 2s_k) < 1,$$

so

$$\| A_k^{-1}\mathcal{L}'(x_0) \| \leq \frac{1}{1 - \frac{L_0}{2}(t_k + 2s_k)}. \tag{3.7}$$

Hence, by (2.1) and (3.5)-(3.7), we have

$$\| x_{k+1} - y_k \| \leq \frac{2L(s_k - t_k)(s_k - t_k)}{1 - \frac{L_0}{2}(t_k + 2s_k)} = t_{k+1} - s_k$$

and

$$\| x_{k+1} - x_0 \| \leq \| x_{k+1} - y_k \| + \| y_k - x_0 \|$$

$$\leq t_{k+1} - s_k + s_k - t_0 = t_{k+1} \leq t^*.$$

Hence, $x_{k+1} \in U[x_0, t^*]$ and (3.2) holds for $n = 0$. Hence, iterate y_{k+1} is well defined (by (3.4) for $u = x_{k+1}$). It follows from method (1.2) that

$$\mathcal{L}(x_{k+1}) = \mathcal{L}(x_{k+1}) - \mathcal{L}(x_k) + \mathcal{L}(x_k)$$

$$= \mathcal{L}(x_{k+1}) - \mathcal{L}(x_k) - \frac{1}{4} A_k(x_{k+1} - x_k)$$

$$= (b_k - \frac{1}{4} A_k)(x_{k+1} - x_k),$$

where $b_k = \int_0^1 \mathcal{L}'(x_k + \theta(x_{k+1} - x_k))d\theta$. We need an estimate:

$$4b_k - A_k = 4\int_0^1 \mathcal{L}'(x_k + \theta(x_{k+1} - x_k))d\theta - 3\mathcal{L}'\left(\frac{2x_k + y_k}{3}\right) - \mathcal{L}'(y_k)$$

$$= 3[\int_0^1 \mathcal{L}'(x_k + \theta(x_{k+1} - x_k))d\theta - \mathcal{L}'\left(\frac{2x_k + y_k}{3}\right)]$$

$$+ \int_0^1 \mathcal{L}'(x_k + \theta(x_{k+1} - x_k))d\theta - \mathcal{L}'(y_k).$$

So,

$$\| \mathcal{L}'(x_0)^{-1} \mathcal{L}'(x_{k+1}) \| \leq \frac{1}{4}\left\{3L\int_0^1 \| x_k - \left(\frac{2x_k + y_k}{3} + \theta(x_k - x_k)\right) \| d\theta\right.$$

$$\left. + L\int_0^1 \| x_k + \theta(x_{k+1} - x_k) - y_k \| d\theta\right\} \| x_{k+1} - x_k \|$$

$$\leq \frac{L}{2}(2(s_k - t_k) + (t_{k+1} - t_k))(t_{k+1} - t_k), \tag{3.8}$$

$$\| y_{k+1} - x_{k+1} \| \leq \frac{L(2(s_k - t_k) + (t_{k+1} - t_k))(t_{k+1} - t_k)}{2(1 - L_0 t_{k+1})} = s_{k+1} - t_{k+1}$$

and

$$\| y_{k+1} - x_0 \| \leq \| y_{k+1} - x_{k+1} \| + \| x_{k+1} - x_0 \|$$

$$\leq s_{k+1} - t_{k+1} + t_{k+1} - t_0 = s_{k+1} \leq t^*.$$

These computations complete the induction for (3.1) and (3.2). Sequence $\{t_k\}$ is fundamental as convergent. Then, so is $\{x_k\}$ (in a Banach space T_1). Hence, there exists $x^* \in U[x_0, t^*]$ such that $\lim_{k \to \infty} x_k = x_k = x^*$. Then, by letting $k \to \infty$ in (3.8) we get $F(x^*) = 0$, where we also used the continuity of F. □

Remark 3.2. The limit point t^* can be replaced by $\frac{1}{L_0}$ and $\frac{\eta}{1 - \alpha}$ given in closed form in (h5) under the conditions of Lemma 2.1 and Lemma 2.3, respectively.

The uniqueness of the solution result follows.

Proposition 3.3. *Assume*:

(1) *There exists element* $x^* \in U(x_0, \rho_0) \subset D$ *for some* $\rho_0 > 0$ *which is a simple solution for equation* $\mathcal{L}(x) = 0$.

(2) *Condition* (H2) *holds*.

(3) *There exists* $\rho_1 \geq \rho_0$ *such that*

$$\frac{L_0}{2}(\rho_0 + \rho_1) < 1. \tag{3.9}$$

Define $D_2 = D \cap U[x, \rho_1]$. *Then, the element* x^* *is the only solution of equation* $\mathcal{L}(x) = 0$ *in the set* D_2.

