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HARMONIC CENTRALITY IN SOME GRAPH FAMILIES

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Abstract

One of the more recent measures of centrality in social network analysis is the normalized harmonic centrality. A variant of the closeness centrality, harmonic centrality sums the inverse of the geodesic distances of each node to other nodes where it is 0 if there is no path from one node to another. It is then normalized by dividing it by $m-1$, where m is the number of nodes of the graph. In this paper, we present notions regarding the harmonic centrality of some important classes of graphs.

1. Introduction

In graph theory and social network analysis, the notion of centrality is based on the importance of a node in a graph. In 1978, Freeman [3] expounded on the concept of centrality being an important attribute of social networks and its characteristics relate to other important properties and processes. Rodrigues [6], however, discussed centrality as not having a formal definition and may not be unique. A street corner in an urban network may be considered central when it is the most accessed part of the map. While in a

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social network, a celebrity or politician is considered central because she can easily share information with her millions of followers with a simple click of a button. Since there is no agreed upon definition of centrality, several measures have been proposed, each with its strengths and qualities.

The most common measures of centrality include degree centrality, closeness centrality, betweenness centrality, eigenvector centrality, and Page Rank centrality. Wasserman and Faust [2] categorized these measures according to whether they are for non-directional relations or directional relations. Some authors classified them according to whether they are degree-based or shortest path-based types of centrality measures.

One of the more recent measures of centrality is Harmonic Centrality. Introduced in 2000 by Marchiori and Latora [4], it is a variant of closeness centrality. While closeness centrality of a vertex u is defined as the reciprocal of the average length of the shortest path between u and all other vertices in G, harmonic centrality reverses this and measures the sum of the reciprocals of the distances of u from each vertex in G. It was invented to solve the problem of dealing with unconnected graphs by equating the reciprocal to 0 if there is no path from one node to another. The harmonic centrality is normalized by dividing it by $m - 1$, where m is the number of nodes in the graph. For related works with closeness and betweenness centrality of some graph families, see [1], [5], and [7].

In this paper, we present notions regarding the harmonic centrality of some important classes of graphs.

2. Preliminaries

For formality, we provide some definitions of the main concepts discussed in this paper.

Definition 2.1 (Harmonic Centrality of a Graph)**.** Let $G = (V(G), E(G))$ be a nontrivial graph of order m. If $u \in V(G)$, then the harmonic centrality of vertex u is given by the expression

$$\mathcal{H}_G(u) = \frac{\mathcal{R}_G(u)}{m - 1}$$

where $\mathcal{R}_G(u) = \sum_{u \neq x} \frac{1}{d(u, x)}$ is the sum of the reciprocals of the shortest distances $d(u, x)$ in G between vertices u and x, for all $x \neq u$, with $\frac{1}{d(x, u)} = 0$ in case there is no path from u to x in G.

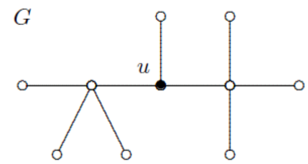

Figure 1. A caterpillar graph G with $u \in V(G)$, where $\mathcal{H}_G(u) = \frac{2}{3}$.

Definition 2.2 (Harmonic Number H_n).

The n-th harmonic number H_n is the sum of the reciprocals of the first n natural numbers, that is

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

In this paper, the harmonic centrality of some special graphs such as path P_m, cycle C_m, fan F_m, wheel W_m, complete bipartite graph $K_{m,n}$, ladder L_m, crown Cr_m, prism Y_m, star S_m, book B_m, and helm graph H_m are derived. Each graph considered is simple, finite, and undirected.

The path P_m of order m is a graph with distinct vertices $a_1, a_2, \ldots, a_m$ and edges $a_1a_2, a_2a_3, \ldots, a_{m-1}a_m$. The cycle C_m of order $m \geq 3$ is a graph with distinct vertices $a_1, a_2, \ldots, a_m$ and edges $a_1a_2, a_2a_3, \ldots, a_{m-1}a_m, a_ma_1$. ama1. The fan F_m of order $m+1$, where $m \geq 3$, is formed by adjoining one vertex u_0 to each vertex of path $P_m = [u_1, u_2, \ldots, u_m]$. Figure 2 shows the skeletal graph for path, cycle, and fan graphs.

