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CLOSED SUPPORT INDEPENDENCE NUMBER OF SOME STANDARD GRAPHS UNDER ADDITION AND MULTIPLICATION

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Abstract

In this paper, closed support independence number of a set and closed support independence number of a graph under addition and multiplication are introduced. Closed support independence number of the path, star, complete graph, wheel, cycle, complete bipartite graph and bistar under addition and multiplication are studied.

I. Introduction

Graphs considered in this paper are finite, undirected and without loops or multiple edges. Let $G = (V, E)$ be a graph with p vertices and q edges.

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Terms not defined here are used in the sense of Harary [4]. For each vertex $v \in V,$ the open neighborhood of v is the set $N(v)$ containing all the vertices u adjacent to v and the closed neighborhood of v is the set $N(v) \cup \{v\}$ [6]. The degree of a vertex $v \in (G)$ is the number of edges of G incident with v and is denoted by $\deg_G(v)$ or $\deg v.$ In a graph G, an independent set is a subset S of $V(G)$ such that no two vertices in S are adjacent. A maximum independent set is an independent set of maximum size [5].

Recently the concept of closed support of a graph under addition was introduced by Balamurugan et al. [1] and further studied in [2]. Closed support of a graph under multiplication was introduced in [3]. Motivated by these definitions, the concept of closed support independence number of a graph under addition and multiplication are introduced.

In this paper, closed support independence number of a set under addition, closed support independence number of a graph G under addition, closed support independence number of a set under multiplication and closed support independence number of a graph G under multiplication are introduced. Closed support independence number of some standard graphs under addition and multiplication are studied.

The following definitions are necessary for the present study.

Definition 1.1. Let $G = (V, E)$ be a graph. A subset S of V is called an independent set of G of no two vertices in S are adjacent in G.

Definition 1.2. An independent set 'S' is maximum in G if G has no independent set S' with $|S'| > |S|$.

Definition 1.3. The number of vertices in a maximum independent set of G is called the independence number of G and is denoted by $\alpha(G)$.

Definition 1.4 [1]**.** Let $G = (V, E)$ be a graph. A closed support of a vertex v under addition is defined by $\sum_{u \in N[v]} \deg u$ and is denoted by supp $[v]$.

Definition 1.5 [1]**.** Let $G = (V, E)$ be a graph. A closed support of the graph G under addition is defined by $\sum_{u \in V[G]} \text{supp}(v)$ and is denoted by supp $[G]$.

Definition 1.6 [3]. Let $G = (V, E)$ be a graph. A closed support of a vertex v under multiplication is defined by $\prod_{u \in N[v]} \deg u$ and is denoted by $mult[v]$.

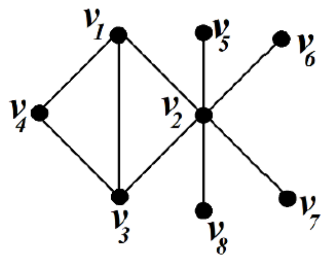
Definition 1.7 [3]. Let $G = (V, E)$ be a graph. A closed support of the graph G under multiplication is defined by $\prod_{u \in V[G]} mult[v]$ and is denoted by $mult[G]$.

II. Main Results

Definition 2.1. Let $G = (V, E)$ be a graph. Let S denote the maximum independent set of G. Closed support independence number of the set S under addition, denoted by $\text{supp}\, S^{+}[G]$, is defined by $\text{supp}\, S^{+}[G] = \sum_{v \in S} \text{supp}\,[v]$. Closed support independence number of G under addition, denoted by $\text{supp}\, \alpha^{+}[G]$, is defined by $\text{supp}\, \alpha^{+}[G] = \max\{\text{supp}\, S_i^{+}[G]; i \geq 1\}$.

Definition 2.2. Let $G = (V, E)$ be a graph. Let S denote the maximum independent set of G. Closed support independence number of the set S under multiplication, denoted by $\text{supp}\, S^{\times}[G]$, is defined by $\text{supp}\, S^{\times}[G] = \prod_{v \in S} mult[v]$. Closed support independence number of G under multiplication, denoted by $\text{supp}\, \alpha^{\times}[G]$, is defined by $\text{supp}\, \alpha^{\times}[G] = \max\{mult\, S_i^{\times}[G]; i \geq 1\}$.

