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OPTIMIZATION OF THE CRAMER LUNDBERG MODEL BASED VALUE FUNCTION OF REINSURANCE WITH RANDOM CLAIMS USING POLICY ITERATION METHOD

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Abstract

Consider an insurance company who experiences two opposing cash flows incoming cash premiums and outgoing claims that is also known as classical risk process. Suppose the classical risk process of the company satisfies Cramér-Lundberg model. If insurance company purchase reinsurance and wish to maximize its surplus level over time. In this model, we consider expectation of surplus process until ruin time with dynamic reinsurance. The maximum of expected surplus is called Value function. It is bounded, monotone, continuity and satisfies Hamilton Jacobi Bellman (HJB) partial differential equation. We apply the policy iteration method to find the maximum the surplus level and corresponding dynamic reinsurance strategy called optimal dynamic reinsurance strategy. We apply the method in proportional, excess of loss and quota share and stop loss reinsurance problems.

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1. Introduction

Many papers model optimal reinsurance or optimal investment solving various issues in risk theory. In these models, the insurer takes reinsurance and invest its capital in the insurance market. Some models use stochastic control theory and related methods to minimize the probability of ruin or the maximum expected utility of returning surplus. Taksar and Markussen [1] considered the optimal reinsurance policy which minimizes the ruin probability of the cedent. Bai and Guo [2] model the problem of maximizing the expected exponential utility of terminal surplus in proportional reinsurance. Asmussen et al. [3] models present the dynamic method of excess-of-loss reinsurance retention level and the dividend distribution policy for the purpose of maximizing the expected present value of the dividends. Irgens and Paulsen [4] presents a model for optimal reinsurance and investment strategy with a jump diffusion process in risk market. Arian Cani and Stefan Thonhauser [5] developed a model of dynamic reinsurance and optimal strategy that maximizes the surplus.

In this model, for a diffusion risk model based on the dynamic reinsurance, we will apply policy iteration procedure to optimize the cost function associated to the Hamilton Jacobi Bellman Equation.

Let (Ω, F, P) be a probability space. In 1903, Cramér-Lundberg introduced an equation in ruin theory. It is also called classical compound-Poisson risk model.

$$X_t^u = y + \alpha t - S(t)\ t \geq 0, \tag{1}$$

where

$$S(t) = \sum_{n=1}^{N(t)} Z_n. \tag{2}$$

Where y denotes insurer's surplus at time $t = 0$ and α is a constant rate of premiums arrival. Customers' claim is denoted by Z_n, follows Poisson process with intensity λ. Both the premium arrival and claims are independent and identically distributed non-negative random variables with distribution $F(z)$ and mean μ, where $N(t)$ is the number of claims in time t.

Time of Ruin: The time of ruin is one of interesting problems in classical ruin theory. The time of ruin is the first time that the surplus becomes negative. Let $X^u = (X_u^t)_{t \geq 0}$ be the surplus process and τ_y^u be the time of ruin to any strategy u then,

$$\tau_y^u = \inf \{t \geq 0 : X_t^u < 0 \,|\, X_0^u = y\}. \tag{3}$$

Probability of Ruin: The probability of ruin $\psi(y)$ is defined as a function of initial capital $x \geq 0$.

$$\psi(y) = P\{\tau_y < \infty\}. \tag{4}$$

Return Function: Let $\delta > 0$ a discount rate then the expected value of the surplus corresponding to a strategy u is called return function. It is defined as follows

$$V^u(y) = E_y\left[\int_0^{\tau_y^u} e^{-\delta t} X_t^u\right] dt. \tag{5}$$

Value Function: We would like to get return function with maximum surplus i.e, our problem is to find

$$V(y) = \sup_{u \in U} V^u(y) \tag{6}$$

$V(y)$ is called Value function.

In order to apply Markov model assume that $P \sim Exp(\delta)$ then equivalently to equation

$$V^u(y) = E_y[X_P^u 1_{\{P < \tau_y^u\}}]. \tag{7}$$

Reinsurance: Reinsurance is a method of sharing a part of loss and premiums of a insurer company by another company. Insurance company purchase reinsurance. Reinsurance allows insurance companies to remain solvent after major claims events, such as major disasters like hurricanes and wildfires. Reinsurance has a roles in risk management, tax mitigation and other reasons. The company that purchases the reinsurance policy is called a ceding company or cedent or cedant. The company issuing the reinsurance policy is known as as the reinsurer. Cedent company pays a premium to the reinsurer company, who in exchange pays a part of the claims incurred by Cedent.