The wheel graph W_m of order $m+1, m > 3$, formed by adjoining one vertex u_0 to each vertex of cycle $C_m = [u_1, u_2, \ldots, u_m]$. The complete bipartite graph $K_{m,\,n}$ where both $m, n \geq 2$, $V(K_{m,\,n}) = \{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_2, \ldots, v_n\}$, $E(K_{m,\,n}) = \{u_i v_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. The ladder L_m, of order $2m$, formed as the Cartesian product of a path graph $P_m = [u_1, u_2, \ldots, u_m]$ with the path graph $P_2 = [v_1, v_2]$. Figure 3 shows the skeletal diagrams for a wheel, complete bipartite graph, and a ladder graph.

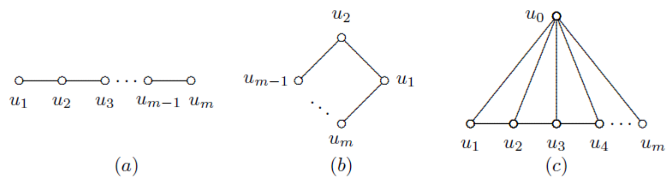

Figure 2. (a) Path P_m, (b) Cycle C_m, and (c) Fan F_m.

The crown graph Cr_m of order $2m$ with $V(Cr_m) = \{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_2, \ldots, v_n\}$ and whose edges are formed by adjoining u_i to v_j whenever $i \neq j$. The prism graph Y_m, of order $2m$ with $m \geq 3$, formed as the Cartesian product of a cycle graph $C_m = [u_1, u_2, \ldots, u_m]$ with the path graph $P_2 = [v_1, v_2]$. The star graph S_m, of order $m + 1$, $m > 1$, formed by adjoining m isolated vertices u_i, $1 \leq i \leq m$, to a single vertex u_0. Figure 4 shows the skeletal diagrams for a crown, prism, and star graph.

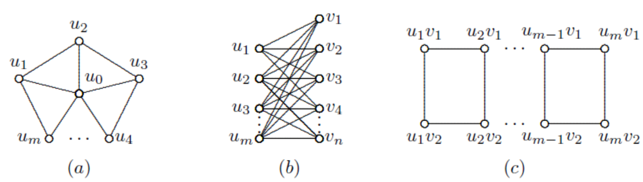

Figure 3. (a) Wheel graph W_m, (b) Complete bipartite graph $K_{m,\,n}$, and (c) Ladder L_m.

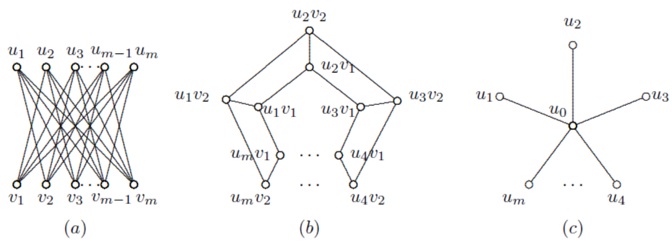

Figure 4. (a) Crown Cr_m, (b) Prism graph Y_m, and (c) Star S_m.

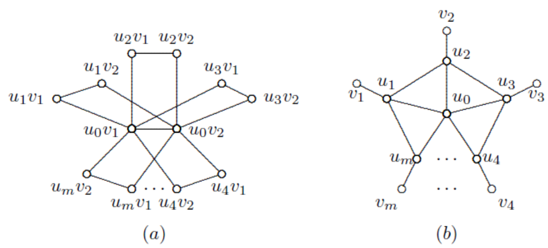

Figure 5. (a) Book B_m, and (b) Helm graph H_m.

The book graph B_m of order $2(m+1)$, formed as the Cartesian product of a star graph S_m (with center vertex u_0) with path graph $P_2 = [v_1, v_2]$. The helm graph H_m, $m \geq 3$, is obtained by adjoining a pendant vertex at each node of the m-ordered cycle of wheel W_m, with vertices $V(H_m) = [u_0, u_1, \ldots, u_m] \cup [v_1, v_2, \ldots, v_m]$. Figure 5 shows the skeletal diagrams for a book graph and a helm graph.

3. Main Results

In this section, we present some properties of harmonic centrality and their application to some graph families. Graphs considered in this paper are the path P_m, cycle C_m, fan F_m, wheel W_m, complete bipartite graph $K_{m,n}$ ladder L_m, crown Cr_m, prism Y_m, star S_m, book B_m, and helm graph H_m.

Theorem 3.1. *Let G be a nontrivial graph. Then each $u \in V(G)$ satisfies* $0 \leq \mathcal{H}_G(u) \leq 1$.

Proof. Let $u \in V(G)$, then the vertex set $V(G)$ can be partitioned into three subsets, namely, (i) $N_G(u)$, (ii) $V(G) \setminus N[u]$, and (iii) the singleton vertex u.