Proof. Assume there exists $u^* \in D_2$ satisfying $\mathcal{L}(u^*) = 0$. Define linear operator $M = \int_0^1 \mathcal{L}'(u^* + t(x^* - u^*))dt$. Then, in view of (H2) and (3.9), we obtain in turn

$$\| \mathcal{L}'(x_0)^{-1}(M - \mathcal{L}'(x_0)) \| \le L_0 \int_0^1 [(1-\theta)(\| u^* - x_0 \|) + \theta \| x^* - x \|] d\theta$$

$$\le \frac{L_0}{2}(\rho_0 + \rho_1) < 1.$$

Therefore, operator M is invertible. Hence, also using the identity $0 = \mathcal{L}(u^*) - \mathcal{L}(x^*) = M(u^* - x^*)$, we deduce that $u^* = x^*$. □

Remark 3.4. Notice that not all conditions (H) are used in Proposition 3.3. But if they were used, then we can certainly set $\rho_0 = t^*$.

4. Numerical Experiment

We verify the conditions of Lemma 2.1 for an example in this section.

Example 4.1. Let us consider a scalar function $\mathcal{L}$ defined on the set $D = U[u_0, 1-s]$ for $s \in (0, 1)$ by

$$\mathcal{L}(x) = x^3 - s.$$

Choose $\gamma = 2$ and $u_0 = 1$. Then, we obtain the estimates $\eta = \frac{1-s}{3}$,

$$| \mathcal{L}'(u_0)^{-1}(\mathcal{L}'(x) - \mathcal{L}'(u_0)) | = | x^2 - u_0^2 |$$

$$\le | x + u_0 || x - u_0 | \le (| x - u_0 | + 2| u_0 |)| x - u_0 |$$

$$= (1 - s + 2)| x - u_0 | = (3 - s)| x - u_0 |,$$

so $L_0 = 3 - s$,

$$| \mathcal{L}'(u_0)^{-1}(\mathcal{L}'(y) - \mathcal{L}'(x)) | = | y^2 - x^2 |$$

$$\le | y + x || y - x | \le (| y - u_0 + x - u_0 + 2u_0 |)| y - x |$$

$$= (| y - u_0 | + | x - u_0 | + 2| u_0 |)|y - x |$$

$$\leq \left(\frac{1}{L_0} + \frac{1}{L_0} + 2\right) | y - x | = 2\left(1 + \frac{1}{L_0}\right) | y - x |,$$

for each $x, y \in, D$ and so $L = 2\left(1 + \frac{1}{L_0}\right)$.

$$| \mathcal{L}'(u_0)^{-1}(\mathcal{L}'(y) - \mathcal{L}'(x)) | = (| y - u_0 | + | x - u_0 | + 2| u_0 |)| y - x |$$

$$\leq (1 - s + 1 - s + 2)| y - x | = 2(2 - s)| y - x |,$$

for each $x, y \in, D$ so $L_1 = (2 - s)^2$. Then, for $s = 0.95$ we have

Table 1. Sequence (2.1) and condition (2.2).

n	1	2	3	4	5	6
t_{n+1}	0.0183	0.0193	0.0193	0.0193	0.0193	0.0193
s_n	0.0167	0.0193	0.0193	0.0193	0.0193	0.0193
$L_0(t_n + 2s_n)$	0.0683	0.1169	0.1189	0.1189	0.1189	0.1189
$L_0 t_{n+1}$	0.0376	0.396	0.0396	0.0396	0.0396	0.0396

Hence, the conditions of Lemma 2.1 hold.

References

[1] I. K. Argyros, On the Newton-Kantorovich hypothesis for solving equations, J. Comput. Math. 169 (2004), 315-332.

[2] I. K. Argyros, Computational theory of iterative methods, Series: Studies in Computational Mathematics, 15, Editors: C. K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[3] I. K. Argyros, Convergence and Applications of Newton-type Iterations, Springer Verlag, Berlin, Germany, (2008).

[4] I. K. Argyros and S. Hilout, Weaker conditions for the convergence of Newton's method, J. Complexity 28 (2012), 364-387.

[5] I. K. Argyros and S. Hilout, On an improved convergence analysis of Newton's method, Applied Mathematics and Computation 225 (2013), 372-386.

[6] I. K. Argyros, Unified convergence criteria for iterative banach space valued methods with applications, Mathematics 9(16) (2021), 1942.

[7] R. Behl, P. Maroju, E. Martinez and S. Singh, A study of the local convergence of a fifth order iterative method, Indian J. Pure Appl. Math. 51(2) (2020), 439-455.

[8] M. A. Noor and M. Waseem, Some iterative methods for solving a system of nonlinear equations, Comput. Math. Appl. 57(1) (2009), 101-106.

[9] A. A. Magrenan and J. M. Gutierrez, Real dynamics for damped Newton's method applied to cubic polynomials, J. Comput. Appl. Math. 275 (2015), 527-538.

[10] J. M. Ortega and W. C. Rheinboldt, Iterative solution of nonlinear equations in several variables, volume 30 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (2000), Reprint of the 1970 original.

[11] A. M. Ostrowski, Solution of equations and systems of equations, Second edition, Pure and Applied Mathematics, vol. 9, Academic Press, New York-London, (1966).

[12] M. S. Petkovic, B. Neta, L. D. Petkovic and J. Dzunic, Multipoint methods for solving nonlinear equations: a survey, Appl. Math. Comput. 226 (2014), 635-660.

[13] J. F. Traub, Iterative methods for the solution of equations, Prentice-Hall, New Jersey, U. S. A., (1964).