Types of reinsurance:

(a) **Proportional:** Proportional reinsurance is a type of reinsurance where one or more reinsurers take a stated percentage share of each policy that an insurer issues then the reinsurer will receive that stated percentage of the premiums and will pay the stated percentage of claims in exchange the reinsurer will allow a ceding commission to the insurer to cover the costs incurred by the insurer

Example: Quota share

(b) Non proportional: A non-proportional reinsurance is another type of reinsurance where the reinsurer only pays to the insurer if the total claims occurred in given period is more than a fixed amount, which is called the retention.

Example: Excess of loss, stop loss

Hamilton Jacobi Equation: When time is discrete the corresponding equation is known as the Bellman equation which was developed in the 1950s by Richard Bellman. It belongs to the theory of dynamic programming. When time is continuous it is called Hamilton-Jacobi equation by William Rowan Hamilton and Carl Gustav Jacob Jacobi. The Hamilton-Jacobi-Bellman (HJB) equation is a partial differential equation. The solution of HJB equation determine the minimum cost for a given dynamical system with an associated cost function. Its solutions are called value functions. The HJB method can be also applied to stochastic systems. It has many applications in optimizing control systems.

Policy Iteration Method: The policy iteration method is an algorithm that manipulates the policy directly instead finding it indirectly via the optimal value function. It determines the value function of a policy. It executes a policy and finds the expected infinite discounted reward that will be gained. It can be obtained by solving a system of linear equations. Once we know the value of each state under the current policy, we look whether value could be improved by changing the first action taken If yes, we change the policy to take the new action whenever it is in that situation. This step must improve the performance of the policy. When no improvements are possible are possible, then the policy must be optimal.

2. Literature Review

This study is related to problems of optimization in reinsurance. Reinsurance is an insurance for insurers that is the transfer of risk from a direct insurer the cadent to a second insurance carrier the reinsurer. Insurer passes some of it premium income to a reinsurer who covers certain proportion of the claims that occur. In the literature it is proved that reinsurance would be a good method for sharing our losses or profits. It reduces risk for cedent and also reduces the probability of a direct insurer's ruin.

In this literature, we optimize the reinsurance that is an interesting research topic in the areas of insurance mathematics. Here the objective is to determine value function corresponding to which strategy it is optimal. The method is used here is iteration procedure. That solve the Hamilton Jacobi Equation. And later 4 we apply the method in Excess of loss and problems. an optimal insurance arrangement for the insurer company with some constraint from the reinsurer company. Many new concepts and methodologies for optimal reinsurance has been studied for everyone's perspective. Researchers introduce the optimum reinsurance strategies. Some researchers introduced the method of obtaining premium and objectives and the risk processes. Borch [6] proved that stop loss reinsurance can be used to minimize the retained loss. Arrow [7] said that stop loss may maximize expected utility of terminal wealth of an insurer. Kaluszka [8] take the combination of stop loss and quota share can be used to find optimal strategies to minimize a cedent's retained risk. Centeno [9] defined the optimum excess of loss retentions for two dependent risks using two objective functions maximizing insurer's expected utility of wealth net of reinsurance with respect to an exponential utility function and maximizing the adjustment coefficient of retained business respectively. The concepts of dynamic reinsurance is a classical problem of maximizing the dividends of an insurer before to ruin in a compound Poisson processes such a model was introduced by Azcue and Muler [10] for general reinsurance schemes. Mnif and Sulem [11] have studied the excess of loss reinsurance optimization problems.

3. Problem Formulation

The surplus process keeps on increasing because of premiums are deposited over time at a constant rate $\alpha > 0$. It decreases since the claims are arriving according to Poisson process $N = (N_t)t \geq 0$ with intensity $\lambda > 0$ The sequence $\{Z_n, n \in N\}$ of claims is a positive independently and identically distributed random variables with a density function $f_z(\cdot)$ With finite mean μ. Both the random variables $\{Z_n, n \in N\}$ and N are independent. The flow denotes by $(F_t)_{t \geq 0}$ represents surplus process. Let E is expectation and conditional expectation $E(\cdot | X_0 = y)$ is denoted by E_y.

Let fix path $\omega \in \Omega$ for comparison of processes starting at different initial surplus x_1 and x_2 the time of ruin, $E_{x_1}(X_{\tau_{x_1}})$ denotes the expected value of the surplus started at x_1 stopped at the time of ruin as if the surplus would have started in $x_2(x_1 > x_2)$ (thinking along the same path).