(i) For the open neighborhood $N_G(u)$,

$$\mathcal{R}_G(u) = \sum_{x \in N_G[u]} \frac{1}{d_G(u, x)} = \deg u$$

(ii) For $V(G) \setminus N[u]$, there will be two cases. Case 1, if $V(G) \setminus N[u] \neq \emptyset$, then this case is the same with the first partition and we obtain

$$\mathcal{H}_G(u) = \frac{\mathcal{R}_G(u)}{m-1} = \frac{\deg u}{m-1} = \frac{m-1}{m-1} = 1$$

For Case 2, if $V(G) \setminus N[u] \neq \emptyset$, then,

$$\begin{aligned}\mathcal{R}_G(u) &= \sum_{x \in N_G(u)} \frac{1}{d_G(u, x)} + \sum_{x \in V(G) \setminus N_G[u]} \frac{1}{d_G(u, x)} \\ &< \deg_G(u, x) + 1[m - 1 - \deg_G(u)] \\ &= m - 1\end{aligned}$$

Thus, from this case we obtain

$$\mathcal{H}_G(u) = \frac{\mathcal{R}_G(u)}{m-1} \leq \frac{m-1}{m-1} = 1$$

(iii) To complete graph G, we consider the singleton vertex u. Note that $\mathcal{H}_G(u) = 0$ wherever u is not adjacent to any vertex in G.

Thus, combining these cases we get $0 \leq \mathcal{H}_G(u) \leq 1$, for all $u \in V(G)$. ■

Theorem 3.2. *Let G be a nontrivial graph of order m and let $u \in V(G)$. Then $\mathcal{H}_G(u) = 1$ if and only if $\deg_G(u) = m - 1$.*

Proof. If $\mathcal{H}_G(u) = 1$, then $\frac{1}{m-1} \sum_{x \in N_G[u]} \frac{1}{d(x, u)} = 1$ and $\mathcal{R}_G(u) = \sum_{x \in N_G[u]} \frac{1}{d(x, u)} = m - 1$. This result can only happen whenever $\deg(u) = m - 1$, which means that vertex u is adjacent and has a distance of 1 with all other vertices in G.

Assuming that $\deg(u) \neq m-1$, then there exists at least one vertex x of G not adjacent to u. For each $x \in V(G) \setminus N_G[u]$, we have $\frac{1}{d(x, u)} < 1$.

$$\text{Now, } \mathcal{R}_G(u) = \sum_{x \in N_G[u]} \frac{1}{d_G(u, x)} + \sum_{x \in V(G) \setminus N_G[u]} \frac{1}{d_G(u, x)}$$

$$< \deg_{(G)}(u, x) + 1[m - 1 - \deg_G(u)]$$

$$= m - 1. \qquad \blacksquare$$

Thus, $\mathcal{R}_G(u) < m-1$ so that $\mathcal{H}_G(u) \frac{\mathcal{R}_G(u)}{m-1} \neq 1$.

Corollary 3.3. *Let G be a nontrivial connected graph of order m. Then $\mathcal{H}_G(u) = 1$ for every $u \in V(G)$ if and only if $G = K_m$, where K_m is the complete graph of order m.*

Proof. This follows from Theorem 3.2 since the degree of each vertex of a complete graph is $m-1$. ∎

Theorem 3.4. *For the path $P_m = [u_1, u_2, \ldots, u_m]$ of order $m \geq 2$, the harmonic centrality of any vertex u_i, $1 \leq i \leq m$, is given by*

$$\mathcal{H}_{P_m}(u_i) = \begin{cases} \dfrac{H_{m-1}}{m-1} & \text{if } i = 1 \text{ or } i = m \\ \dfrac{H_{i-1} + H_{m-i}}{m-1} & \text{if } 1 < i < m. \end{cases}$$

Proof. Consider a path $P_m = [u_1, u_2, \ldots, u_m]$ of order $m \geq 2$, then $\mathcal{R}_{P_m}(u_1) = \mathcal{R}_{P_m}(u_m) = 1 + \frac{1}{2} + \ldots + \frac{1}{m-1} = \sum_{k=1}^{m-1} \frac{1}{k} = H_{m-1}$.