Example 2.3. Consider the following graph G.



In G, $\deg v_1 = 3$, $\deg v_2 = 6$, $\deg v_3 = 3$, $\deg v_4 = 2$, and $\deg v_5 = \deg v_6$ $\deg v_7 = \deg v_8 = 1$.

Maximum independent sets of G are $\{S_1, S_2, S_3\}$, where $S_1 = \{v_1, v_5, v_6, v_7, v_8\}$, $S_2 = \{v_3, v_5, v_6, v_7, v_8\}$ and $S_3 = \{v_4, v_5, v_6, v_7, v_8\}$.

Closed support independence number of G under addition.

Consider the set S_1

$$\text{supp}\,[v_1] = \sum_{u \in N[v_1]} \deg u = \deg v_2 + \deg v_3 + \deg v_4 + \deg v_1 = 14$$

$$\text{supp}\,[v_5] = \sum_{u \in N[v_5]} \deg u = \deg v_2 + \deg v_5 = 7$$

$$\text{supp}\,[v_6] = \sum_{u \in N[v_6]} \deg u = \deg v_2 + \deg v_6 = 7$$

$$\text{supp}\,[v_7] = \sum_{u \in N[v_7]} \deg u = \deg v_2 + \deg v_7 = 7$$

$$\text{supp}\,[v_8] = \sum_{u \in N[v_8]} \deg u = \deg v_2 + \deg v_8 = 7$$

Hence $\text{supp}\, S_1^+[G] = \sum_{v \in S_1} \text{supp}\,[v\} = 42$

Consider the set S_2

$$\text{supp}\,[v_3] = \sum_{u \in N[v_3]} \deg u = \deg v_1 + \deg v_2 + \deg v_4 + \deg v_3 = 14$$

$$\operatorname{supp}[v_5] = 7,\ \sup p[v_6] = 7,\ \operatorname{supp}[v_7] = 7 \text{ and } \operatorname{supp}[v_8] = 7$$

Hence $\operatorname{supp} S_2^+[G] = \sum_{v \in S_2} \operatorname{supp}[v] = 42$

Consider the set S_3

$$\operatorname{supp}[v_4] = \sum_{u \in N[v_4]} \deg u = \deg v_1 + \deg v_3 + \deg v_4 = 8$$

$$\operatorname{supp}[v_5] = 7,\ \operatorname{supp}[v_6] = 7,\ \operatorname{supp}[v_7] = 7 \text{ and } \operatorname{supp}[v_8] = 7$$

Hence $\operatorname{supp} S_3^+[G] = \sum_{v \in S_3} \operatorname{supp}[v] = 36$

Therefore $\operatorname{supp} \alpha^+[G] = \max\{\operatorname{supp} S_1^+[G], \operatorname{supp} S_2^+[G], \operatorname{supp} S_3^+[G]\}$

$$= 42.$$

Closed support independence number of G under multiplication.

Consider the set S_1

$$mult[v_1] = \prod_{u \in N[v_1]} \deg u = \deg v_2 \times \deg v_3 \times \deg v_4 \times \deg v_1 = 108$$

$$mult[v_5] = \prod_{u \in N[v_5]} \deg u = \deg v_2 \times \deg v_5 = 6$$

$$mult[v_6] = \prod_{u \in N[v_6]} \deg u = \deg v_2 \times \deg v_6 = 6$$

$$mult[v_7] = \prod_{u \in N[v_7]} \deg u = \deg v_2 \times \deg v_7 = 6$$

$$mult[v_8] = \prod_{u \in N[v_8]} \deg u = \deg v_2 \times \deg v_8 = 6$$

Hence $\operatorname{supp} S_1^{\times}[G] = \prod_{v \in S_1} mult[v] = 139968$

Consider the set S_2

$$mult[v_3] = \prod_{u \in N[v_3]} \deg u = \deg v_1 \times \deg v_2 \times \deg v_4 \times \deg v_3 = 108$$

$$mult\,[v_5] = 6,\ mult\,[v_6] = 6,\ mult\,[v_7] = 6 \text{ and } mult\,[v_8] = 6$$