Let $\theta = \inf \{t \geq 0 | X_t < x_1 - x_2\}, E_x(X_{\tau_{x_2}}) = E_x(X_\theta)$ let the premium intensity satisfying the net-profit condition $\alpha > \lambda\mu$. expected Based on value premium principle we have $\alpha = (1 + \eta)\lambda\mu$ where safety loading $\eta > 0$.

Let a reinsurance $r : [0, \infty) \to [0, \infty)$ is a monotone increasing function satisfying the inequality $0 \leq r(z) \leq z$ i.e. r is the retention function means a claim of size z, the amount $r(z)$ is paid by the insurer and $z - r(z)$ is paid by the reinsurer. Assume that $r(z, u)$ is continuous and in r, and increasing z. then $r : [0, \infty) \times U \to R^+$ with $0 \leq r(z, u) \leq z$ then $r(\cdot, u) \in R$. $R = \{r(\cdot, u) \in R | u \in U, 0 \leq (z, u) \leq z\}$.

Where R represents set of schemes parameterized by a control parameter $u \in U$.

The inverse of $r(z, u)$ is $\rho(y, u)$ in the z exists since r is monotonic. Let a deterministic premium function $L^1(\Omega, P) \to [0, \infty)$ is represented by $\pi(z - r(z, u))$ for a fix $u \in U$ and at any time t, $\lambda\pi(z - r(z, u))$. Now the premium income of insurer is given by

$$\alpha(u) = \alpha - \lambda\pi(z - r(Z, u)). \tag{8}$$

In the sequel, we shall always assume that (u) is continuous and that full reinsurance leads to a negative premium income $\alpha < \lambda\pi(z)$.

The premium function π may be based on the expected value principle, $\pi(z - r(z, u)) = (1 + \theta)E[z - r(z, u)]$ where $\theta > \eta$ denotes the safety loading of the reinsurer, or on the variance principle,

$$\pi(z - r(z, u)) = E[z - r(z, u)] + \beta Var[z - r(z, u)] \tag{9}$$

for $\beta Var[z] > \eta\mu$.

For the proportional reinsurance $r(z, u) = uz$ where $u \in U' = [0, 1]$ and for the excess-of-loss reinsurance $r(z, u) = \min(z, u)$ and $u \in U' = [0, \infty]$.

Here infinite retention level implies no reinsurance. Since we have positive premium rate so

$$U = \{u \in U' \mid (u) \geq 0\}.$$

The meaning of a dynamic reinsurance strategy is that at each time step t, the insurer selects a control $u = u_t \in U$ with which reinsurance scheme $r(\cdot, u)$ has to select. Suppose at time step t, a claim arrives then the insurer has to give $r(z, u_t)$ and the reinsurer has to give $z - r(z, u_t)$ due to which insurer give premium at a rate $\lambda\pi(z - r(z, u_t)$ to the reinsurer a reinsurance. So, the complete formulation of the problem can be written as,

$$X_t^u = x + \int_0^t \alpha - \lambda\pi(Z - r(z, u_s))]ds - \sum_{n=1}^{N_t} r(z_n - u_{t_n}). \tag{10}$$

Rogers and Williams the integral $\int_0^t \alpha(u_s)ds$ is Lebesgue integral. The process X_u follows Poisson process and is continuous between its jump times, the process is right continuous with existing limits from the left so the X_u is measurable in t. Therefore $\int_0^t X_s\, ds$ Lebesgue integral.

4. Results on Value Function

$V(y)$ **is bounded:**

Theorem 1. (A) (Upper Bound): $V(y) \leq \frac{y}{\delta} + \frac{\alpha}{\delta^2}$ for $y \geq 0$

(B) (Lower Bound): $V(y) \geq \frac{y}{\delta} - \frac{\lambda\pi(z) - \alpha}{\delta^2}\left[1 - \frac{-\delta y}{e^{\lambda\pi z - \alpha}}\right]$ *for* $y \geq 0.$

Proof (A): For any given strategy $u = \{u_t\}t \geq 0$, $\alpha(u_t) \leq \alpha$ for all $t \geq 0.$ So the equation from the equation (1) implies $X_t^{u_t} \leq y + \alpha t$ for all $t \geq 0.$ So the supremum over all strategies u implies

$$V^u(y) \leq \int_0^\infty e^{-\delta t}(y + \alpha t)dt = \frac{y}{\delta} + \frac{\alpha}{\delta^2} \Rightarrow V^u(y) \leq \frac{y}{\delta} + \frac{\alpha}{\delta^2}. \tag{11}$$