$$\mathcal{R}_G(u) = \sum_{j=1}^{m} \frac{1}{d_{P_m}(u_i, u_j)}$$

$$= \sum_{j=1}^{i-1} \frac{1}{d_{P_m}(u_i, u_j)} + \sum_{j=1}^{m} \frac{1}{d_{P_m}(u_i, u_j)}$$

$$= \left[\frac{1}{i-1} + \frac{1}{i-2} + \ldots + \frac{1}{3} + \frac{1}{2} + 1\right] + \left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m-i}\right]$$

$$= \sum_{j=1}^{i-1} \frac{1}{j} + \sum_{j=1}^{m-i} \frac{1}{j}$$

$$= H_{i-1} + H_{m-1}$$

Thus,

$$\mathcal{H}_{P_m}(u_i) = \begin{cases} \dfrac{H_m}{m-1} & \text{if } i = 1 \text{ or } i = m \\ \dfrac{H_{i-1} + H_{m-i}}{m-1} & \text{if } 1 < i < m. \end{cases}$$ ■

Theorem 3.5. *For the cycle* $C_m = [u_1, u_2, \ldots, u_m, u_1]$, *the harmonic centrality of any vertex* u_i, $1 \leq i \leq m$ *is given by*

$$\mathcal{H}_{C_m}(u_i) = \begin{cases} \dfrac{2}{m-1}(H_{\frac{m-1}{2}}) & \textit{if } m \textit{ is odd} \\ \dfrac{2}{m-1}(H_{\frac{m-2}{2}} + \dfrac{1}{m}) & \textit{if } m \textit{ is even.} \end{cases}$$

Proof. To prove this theorem, we need to consider the structure of a cycle C_m. For when m is odd, a vertex u's distances from each x will be the same on both sides and $\mathcal{R}_{C_m}(u_i)$ can be computed as follows:

$$\mathcal{R}_{C_m}(u_i) = \sum_{x \in V(C_m)} \frac{1}{d_{C_m}(u_i, x)} = 2\left[1 + \frac{1}{2} + \ldots + \frac{1}{\frac{m-1}{2}}\right] = 2H_{\frac{m-1}{2}}$$

On the other hand, if m is even, then a vertex directly opposite u exists and $\mathcal{R}_{C_m}(u_i)$ can be computed as follows:

$$\mathcal{R}_{C_m}(u_i) = \sum_{x \in V(C_m)} \frac{1}{d_{C_m}(u_i, x)} = 2\left[1 + \frac{1}{2} + \ldots + \frac{1}{\frac{m-1}{2}}\right] + \frac{1}{\frac{m}{2}} = 2H_{\frac{m-2}{2}} + \frac{2}{m}$$

Normalizing these two cases form the harmonic centrality of cycle graph

C_m. ∎

Theorem 3.6. *For the fan graph F_m of order $m+1$, $m > 2$, formed by adjoining one vertex u_0 to each vertex of path $P_m = [u_1, u_2, \ldots, u_m]$, the harmonic centrality of any vertex $u_i \in V(F_m)$ is given by*

$$\mathcal{H}_{F_m}(u_i) = \begin{cases} 1 & \text{if } i = 0 \\ \dfrac{m+2}{2m} & \text{if } i = 1 \text{ or } i = m \\ \dfrac{m+3}{2m} & \text{if } 1 < i < m. \end{cases}$$

Proof. Given that u_0 is adjacent to each vertex of path P_m, then by Theorem 3.2, $\mathcal{H}_{F_m}(u_0) = 1$. For $u_1, u_m \in V(F_m)$, we have

$$\mathcal{R}_{F_m}(u_1) = \mathcal{R}_{F_m}(u_m) = \sum_{x \in V(F_m)} \frac{1}{d_{F_m}(u_i, x)}$$

$$= \frac{1}{d(u_1, u_0)} + \frac{1}{d(u_1, u_2)} + \frac{1}{d(u_1, u_3)} + \ldots + \frac{1}{d(u_1, u_m)}$$

$$= 1 + 1 + \underbrace{\frac{1}{2} + \ldots + \frac{1}{2}}_{m-2 \text{ addends}}$$

$$= 2 + \frac{1}{2}(m-2)$$

$$= \frac{m+2}{2}$$

For $u_i \in V(F_m)$, if $1 < i < m$, let us consider u_2 we have

$$\mathcal{R}_{F_m}(u_i) = \sum_{x \in V(F_m)} \frac{1}{d_{F_m}(u_i, x)}$$

$$= \frac{1}{d(u_2, u_0)} + \frac{1}{d(u_2, u_1)} + \frac{1}{d(u_2, u_3)} + \ldots + \frac{1}{d(u_2, u_m)}$$

$$= 1 + 1 + 1 + \underbrace{\frac{1}{2} + \ldots + \frac{1}{2}}_{m-3 \text{ addends}}$$

$$= 3 + \frac{1}{2}(m - 3)$$

$$= \frac{m + 3}{2}$$

To normalize, we divide $\mathcal{R}_{F_m}(u)$ by m because fan graph F_m has an order of $m + 1$. ∎