Hence $\text{supp } S_2^{\times}[G] = \prod_{v \in S_2} mult\,[v] = 139968$

Consider the set S_3

$$mult\,[v_4] = \prod_{u \in N[v_4]} \deg u = \deg v_1 \times \deg v_3 \times \deg v_4 = 18$$

$$mult\,[v_5] = 6,\ mult\,[v_6] = 6,\ mult\,[v_7] = 6 \text{ and } mult\,[v_8] = 6$$

Hence $\text{supp } S_3^{\times}[G] = \prod_{v \in S_3} mult\,[v] = 23328$

Therefore $\text{supp } \alpha^{\times}[G] = \max\{\text{supp } S_1^{\times}[G], \text{supp } S_2^{\times}[G], \text{supp } S_3^{\times}[G]\}$

$$= 139968$$

Theorem 2.4. *Let* $G = K_{1,n}$ *where* $n \geq 1$ *be a star. Then* $\text{supp } \alpha^{+}[G] = n(n+1)$ *and* $\text{supp } \alpha^{\times}[G] = n^n$.

Proof. Let $G = K_{1,n}$ where $n \geq 1$. Let $v, v_1, v_2, \ldots, v_n$ be the vertices of G where v is the central vertex and $v_1, v_2, \ldots, v_n$ are the pendant vertices.

Then $\deg v = n, \deg v_i = 1; 1 \leq i \leq n$. $S = \{v_1, v_2, \ldots, v_n\}$ is the unique maximum independent set of G.

$$\text{supp }[v_1] = \sum_{u \in N[v_1]} \deg u = \deg v + \deg v_1 = n + 1$$

Similarly $\text{supp }[v_2] = \text{supp }[v_3] = \ldots = \text{supp }[v_n] = n + 1$.

Hence $\text{supp } \alpha^{+}[G] = \sum_{u \in S} \text{supp }[u]$

$$= n(n+1).$$

Similarly $\text{supp } \alpha^{\times}[G] = \prod_{u \in S} mult\,[u]$

$$= n^n.$$

Theorem 2.5. *Let* $G = P_n$ *where* $n > 1$ *be a path on n vertices. Then*

$$\operatorname{supp} \alpha^{+}[G] = \begin{cases} 3n-3 \text{ if } n \text{ is odd} \\ 3n-4 \text{ if } n \text{ is even} \end{cases} \text{ and } \operatorname{supp} \alpha^{\times}[G] = \begin{cases} 2^{\left(\frac{3n-5}{2}\right)} \text{ if } n \text{ is odd} \\ 2^{\left(\frac{3n-6}{2}\right)} \text{ if } n \text{ is even} \end{cases}$$

Proof. Let $G = P_n$ where $n > 1$. Let $v_1, v_2, \ldots, v_n$ be the vertices of G.

Then $\deg v_1 = \deg v_n = 1$ and $\deg v_i = 2$ for all $i = 2, 3, \ldots, n-1$.

Case (i). Suppose n is odd.

$S = \{v_1, v_3, v_5, \ldots, v_n\}$ is the unique maximum independent set of G.

$$\operatorname{supp}[v_1] = \sum_{v \in N[v_1]} \deg v = \deg v_2 + \deg v_1 = 3$$

Similarly $\operatorname{supp}[v_n] = 3$

$$\operatorname{supp}[v_3] = \sum_{v \in N[v_3]} \deg v = \deg v_2 + \deg v_4 + \deg v_3 = 6$$

Similarly $\operatorname{supp}[v_5] = \operatorname{supp}[v_7] = \ldots = \operatorname{supp}[v_{n-2}] = 6$

Hence $\operatorname{supp} \alpha^{+}[G] = \sum_{v \in S} \operatorname{supp}[v]$

$$= 3 + 6\left(\frac{n-3}{2}\right) + 3$$

$$= 3n - 3$$

$$mult[v_1] = \prod_{v \in N[v_1]} \deg v = \deg v_2 \times \deg v_1 = 2$$

Similarly $mult[v_n] = 2$

$$mult[v_3] = \prod_{v \in N[v_3]} \deg v = \deg v_2 \times \deg v_4 \times \deg v_3 = 8$$

Similarly $mult[v_5] = mult[v_7] = \ldots = mult[v_{n-2}] = 8$

Hence $\operatorname{supp} \alpha^{\times}[G] = \prod_{u \in S} mult[u]$

$$= 2 \times 8^{\left(\frac{n-3}{2}\right)} \times 2$$

$$= 2^{\left(\frac{3n-5}{2}\right)}.$$

Case (ii). Suppose n is even. There are two subcases.