Proof (B): Suppose strategy u_0 corresponds to buy full reinsurance until the time of ruin so a respected reserve X_{u_0}.

$$X_t^{u_0} = y + (\alpha - \lambda\pi(z))t \text{ and time of ruin } \tau_y^{u_0} = \frac{y}{\lambda\pi(z) - \alpha}.$$

Integrating

$$V^{u_0}(y) = \int_0^{\frac{y}{\lambda\pi(z)-\alpha}} e^{-\delta t}(y + \alpha t)dt = \frac{y}{\delta} - \frac{\lambda\pi(z) - \alpha}{\delta^2}[1 - e^{\frac{-\delta y}{\lambda\pi(z)-\alpha}}]$$

$$\Rightarrow V(y) \geq \frac{y}{\delta} - \frac{\lambda\pi(z) - \alpha}{\delta^2}[1 - e^{\frac{-\delta y}{\lambda\pi(z)-\alpha}}]. \tag{12}$$

5. Value Function is Monotonic and Absolutely Continuous

Theorem 2. *For* $y_1 > y_2 \geq 0$, *the value function satisfies*:

(a) $V(y_1) - V(y_2) \leq \frac{y_1 - y_2}{\delta} + C(y, y)V(y_1 - y_2)$, *where* $C(y_1, y) \to 0$ *as* $| y_1 - y | \to 0$

(b) $V(y_1) - V(y_2) \geq \frac{y_1 - y_2}{\delta + \lambda}$.

Proof (a): For a given and $\in_1 > 0$ for any given $y_1 > 0$. Consider a $\in$-optimal strategy u

$$V^u(y_1) \leq E_{y_1}\left[\int_0^{\tau_{y_1}^u} e^{-\delta t} X_t^u dt\right] + \in_1 .$$

Similarly for initial capital x_2,

$$V^u(y) \leq E_{y_2}\left[\int_0^{\tau_{y_2}^u} e^{-\delta t} X_t^u dt\right] + \in_2 .$$

Where E_y is starting value of the process.

$$\Rightarrow V(y_1) - V(y_2) \leq E_{y_1}\left[\int_0^{\tau_{y_1}^u} e^{-\delta t} X_t^u dt\right] - E_{y_2}\left[\int_0^{\tau_{y_2}^u} e^{-\delta t} X_t^u dt\right] + \in .$$

Consider a path $E = \{\omega \in \Omega \,|\, \tau_{y_1}^u(\omega) = \tau_y^u(\omega)$. On E reserves started in y_1 and y_2 move parallel with a distance $y_1 - y > 0$ and get ruined at the same point in time. Therefore,

$$V(y_1) - V(y_2) \leq E_{y_1}\left[\int_0^{\tau_{y_2}^u} e^{-\delta t} X_t^u dt\right] - E_{y_2}\left[\int_0^{\tau_{y_2}^u} e^{-\delta t} X_t^u dt\right]$$

$$+ E_{y_1}\left[1_{E_c} \int_{\tau_{y_2}^u}^{\tau_{y_1}^u} e^{-\delta t} X_t^u dt\right] + \in$$

$$\leq \frac{y_1 - y_2}{\delta} + E_{y_1}\left[1_{E_c}\int_{\tau^u_{y_2}}^{\tau^u_{y_1}} e^{-\delta t} X_t^u dt\right]$$

$$\leq \frac{y_1 - y_2}{\delta} + E_{y_1}[1_{E_c}]V(y_1 - y_2)] + \in$$

second inequality is due to

$$E_{y_1}\left[\int_0^{\tau^u_{y_2}} e^{-\delta t} X_t^u dt\right] - E_{y_2}\left[\int_0^{\tau^u_{y_2}} e^{-\delta t} X_t^u dt\right] = \left[\int_0^{\tau^u_{y_2}} e^{-\delta t}(y_1 - y_2)dt\right]$$

$$\leq \int_0^{\infty} e^{-\delta t}(y_1 - y_2)dt.$$

Since for the last inequality $X^u_{\tau^u_{y_2}} \leq y_1 - y_2$.