Theorem 3.7. *For the wheel graph* W_m *of order* $m + 1$, $m > 3$, *formed by adjoining one vertex* u_0 *to each vertex of cycle* $C_m = [u_1, u_2, \ldots, u_m]$, *the harmonic centrality of any vertex* $u_i \in V(W_m)$ *is given by*

$$\mathcal{H}_{W_m}(u_i) = \begin{cases} 1 & \text{if } i = 0 \\ \dfrac{m + 3}{2m} & \text{if } 1 \leq i \leq m. \end{cases}$$

Proof. Given that u_0 is adjacent to each vertex of cycle C_m, then by Theorem 3.2, $\mathcal{H}_{W_m}(u_0) = 1$. For $u_i \in V(F_m)$, if $1 \leq i \leq m$, we have

$$\mathcal{R}_{W_m}(u_i) = \sum_{x \in V(W_m)} \frac{1}{d_{W_m}(u_i, x)}$$

$$= \frac{1}{d(u_i, u_0)} + \frac{1}{d(u_i, u_{i-1})} + \frac{1}{d(u_i, u_{i+1})} + \ldots + \frac{1}{d(u_i, u_m)}$$

$$= 1 + 1 + 1 + \underbrace{\frac{1}{2} + \ldots + \frac{1}{2}}_{m-3 \text{ addends}}$$

$$= 3 + \frac{1}{2}(m - 3)$$

$$= \frac{m + 3}{2}$$

We normalize the wheel harmonic centrality by dividing it my m since W_m has $m + 1$ vertices. ∎

Theorem 3.8. *For the complete bipartite graph* $K_{m,\,n}$ *where both* $m,\, n \geq 0,\, V(K_{m,\,n}) = \{u_1,\, u_2,\, \ldots,\, u_m\} \cup \{v_1,\, v_2,\, \ldots,\, v_n\},\, E(K_{m,\,n}) =$ $\{u_i v_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$, *the harmonic centrality of vertices* u_i *and* v_j *are given by*

$$\mathcal{H}_{K_{m,\,n}}(u_i) = \frac{m + 2n - 1}{2(m + n - 1)} \text{ and } \mathcal{H}_{K_{m,\,n}}(v_j) = \frac{m + 2n - 1}{2(m + n - 1)}.$$

Proof. Considering the specific structure of $K_{m,\,n}$ and the way the partite sets $\{u_1,\, u_2,\, \ldots,\, u_m\}$ and $\{v_1,\, v_2,\, \ldots,\, v_n\}$ are arranged, for $u_i \in V_1(K_{m,\,n})$, we have

$$\mathcal{H}_{K_m}(u_i) = \frac{1}{m + n - 1}\left(\sum_{y \in V_2(K_{m,\,n})} \frac{1}{d_{K_{m,\,n}}(u_i,\, y)} + \sum_{x \in V_1(K_{m,\,n})} \frac{1}{d_{K_{m,\,n}}(v_j,\, x)}\right)$$

$$= \frac{1}{m + n - 1}\left((m - 1)\left(\frac{1}{2}\right) + n\right)$$

$$= \frac{m + 2n - 1}{2(m + n - 1)}$$

As for $v_j \in V_2(K_{m,\,n})$, we have

$$\mathcal{H}_{K_m}(v_i) = \frac{1}{m + n - 1}\left(\sum_{y \in V_2(K_{m,\,n})} \frac{1}{d_{K_{m,\,n}}(u_i,\, y)} + \sum_{x \in V_1(K_{m,\,n})} \frac{1}{d_{K_{m,\,n}}(v_j,\, x)}\right)$$

$$= \frac{1}{m + n - 1}\left(m(n - 1)\left(\frac{1}{2}\right) + n\right)$$

$$= \frac{2m + 2n - 1}{2(m + n - 1)}. \qquad \blacksquare$$

Theorem 3.9. *For the ladder graph* L_m *of order* $2m$, *formed as the Cartesian product of a path graph* $P_m = [u_1,\, u_2,\, \ldots,\, u_m]$ *with the path graph* $P_2 = [v_1,\, v_2]$, *the harmonic centrality of any vertex* $(u_i,\, v_j)$ *given by*

$$\mathcal{H}_{L_m}(u_i, v_j) = \begin{cases} \frac{1}{2m-1}\left(2H_{m-1} + \frac{1}{m}\right) & \text{for } i = 1 \text{ or } i = m, 1 \le j \le 2 \\ \frac{1}{2m-1}\left[2(H_{i-1} + H_{m-i}) + \frac{1}{i} + \frac{1}{m-i+1} - 1\right] & \text{for } 1 < i < m, 1 \le j \le 2. \end{cases}$$