Subcase (a). In this case, we consider the two maximum independent sets with only one pendant vertex.

Without loss of generality let $S_1 = \{v_1, v_3, v_5, \ldots, v_{n-1}\}$.

Proof is similar for the other set.

$$\text{supp}\,[v_1] = \sum_{v \in N[v_1]} \deg v = \deg v_2 + \deg v_1 = 3$$

$$\text{supp}\,[v_3] = \sum_{v \in N[v_3]} \deg v = \deg v_2 + \deg v_4 + \deg v_3 = 6$$

Similarly $\text{supp}\,[v_5] = \text{supp}\,[v_7] = \ldots = \text{supp}\,[v_{n-3}] = 6$

$$\text{supp}\,[v_{n-1}] = \sum_{v \in N[v_{n-1}]} \deg v = \deg v_{n-2} + \deg v_n + \deg v_{n-1} = 5$$

Hence $\text{supp}\, S_1^{+}[G] = \sum_{v \in S} \text{supp}\,[v]$

$$= 3 + 6\left(\frac{n-4}{2}\right) + 5$$

$$= 3n - 4$$

As before, $mult\,[v_1] = mult\,[v_n] = 2$ and $mult\,[v_3] = mult\,[v_5] = \ldots = mult\,[v_{n-3}] = 8$

$$mult\,[v_{n-1}] = \prod_{v \in N[v_{n-1}]} \deg v = \deg v_{n-2} \times \deg v_n \times \deg v_{n-1} = 4$$

Hence $\text{supp}\, S_1^{\times}[G] = \prod_{u \in S_1} mult\,[u]$

$$= 2 \times 8^{\left(\frac{n-4}{2}\right)} \times 4 = 2^{\left(\frac{3n-6}{2}\right)}$$

Subcase (b). In this case, we consider all the maximum independent sets with both the pendant vertices.

Without loss of generality let $S_2 = \{v_1, v_3, v_5, \ldots, v_{n-3}, v_n\}$.

Proof is similar for the other set.

As before, $\text{supp}[v_1] = \text{supp}[v_n] = 3$ and $s\text{upp}[v_3] = s\text{upp}[v_5] = \ldots = s\text{upp}[v_{n-3}] = 6$

Hence $\text{supp}\, \text{S}_2^+[G] = \sum_{v \in S_2} \text{supp}[v]$

$$= 3 + 6\left(\frac{n-4}{2}\right) + 3$$

$$= 3n - 6$$

Therefore $\text{supp}\, \alpha^+[G] = \max\{\text{supp}\, S_i^+[G]; i \geq 1\}$

$$\text{supp}\, \alpha^+[G] = \max\{\text{supp}\, S_1^+[G], \text{supp}\, S_2^+[G]\}$$

$$= \max\{3n - 4, 3n - 6\}$$

$$= 3n - 4$$

Similarly $\text{supp}\, \text{S}_2^+[G] = \prod_{u \in S_2} mult[u]$

$$= 2 \times 8^{\left(\frac{n-4}{2}\right)} \times 2$$

$$= 2^{\left(\frac{3n-8}{2}\right)}$$

Therefore $\text{supp}\, \alpha^\times[G] = \max\{mult\, S_1^\times[G], mult\, S_2^\times[G]\}$

$$= \max\{2^{\left(\frac{3n-6}{2}\right)}, 2^{\left(\frac{3n-8}{2}\right)}\}$$

$$= 2^{\left(\frac{3n-6}{2}\right)}$$

Theorem 2.6. *Let $G = C_n$ where $n \geq 3$ be a cycle on n vertices. Then*

$$\text{supp}\, \alpha^{+}[G] = \begin{cases} 3n - 3 & \text{if } n \text{ is odd} \\ 3n & \text{if } n \text{ is even} \end{cases} \text{ and}$$

$$\text{supp}\, \alpha^{\times}[G] = \begin{cases} 8^{\left(\frac{n-1}{2}\right)} & \text{if } n \text{ is odd} \\ 8^{\left(\frac{n}{2}\right)} & \text{if } n \text{ is even} \end{cases}.$$

Proof. Let $G = C_n$ where $n \geq 3$. Let $v_1, v_2, \ldots, v_n$ be the vertices of G.