Let $\theta = \inf\{t \geq 0 : X_t^u < y_1 - y_2\}$ and $C(y_1, y_2) = E[1_{E_c}]$ $= P\{\tau^u_{y_2} < \tau^u_{y_1}\} = P_y\{\theta < \tau^u_y\}$

$$\Rightarrow C(y_1, y_2) \to 0 \text{ as } | y_1 - y_2 |$$

$$\Rightarrow V(y_1) - V(y_2) \leq \frac{y_1 - y_2}{\delta} + C(y_1, y_2)V(y_1, y_2).$$

Proof (b). Let $y_2 \geq 0$ and $\in > 0$ be given. Let strategy $\hat{u}$ such that $V^{\hat{u}}(y_2) + \in \geq V(y_2)$. For $y_1 > y_2$ we have, $V(y_1) - V(y) \geq E_{y_1}\left[\int_0^{\tau^{\hat{u}}_{y_1}} e^{-\delta t} X_t^{\hat{u}} dt\right]$ $- E_{y_2}\left[\int_0^{\tau^{\hat{u}}_{y_2}} e^{-\delta t} X_t^{\hat{u}} dt\right] - \in$.

Let $E = \{\tau^{\hat{u}}_{y_1} = \tau^{\hat{u}}_{y_2}\}$. If T_1 first claim then

$$V(y_1) - V(y) \geq E_{y_1}\left[\int_0^{\tau^{\hat{u}}_{y_1}} e^{-\delta t} X_t^{\hat{u}} dt\right] - E_{y_2}\left[\int_0^{\tau^{\hat{u}}_{y_2}} e^{-\delta t} X_t^{\hat{u}} dt\right]$$

$$+ E_{y_1}\left[1_{E_c}\int_{\tau_{y_2}^{\hat{u}}}^{\tau_{y_1}^{\hat{u}}} e^{-\delta t} X_t^{\hat{u}} dt\right] - \in$$

$$\geq E\left[\int_0^{T_1} e^{-\delta t}(y_1 - y_2)\right] - \in = \frac{y_1 - y_2}{\delta + \lambda} - \in .$$

Since $\in$ is arbitrary

$$\Rightarrow V(y_1) - V(y_2) \geq \frac{y_1 - y_2}{\delta + \lambda}.$$

Theorem 3. *Value function is Lipschitz continuous.*

Proof. For a given $y_1 > 0$ and $\in > 0$, we have

$$V(y) \leq E_{y_1}\left[\int_0^{\tau_{y_1}^u} e^{-\delta t} X_t^u dt\right] + \in .$$

Let $u \in U$ such that $\alpha(u) > \lambda E(r(z, u)) > 0$ and $\theta_{x_1} = \inf\{t \geq 0 : X_t^u \geq y_1$ and $X_0^u = y_2\}$. Define strategy $u^{y_2} = (u_t^{y_2}) \geq 0$ for initial capital y_2, such that $0 \leq y_2 \leq y_1$, by $u_t^{y_2} = u$ for $0 \leq t \leq \theta_{y_1}$ and $u_t^{y_2} = u_{t-\theta_{y_1}}^y$ for $t \geq \theta_{y_1}$. If the first claim occurs at $T_1 > \frac{y_1 - y}{\alpha(u)}$, then level y_1 is directly reached from level y_2. We have,

$$V(y_2) \geq E_{y_2}\left[\int_0^{\tau_{y_2}^{u^{y_2}}} e^{-\delta t} X_t^{u^{y_2}} dt\right]$$

$$\geq P\left(T_1 > \frac{y_1 - y_2}{\alpha(u)}\right)\left(\int_0^{\frac{y_1 - y_2}{\alpha(u)}} e^{-\delta t}[y_2 + \alpha(u)t\right)dt + e^{-\delta\frac{y_1 - y_2}{\alpha(u)}}(V(y_1) - \in).$$

Since $y_1 \geq y_2 \geq 0 \Rightarrow 0 \leq V(y_1) - V(y_2) \Rightarrow$

$$\leq V(y_1)\left(1 - e^{-\delta\frac{y_1 - y_2}{\alpha(u)}}\right) - e^{-\lambda\frac{y_1 - y_2}{\alpha(u)}}\left(\frac{\alpha + \delta y_2 - (\alpha + \delta y_1)e^{-\delta\frac{x_1 - x_2}{\alpha(u)}}}{\delta^2}\right) + \in$$

$$= (V(y_1)\frac{\delta+\lambda}{\alpha(u)}(y_1 - y_2) + O(y_1 - y_2)^2) + \frac{y_1}{\alpha(u)}(y_1 - y_2) + O((y_1 - y_2)^2) + \in$$

$\Rightarrow V$ is lipschitz Continuous.