Proof. Considering the structure of a ladder graph, we can group the vertices according the paths they belong to, that is, $[(u_1, v_1), (u_2, v_1), \ldots, (u_m, v_1)] \in V_1(L_m)$ and $[(u_1, v_2), (u_2, v_2), \ldots, (u_m, v_2)] \in V_2(L_m)$, so for (u_1, v_j) and (u_m, v_j) we have

$$\begin{aligned} \mathcal{R}_{L_m}(u_1, v_j) = \mathcal{R}_{L_m}(u_m, v_j) & \\ &= \sum_{x \in V_1(L_m)} \frac{1}{d_{L_m}((u_i, v_1), x)} + \sum_{x \in V_2(L_m)} \frac{1}{d_{L_m}((u_i, v_1), x)} \\ &= \frac{1}{d((u_1, v_1), (u_2, v_1))} + \frac{1}{d((u_1, v_1), (u_3, v_1))} + \ldots \\ &\quad + \frac{1}{d((u_1, v_1), (u_{m-1}, v_2))} + \frac{1}{d((u_1, v_1), (u_m, v_2))} \\ &= 1 + \frac{1}{2} + \ldots + \frac{1}{m-1} + 1 + \frac{1}{2} + \ldots + \frac{1}{m-1} + \frac{1}{m} \\ &= 2\sum_{k=1}^{m-1} \frac{1}{k} + \frac{1}{m} \\ &= 2H_{m-1} + \frac{1}{m} \end{aligned}$$

As for u_i, $1 < i < m$,

$$\begin{aligned} \mathcal{R}_G(u) &= \sum_{x \in V_1(L_m)} \frac{1}{d_{L_m}((u_i, v_j), x)} + \sum_{x \in V_2(L_m)} \frac{1}{d_{L_m}((u_i, v_j), x)} \\ &= \sum_{j=1}^{i-1} \frac{1}{d_{L_m(V_1)}(u_i, u_j)} + \sum_{j=i+1}^{m} \frac{1}{d_{L_m(V_1)}(u_i, u_j)} \end{aligned}$$

$$+\sum_{j=1}^{i-1}\frac{1}{d_{L_m(V_2)}(u_i, u_j)}+\sum_{j=i+1}^{m}\frac{1}{d_{L_m(V_2)}(u_i, u_j)}$$

$$=\left[\frac{1}{i-1}+\frac{1}{i-2}+\ldots+\frac{1}{3}+\frac{1}{2}+1\right]+\left[1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m-i}\right]$$

$$+\left[\frac{1}{i}+\frac{1}{i-1}+\ldots+\frac{1}{3}+\frac{1}{2}\right]+\left[1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m-i+1}\right]$$

$$=\left[\frac{1}{i-1}+\frac{1}{i-2}+\ldots+\frac{1}{3}+\frac{1}{2}+1\right]+\left[1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m-i}\right]$$

$$+\left[\frac{1}{i-1}+\frac{1}{i-2}+\ldots+\frac{1}{3}+\frac{1}{2}+1\right]+\left[1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m-i}\right]$$

$$+\frac{1}{i}+\frac{1}{m-i+1}-1$$

$$=2(H_{i-1}+H_{m-i})+\frac{1}{i}+\frac{1}{m-i+1}-1$$

Thus, after normalizing we get

$$\mathcal{H}_{L_m}(u_i, v_j)=\begin{cases}\frac{1}{2m-1}\left(2H_{m-1}+\frac{1}{m}\right) & \text{for } i=1 \text{ or } i=m, 1\le j\le 2\\ \frac{1}{2m-1}\left[2(H_{i-1}+H_{m-i})+\frac{1}{i}+\frac{1}{m-i+1}-1\right] & \text{for } 1<i<m, 1\le j\le 2.\end{cases}$$

■

Theorem 3.10. *For the crown graph Cr_m of order $2m$ with $V(Cr_m)=\{u_1, u_2, \ldots, u_m\}\cup\{v_1, v_2, \ldots, v_n\}$ and whose edges are formed by adjoining u_i to v_j whenever $i\neq j$, the harmonic centrality of vertices u_i and v_j is given by*

$$\mathcal{H}_{Cr_m}(u_i)=\mathcal{H}_{Cr_m}(v_j)=\frac{9m-7}{12m-6}$$

Proof. Considering the structure of a crown graph Cr_m, the geodesic distance of u_i to other u's is 2. While distance between u_i and v_j is 3 whenever $i=j$. On the other hand, u_i is adjacent to v_j, whenever $i\neq j$.