Then $\deg v_i = 2$ for all $i = 1, 2, 3, \ldots, n$.

Case (i). Let n be odd.

Consider the maximum independent set $S = \{v_1, v_3, v_5, \ldots, v_{n-2}\}$.

Proof is similar for the other set.

$$\text{supp}\,[v_1] = \sum_{v \in N(v_1)} \deg v = \deg v_2 + \deg v_n + \deg v_1 = 6$$

Similarly $\text{supp}\,[v_3] = \text{supp}\,[v_5] = \ldots = \text{supp}\,[v_{n-2}] = 6$

$$\text{Hence } \text{supp}\, \alpha^{+}[G] = 6\left(\frac{n-1}{2}\right)$$

$$= 3n - 3$$

$$mult\,[v_1] = \prod\nolimits_{v \in N[v_1]} \deg v = \deg v_2 \times \deg v_n \times \deg v_1 = 8$$

Similarly $mult\,[v_3] = mult\,[v_5] = \ldots = mult\,[v_{n-2}] = 8$

$$\text{Hence } \text{supp}\, \alpha^{\times}[G] = \prod\nolimits_{v \in S} mult\,[v]$$

$$= 8^{\left(\frac{n-1}{2}\right)}.$$

Case (ii). Let n be even.

$S_1 = \{v_1, v_3, v_5, \ldots, v_{n-1}\}$ and $S_2 = \{v_2, v_4, v_6, \ldots, v_n\}$ are two maximum independent sets of G. Consider the set S_1.

Proof is similar for the set S_2.

As before $\operatorname{supp}[v_1] = \operatorname{supp}[v_3] = \operatorname{supp}[v_5] = \ldots = \operatorname{supp}[v_{n-1}] = 6$

Hence $\operatorname{supp}\alpha^{+}[G] = 6\left(\frac{n}{2}\right)$

$$= 3n.$$

Similarly $\operatorname{supp}\alpha^{\times}[G] = \prod_{v \in S_1} mult[v]$

$$= 8^{\left(\frac{n}{2}\right)}.$$

Theorem 2.7. *Let $G = K_n$ where $n \geq 1$ be the complete graph on n vertices. Then $\operatorname{supp}\alpha^{+}[G] = n(n-1)$ and $\operatorname{supp}\alpha^{\times}[G] = (n-1)^n$.*

Proof. Let $G = K_n$ where $n \geq 1$. Let $v_1, v_2, \ldots, v_n$ be the vertices of G. Then $\deg v_i = n - 1$ for all $i = 1, 2, 3, \ldots, n$. $S_i = \{v_i\}; 1 \leq i \leq n$ are maximum independent sets of G.

Consider the set S_1. Proof is similar for other sets.

$$\operatorname{supp}[v_1] = \sum_{v \in N[v_1]} \deg v = \deg v_2 + \deg v_3 + \ldots + \deg v_n + \deg v_1$$

$$= n(n-1)$$

Hence $\operatorname{supp}\alpha^{+}[G] = n(n-1)$

Similarly $\operatorname{supp}\alpha^{\times}[G] = \prod_{v \in S_1} mult[v]$

$$= (n-1)^n.$$

Theorem 2.8. *Let $G = K_{m,n}$ where $m \geq n,\ m, n \geq 1$ be a complete bipartite graph. Then $\operatorname{supp}\alpha^{+}[G] = m(mn+n)$ and $\operatorname{supp}\alpha^{\times}[G] = (nm^n)^m$.*

Proof. Let $G = K_{m,n}$ where $m \geq n$ be a complete bipartite graph with the bipartition (X, Y) where $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

Then $\deg x_i = n; 1 \leq i \leq m$, and $\deg y_i = m; 1 \leq i \leq n$. $S = \{x_1, x_2, \ldots, x_m\}$ is the unique maximum independent set of G.

$$\text{supp}[x_1] = \sum_{u \in N[x_1]} \deg u$$

$$= \deg x_1 + \sum_{j=1}^{n} \deg y_j$$

$$= mn + n$$

Similarly $\text{supp}[x_2] = \text{supp}[x_3] = \ldots = \text{supp}[x_m] = mn + n$.