Define a function $g(y) = \begin{cases} \frac{y}{\delta}, & y \geq 0, \\ 0, & y < 0 \end{cases}$

Let $L(g(y)) = \alpha g'(y) + \lambda(\int_0^y g(y-z)dF_z(z) - g(y)) > 0$ be an operator that generates process X, then define $k(y) = Lg((y)) - \delta g(y) + y,\ y \geq 0$

$$= y + \frac{\alpha}{\delta}(\delta + \lambda)\frac{y}{\delta} + \lambda\int_0^y \frac{y-z}{\delta}dF_z$$

$$= \frac{\alpha}{\delta} + \frac{\lambda}{\delta}(y(F_z(z) - 1) - \frac{\lambda}{\delta}\int_0^y zdF_z(z).$$

Differentiating with respect to y

$$K'(y) = \frac{\lambda}{\delta}(F_z(z) - 1) \leq 0 \text{ for all } y \geq 0,$$

$\Rightarrow K$ is monotone decreasing. Determination of the boundary values, $K(0) = \frac{\alpha}{\delta} > 0$ and

$$\lim y \to \infty K(y) = \frac{\alpha - \lambda\mu}{\delta} > 0$$

$\Rightarrow K$ is strictly positive.

Theorem 4. $V(y) \geq \frac{y}{\delta} + \frac{\alpha - \lambda\mu}{\delta(\delta + \lambda)}.$

Proof. Since $g(y)$ is differentiable using Dynkin's formula

$$E_y(e^{-\delta t\Lambda\tau}g(x_{t\Lambda\tau})) = g(y) + E_y\left(\int_0^{t\Lambda\tau} e^{-\delta s}[Lg(X_s) - \delta g(X_s)]\right)ds.$$

Applying the fact that $Lg(X_s) - \delta g(X_s) \geq -X_s + \frac{\alpha - \lambda\mu}{\delta(\delta + \lambda)}$

$$\Rightarrow E_y(e^{-\delta t\Lambda\tau}g(X_{t\Lambda\tau}))+E_y\left(\int_0^{t\Lambda\tau}e^{-\delta s}g(X_s)\right)ds\geq g(y)+E_y\left(\int_0^{t\Lambda\tau}e^{-\delta s}\left[\frac{\alpha-\lambda\mu}{\delta(\delta+\lambda)}\right]\right)ds$$

$$\geq g(y)+E_y\left(\int_0^{t\Lambda T_1}e^{-\delta s}\left[\frac{\alpha-\lambda\mu}{\delta(\delta+\lambda)}\right]\right)ds.$$

Since $g(X_{t\Lambda\tau})$ is bounded and monotone convergence

$$E_y\left(\int_0^{\tau}e^{-\delta s}X_s ds\right)\geq g(y)+\frac{\alpha-\lambda\mu}{\delta(\delta+\lambda)}$$

$$\Rightarrow V(y)\geq E_y\left(\int_0^{\tau}e^{-\delta s}X_s\,ds\right)\geq\frac{y}{\delta}+\frac{\alpha-\lambda\mu}{\delta(\delta+\lambda)}. \tag{13}$$

6. Bounds for Optimum of V^u

$$\frac{y}{\delta}+\frac{\alpha-\lambda\mu}{\delta(\delta+\lambda)}\leq V(y)\leq\frac{y}{\delta}+\frac{\alpha}{\delta^2}\; y\geq 0 \tag{14}$$

and $\alpha,\ \delta,\ \lambda,\ \mu$ are fixed constants. So

where $V^u(y)=\frac{y}{\delta}+C\Rightarrow\frac{\alpha-\lambda\mu}{\delta(\delta+\lambda)}\leq C\leq\frac{\alpha}{\delta^2}$ for all $y\geq 0$. Values of C are range of optimum of V^u.

Theorem 5. *The value function V satisfies Hamilton Jacobi equation (HJB).*

$$\sup_{u\in U}\left\{y+\alpha(u)V'(y)-(\delta+\lambda)V(y)+\lambda\int_0^{\rho(y,\,u)}V(y-r(z,\,u))dF_z(z)\right\}=0.$$

Proof. In the dynamic programming methods of solving optimization problems f or every F_t and a stopping time $S\geq 0$. Value function can be obtained from the following equation

$$V(y)=\sup_{u\in U}E_y\left[\int_0^{\tau_y^u\Lambda S}e^{-\delta t}X_t^u dt+e^{-\delta(\tau_y^u\Lambda S)}V\left(X^u_{\tau_y^u\Lambda S}\right)\right] \tag{15}$$

Step (1). In the first step, we show that right side of equation (15) smaller or equal to zero. Let $y > 0$, $h > 0$ and strategy $u \in U$ be as follows $\tilde{u} = (u_t)t \geq 0$ such that $u_t = u$ in $[0, h]$ and $u_t = \tilde{u}_{t-h}$ for $t > 0$ for some $\tilde{u} \in U$. For small $h > 0$ such that $y + \alpha(u)h > 0$.