Thus,

$$\mathcal{H}_{Cr_m}(u_i) = \mathcal{H}_{Cr_m}(v_j) = \frac{1}{2m-1}\left(\frac{1}{2}(m-1) + 1(m-1) + \frac{1}{3}\right)$$

$$= \frac{1}{2m-1}\left(\frac{9(m-1)+2}{6}\right)$$

$$= \frac{9m-7}{12m-6}.$$

∎

Theorem 3.11. *For the prism graph* Y_m, *of order* $2m$ *and with* $m \geq 3$, *formed as the Cartesian product of a cycle graph* $C_m = [u_1, u_2, \ldots, u_m]$ *with the path graph* $P_2 = [v_1, v_2]$, *the harmonic centrality of any vertex* (u_i, v_j), *is given by*

$$\mathcal{H}_{Y_m}(u_i, v_j) = \begin{cases} \dfrac{1}{2m-1}\left(4H_{\frac{m-1}{2}} + \dfrac{m-3}{m+1}\right) & \text{if } m \text{ is odd}, 1 \leq j \leq 2 \\ \dfrac{1}{2m-1}4\left(H_{\frac{m}{2}} + \dfrac{2}{m+2} - \dfrac{m+2}{m}\right) & \text{if } m \text{ is even}, 1 \leq j \leq 2. \end{cases}$$

Proof. Considering the construction of a prism Y_m as a Cartesian product of a cycle graph $C_m = [u_1, u_2, \ldots, u_m]$ and a path graph $P_2 = [v_1, v_2]$, we can segregate the vertices according to the cycle they belong to. That is, $[(u_1, v_1), (u_2, v_1), \ldots, (u_m, v_1)] \in V_1(Y_m)$ and $[(u_1, v_2), (u_2, v_2), \ldots, (u_m, v_2)] \in V_2(Y_m)$, so if m is odd we have

$$\mathcal{R}_{Y_m}(u_i, v_j) = \sum_{x \in V_1(Y_m)} \frac{1}{d_{Y_m}((u_i, v_1), x)} + \sum_{x \in V_2(Y_m)} \frac{1}{d_{Y_m}((u_i, v_1), x)}$$

$$= \left[1 + 1 + \ldots + \frac{1}{\frac{m-1}{2}} + \frac{1}{\frac{m-1}{2}}\right] + \left[1 + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{\frac{m+1}{2}} + \frac{1}{\frac{m+1}{2}}\right]$$

$$= 4\sum_{k=1}^{\frac{m-1}{2}} \frac{1}{k} + 2\left(\frac{2}{m+1}\right) - 1$$

$$= 4H_{\frac{m-1}{2}} - \frac{m-3}{m+1}$$

Now, if m is even, we have

$$\mathcal{R}_{Y_m}(u_i, v_j) = \sum_{x \in V_1(Y_m)} \frac{1}{d_{Y_m}((u_i, v_1), x)} + \sum_{x \in V_2(Y_m)} \frac{1}{d_{Y_m}((u_i, v_1), x)}$$

$$= \left[1 + 1 + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{\frac{m}{2}}\right] + \left[1 + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{\frac{m+2}{2}}\right]$$

$$= 4\sum_{k=1}^{\frac{m-1}{2}} \frac{1}{k} + \frac{1}{\frac{m+2}{2}} - 1 - \frac{1}{\frac{m}{2}}$$

$$= 4H_{\frac{m}{2}} + \frac{2}{m+2} - \frac{m+2}{m}$$

Normalizing and consolidating these results we get

$$\mathcal{H}_{Y_m}(u_i, v_j) = \begin{cases} \frac{1}{2m-1}\left(4H_{\frac{m-1}{2}} + \frac{3-m}{m+1}\right) & \text{if } m \text{ is odd}, 1 \le j \le 2 \\ \frac{1}{2m-1} 4\left(H_{\frac{m}{2}} + \frac{2}{m+2} - \frac{m+2}{m}\right) & \text{if } m \text{ is even}, 1 \le j \le 2. \end{cases} \qquad \blacksquare$$