Hence $\text{supp}\, \alpha^+[G] = \sum_{u \in S} \text{supp}[u]$

$$= m(mn + n).$$

$$mult[x_1] = \prod_{u \in N[x_1]} \deg u = \deg y_1 \times \deg y_2 \times \ldots \times \deg y_n \times \deg x_1$$

$$= nm^n$$

Similarly $mult[x_2] = mult[x_3] = \ldots = mult[x_m] = nm^n$

Therefore $\text{supp}\, \alpha^\times[G] = \prod_{v \in S} mult[v]$

$$= (nm^n)^m.$$

Theorem 2.9. *Let $G = W_n$ where $n \geq 4$ be a wheel of order n. Then*

$$\text{supp}\, \alpha^+[G] = \begin{cases} \left(\frac{n-1}{2}\right)(n+8) & \text{if } n \text{ is odd} \\ \left(\frac{n-2}{2}\right)(n+8) & \text{if } n \text{ is even} \end{cases} \quad \text{and}$$

$$\text{supp}\, \alpha^\times[G] = \begin{cases} [27(n-1)]^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ [27(n-1)]^{\frac{n-2}{2}} & \text{if } n \text{ is even} \end{cases}$$

Proof. Let $G = W_n$ where $n \geq 4$. Let $v_0, v_1, v_2, \ldots, v_{n-1}$ be the vertices of G where v_0 is the central vertex. Then $\deg v_0 = n - 1$ and $\deg v_i = 3$ for all $i = 1, 2, 3, \ldots, n-1$.

Case (i). Let n be odd.

$\{v_1, v_3, v_5, \ldots, v_{n-2}\}$ and $\{v_2, v_4, v_6, \ldots, v_{n-1}\}$ are the two maximum independent sets of G. Consider the set $S_1 = \{v_1, v_3, v_5, \ldots, v_{n-2}\}$. Proof is similar for the other set.

$$\text{supp}[v_1] = \sum_{v \in N[v_1]} \deg v = \deg v_2 + \deg v_{n-1} + \deg v_0 + \deg v_1 = n + 8$$

Similarly $\text{supp}[v_3] = \text{supp}[v_5] = \ldots = \text{supp}[v_{n-2}] = n + 8$

Hence $\text{supp}\,\alpha^{+}[G] = \sum_{v \in S_1} \text{supp}[v]$

$$= \left(\frac{n-1}{2}\right)(n+8)$$

$$mult[v_1] = \prod_{v \in N[v_1]} \deg v = \deg v_2 \times \deg v_{n-1} \times \deg v_0 \times \deg v_1$$

$$= 3 \times 3 \times (n-1) \times 3 = 27(n-1)$$

Similarly $mult[v_3] = mult[v_5] = \ldots = mult[v_{n-2}] = 27(n-1)$.

$$\text{supp}\,\alpha^{\times}[G] = \prod_{v \in S_1} mult[v]$$

$$= [27(n-1)]^{\frac{n-1}{2}}$$

Case (ii). Let n be even.

$\{v_1, v_3, v_5, \ldots, v_{n-3}\}$ and $\{v_2, v_4, v_6, \ldots, v_{n-2}\}$ are two maximum independent sets.

Consider the maximum independent set $S_2 = \{v_1, v_3, v_5, \ldots, v_{n-3}\}$.

Proof is similar for the other set.

As before, $\text{supp}[v_1] = \text{supp}[v_3] = \text{supp}[v_5] = \ldots = \text{supp}[v_{n-3}] = n + 8$

Hence $\text{supp}\,\alpha^{+}[G] = \sum_{v \in S_2} \text{supp}[v]$

$$= \left(\frac{n-2}{2}\right)(n+8)$$

Similarly $\text{supp}\,\alpha^{\times}[G] = \prod_{v \in S_2} mult\,[v]$

$$= [27(n-1)]^{\frac{n-1}{2}}.$$

Theorem 2.10. *Let* $G = B_{m,n}$ *where* $m \geq n, n \geq 1$ *be a bistar on* $m + n + 2$ *vertices.*