Let T_1 be time of the first claim occurrence and set $S = \min\{T_1, h\}$. Then the equation from (15)

$$0 \geq E_y\left[\int_0^S e^{-\delta t}(y + \alpha(u)t)dt + e^{-\delta t}V(X_S^{\tilde{u}}) - V(y)\right]. \tag{16}$$

Since u is a constant control it applies on the interval $[0, S]$ we can apply Rolski [13] and get that V lies in the domain of the generator where generator A^u is a constantly controlled process X^u Rolski [13] implies

$$A^u(g(y)) = \alpha(u)g'(y) - \lambda g(y) + \lambda \int_0^{\rho(y,\, u)} g(y - r(y,\, z))dF_z(z).$$

Since absolutely continuous map $t \to V(y + \alpha(u)t)$ and the boundary is empty and we proved the integrable condition. So Dynkin formula, and equation (16) implies

$$0 \geq E_y\left[\int_0^S e^{-\delta t}(y + \alpha(u)t)dt + \int_0^S e^{-\delta t}(\alpha(u)V'(y) + \alpha(u)t - (\delta + \lambda)V(y + \alpha(u)t))\right.$$

$$\left. + \lambda \int_0^S V(y + (u)t - r(z,\, u)dF_z(z))dt\right].$$

Divide both sides by h we have

$$0 \geq \frac{1}{h}E_y\left[\int_0^S e^{-\delta t}(y + \alpha(u)t) - (\delta + \lambda)V(y + \alpha(u)t)\right.$$

$$\left. + \lambda \int_0^S V(y + \alpha(u)t - r(z,\, u))dF_z(z)dt\right]$$

$$+ \frac{1}{h}E_y\left[\int_0^S e^{-\delta t}V'(y + \alpha(u)t)dt\right].$$

The first term in the right is Riemann integrable and V is continuous so $h \to 0$ implies

$$0 \geq y - (\delta + \lambda)V(y) - (\delta + \lambda)V(y) + \lambda \int_0^S V(y + \alpha(u)t - r(z, u))dF_z(z))$$

$$+ \lim_{h \to 0} \frac{1}{h} E_y\left[\int_0^S e^{-\delta t}\alpha(u)V'(y + \alpha(u)t)dt\right].$$

Consider the limit

$$\lim_{h \to 0} \frac{1}{h} E_y\left[\int_0^S e^{-\delta t}\alpha(u)V'(y + \alpha(u)t)dt\right]$$

$$= \lim_{h \to 0} \frac{1}{h} e^{-\lambda h}\left[\int_0^h e^{-\delta t}\alpha(u)V'(y + \alpha(u)t)dt\right]$$

$$+ \lim_{h \to 0} + \frac{1}{h}\left[\int_0^h \lambda e^{-\lambda s}\int_0^S e^{-\delta t}\alpha(u)V'(y + \alpha(u)t)dt\,ds\right] = \alpha(u)V'(y)$$

Applying Lebesgue's differentiation concepts and Wheeden and Zygmund [14], since $V'(x)$ is Lebesgue integrable because of bounded and monotonic properties

$$\Rightarrow \frac{1}{h}[\int_0^h \lambda e^{-\lambda s}\int_0^S e^{-\delta t}\alpha(u)V'(y + \alpha(u)t)dt\,ds] = 0.$$

Since $s = 0$ integrand ds is also zero $u \in U$ is arbitrary

$$0 \geq \sup_{u \in U}\{y + \alpha(u)V'(y) - (\delta + \lambda)V(y) + \lambda \int_0^{\rho(y, u)} V(y - r(z, u))dF_z(z)\}. \quad (17)$$