Theorem 3.12. *For the star graph* S_m, *of order* $m+1$ *with* $m > 1$, *formed by adjoining m isolated vertices* $u_i, 1 \le i \le m$, *to a single vertex* u_0, *the harmonic centrality of any vertex* u_i *is given by*

$$\mathcal{H}_{S_m}(u_i) = \begin{cases} 1 & \textit{for } i = 0 \\ \frac{m+1}{2m} & \textit{for } 1 \le i \le m. \end{cases}$$

Proof. By theorem 3.2, $\mathcal{H}_{S_m}(u_0) = 1$ since all other vertices are adjacent to u_0. For $1 \le i \le m$, u_i has a geodesic distance of 1 to u_0 and $\frac{1}{2}$, otherwise, thus we have

$$\mathcal{H}_{S_m}(u_i) = \frac{1}{m}\left(1 + \frac{1}{2}(m-1)\right)$$

$$= \frac{m-1}{2m}. \qquad \blacksquare$$

Theorem 3.13. *For the book graph* B_m *of order* $2(m+1)$, *formed as the Cartesian product of a star graph* $S_m = [u_0, u_1, \ldots, u_m]$ *(with center vertex* u_0 *and order* $m+1$*) with path graph* $P_2 = [v_1, v_2]$, *the harmonic centrality of any vertex* (u_i, v_j), $0 \le i \le m$, $1 \le j \le 2$, *is given by*

$$\mathcal{H}_{B_m}(u_i) = \begin{cases} \dfrac{3m+2}{4m+2} & \text{for } i = 0, 1 \le j \le 2 \\ \dfrac{5(m+2)}{6(2m+1)} & \text{for } 1 \le i \le m, 1 \le j \le 2. \end{cases}$$

Proof. For vertices (u_0, v_1) and (u_0, v_2),

$$\mathcal{H}_{B_m}(u_0, v_j) = \frac{1}{2(m+1)-1}\left[1(m+1) + \frac{1}{2}(m)\right]$$

$$= \frac{3m+2}{4m+2}$$

For vertices $(u_i, v_j), 1 \le i \le m, 1 \le j \le 2$,

$$\mathcal{H}_{B_m}(u_i, v_j) = \frac{1}{2m+1}\left[2(1) + \frac{1}{2}(m) + \frac{1}{3}(m-1)\right]$$

$$= \frac{1}{2m+1}\left(\frac{12 + 3m + 2m - 2}{6}\right) = \frac{5m+10}{12m+6}$$

$$= \frac{5(m+2)}{6(2m+1)}. \qquad \blacksquare$$

Theorem 3.14. *For the helm graph* H_m, *of order* $2m+1$ *with* $m \ge 3$, *obtained by adjoining a pendant vertex to each node of the m-ordered cycle of wheel* W_m, *with vertices* $V(H_m) = \{u_0, u_1, \ldots, u_m\} \cup \{v_1, v_2, \ldots, v_m\}$, *the harmonic centrality of any vertex* u_i, $0 \le i \le m$ *and* v_j, $1 \le j \le m$, *is given by*

$$\mathcal{H}_{H_m}(u_i) = \begin{cases} \dfrac{3}{4} & \textit{for } i = 0 \\ \dfrac{5m+15}{12m} & \textit{for } 1 \le i \le m \end{cases}$$

$$\mathcal{H}_{H_m}(v_i) = \left\{ \dfrac{7m+17}{24m} \quad \textit{for } 1 \le j \le m. \right.$$

Proof. The distance of u_0 to the m-ordered u_i's is 1, while its distance to the m-ordered u_j's is $\frac{1}{2}$. So, $\mathcal{H}_{H_m}(u_0) = \frac{1}{2m}\left(m(1) + m\left(\frac{1}{2}\right)\right) = \frac{3}{4}$. For $u_i, 1 \le i \le m$

$$\mathcal{H}_{H_m}(u_i) = \frac{1}{2m}\left[4(1) + \frac{1}{2}(m-1) + \frac{1}{3}(m-3)\right]$$

$$= \frac{5m+15}{12m}$$

As for $v_j, 1 \le j \le m,$ we have

$$\mathcal{H}_{H_m}(v_j) = \frac{1}{2m}\left[1 + \left(\frac{1}{2}\right)3 + \frac{1}{3}(m-1) + \frac{1}{4}(m-3)\right]$$

$$= \frac{7m+17}{24m}. \qquad \blacksquare$$

4. Conclusion

Harmonic centrality is a useful metric for analyzing graph structures. When compared to other centrality measures, harmonic centrality has the advantage of considering disconnected graphs. We have derived expressions for harmonic centrality of some graph families which are the basic components of larger and more complex networks. This study is therefore helpful for analyzing larger classes of graphs.

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