Then $\text{supp}\,\alpha^{+}[G] = \begin{cases} m(m+2) + n(n+2) & if\ n > 1 \\ m^2 + 3m + 4 & if\ n = 1 \end{cases}$ *and*

$$\text{supp}\,\alpha^{\times}[G] = \begin{cases} (m+1)^m (n+1)^n & if\ n > 1 \\ 2(m+1)^{m+1} & if\ n = 1 \end{cases}$$

Proof. Let $G = B_{m,n}$ where $m \geq n$. Let $u_1, u_2, \ldots, u_m, x, y, v_1, v_2, \ldots, v_n$ be the vertices of G. Then $\deg x = m + 1, \deg y = n + 1$ and $\deg u_i = \deg v_j = 1, 1 \leq i \leq m, 1 \leq j \leq n$.

Case (i). Suppose $n > 1$.

$S = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ is the unique maximum independent set of G.

$$\text{supp}\,[u_1] = \sum_{v \in N[u_1]} \deg v = \deg x + \deg u_1 = m + 2$$

Similarly $\text{supp}\,[u_i] = m + 2\ \forall\ 2 \leq i \leq m$

$$\text{supp}\,[v_1] = \sum_{v \in N[v_1]} \deg v = \deg y + \deg v_1 = m + 2$$

Similarly $\text{supp}\,[v_j] = m + 2\ \forall\ 2 \leq j \leq n$

Hence $\text{supp}\,\alpha^{+}[G] = \sum_{v \in S} \text{supp}\,[v]$

$$= \sum_{i=1}^{m} \text{supp}\,[u_i] + \sum_{j=1}^{n} \text{supp}\,[v_j]$$

$$= m(m+2) + n(n+2)$$

$mult\,[u_1] = \prod_{v \in N[u_1]} \deg v = \deg x \times \deg u_1 = m + 1$

Similarly $mult\,[u_i] = m + 1,\ 1 \leq i \leq m$

$$mult\,[v_1] = \prod_{v \in N[v_1]} \deg v = \deg y \times \deg v_1 = n + 1$$

Similarly $mult\,[v_j] = n + 1,\ 1 \leq j \leq n$

Hence $\operatorname{supp} \alpha^{\times}[G] = \prod_{v \in S} mult\,[v]$

$= (m + 1)^m (n + 1)^n.$

Case (ii). Suppose $n = 1.$ There are two subcases.

Subcase (a). In this case we consider the maximum independent set with all the pendant vertices. Let $S_1 = \{u_1, u_2, \ldots, u_m, v_1\}$

Now put $n = 1$ in case (i)

Hence $\operatorname{supp} S_1^{+}[G] = m^2 + 2m + 3$

Similarly $\operatorname{supp} S_1^{\times}[G] = \prod_{v \in S_1} mult\,[v]$

$$= 2(m + 1)^m$$

Subcase (b). In this case, we consider the maximum independent set $S_2 = \{u_1, u_2, \ldots, u_m, y\}$ as before,

$$\operatorname{supp}\,[u_i] = m + 2\ \forall 1 \leq i \leq m$$

$$\operatorname{supp}\,[y] = \sum_{v \in N[y]} \deg v = \deg x + \deg v_1 + \deg y$$

Hence $\operatorname{supp} S_2^{+}[G] = m^2 + 3m + 4$

Therefore $\operatorname{supp} \alpha^{+}[G] = \max\{\operatorname{supp} S_1^{+}[G], \operatorname{supp} S_2^{+}[G]\}$

$$= \max\{m^2 + 2m + 3,\ m^2 + 3m + 4\}$$

$$= m^2 + 3m + 4$$

As before $mult[u_i] = m + 1, 1 \le i \le m$

$$mult[y] = \prod_{v \in N[y]} \deg v = \deg x \times \deg v_1 \times \deg y = 2(m+1)$$

Hence $\text{supp } S_2^{\times}[G] = \prod_{v \in S_2} mult[v]$

$$= 2(m+1)^{m+1}$$

Therefore $\text{supp } \alpha^{\times}[G] = \max\{mult\, S_1^{\times}[G], mult\, S_2^{\times}[G]\}$

$$= \max\{2(m+1)^m, 2(m+1)^{m+1}\}$$

$$= 2(m+1)^{m+1}.$$

III. Conclusion

In this paper, the closed support independence number of some standard graphs under addition and multiplication are studied. Further studies can be observed for some special graphs.

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