Step (2). In this step we will prove that right hand side of equation (15) is greater than or equal to zero. Let $S = \min\{T_1, h\}$ where $h > 0$ and let $u^1 = (u_t^1)t \geq 0$ be h^2- optimal strategy so the right hand side of (15) can be written as

$$V(y) = \sup_{u \in U} y\left[\int_0^S e^{-\delta t}\left(y + \int_0^t \alpha(u_s)ds\right)dt + e^{-\delta t}V(X_S^u)\right]$$

$$< E_y\left[\int_0^S e^{-\delta t}\left(y + \int_0^t \alpha(u_s^1)ds)\right)dt + e^{-\delta t}V(X_S^{u_s^1}) + h^2 + \epsilon h\right],$$

where $\varepsilon > 0$ is arbitrary. Suppose $T_1 \sim Exp(\lambda)$ now above equation becomes

$$0 < E_y[\int_0^S e^{-\delta t}(y + \int_0^t \alpha(u_s^1)ds)dt + e^{-(\delta+\lambda)h} - 1)E_y[V(y + \int_0^h \alpha(u_s^1)ds)]$$

$$+ E_y\left[\int_0^h \lambda e^{-\lambda t}\int_0^{\rho(y+\int_0^t \alpha(u_s^1)ds,\, u_s^1)} V\left(y + \int_0^t \alpha(u_s^1)\right)ds - r(z, u_s^1)dF_z(z)dt\right.$$

$$+ E_y\left[V\left(y + \int_0^h \alpha(u_s^1)ds - V(y)\right)\right] + h^2 + \epsilon h$$

$$= A + B + C + D + h^2 + \epsilon h \text{ (let)}.$$

Apply dividing by h and dominated convergence Theorem.

$$B : \lim_{h\to 0}\frac{1}{h}(e^{-(\delta+\lambda)h} - 1)E_y\left[V\left(y + \int_0^h \alpha(u_s^1)ds\right)\right] = -(\delta+\lambda)V(y)$$

which follows from continuity of V.

$$C : \lim_{h\to 0}\frac{1}{h}E_y\left[\int_0^h \lambda e^{-\lambda t}\int_0^{\rho\left(y+\int_0^t \alpha(u_s^1)ds,\, u_s^1\right)} V\left(y + \int_0^t \alpha(u_s^1)ds\right) - r(z, u_s^1)dF_z(z)dt\right]$$

$$= \lambda\int_0^{\rho\left(y,\, u_0^1\right)} V(y - r(z, u_0^1)dF_z(z).$$

Applying Wheeden and Zygmund [14] ***D***. Using absolute continuity property of V

$$\lim_{h\to 0}\frac{1}{h}\, y\left[V\left(y + \int_0^h \alpha(u_s^1)ds)\right)V(y)\right]$$

$$= \lim_{h\to 0}\frac{1}{h}E_y\left[\int_0^{\int_0^h \alpha(u_s^1)ds} V'(y+z)dz\right]$$

$$= \lim_{h\to 0}\frac{1}{h}E_y[\int_0^h \alpha(u_s^1)V'(y + \int_0^t \alpha(u_s^1)ds)dt]$$

$$= \alpha(u_s^1)V'(y)a$$

$$A : \lim it_{h\to 0} \frac{1}{h} [\int_0^S e^{-\delta t}(y + \int_0^t \alpha(u_s^1)ds)dt] = y$$

$$0 \leq y + \alpha(u_0^1)V' - (\delta + \lambda)V(y) + \lambda\int_0^{\rho(y,\, u_0^1)} V(y - r(z,\, u_0^1)dF_z(z) + \epsilon. \tag{18}$$

Since ϵ is arbitrary Thus equation (17) and (18) implies Hamilton Jacobi Bellman Equation:

$$\sup_{u\in U}\{y + \alpha(u)V'(y) - (\delta + \lambda)V(y) + \lambda\int_0^{\rho(y,\, u)} V(y - r(z,\, u))dF_z(z)\} = 0. \tag{19}$$

Lower bound of premium $\alpha(u)$:

Since the function $u(x)$ exists such that the supremum equal to zero is obtained. So equation (19) can be written as

$$y + \alpha(u)V'(y) - (\delta + \lambda)V(y) + \lambda\int_0^{\rho(y,\, u)} V(y - r(z,\, u))dF_z(z) = 0 \tag{20}$$

$$V(y) \leq \frac{y}{\delta} + \frac{\alpha}{\delta^2} \Rightarrow \alpha(u(y))V(y) > 0 \Rightarrow \alpha(u(y)) > 0.$$

Let $U = \{u \in U \mid \alpha(u) \geq 0\}$ and V is monotonic

$$V'(y) = \inf_{u\in\overline{U}}\left\{\frac{(\delta + \lambda)V(y) - y - \lambda\int_0^{\rho(y,\, u)} V(y - r(z,\, u))dF_z(z)}{\alpha(u)}\right\}$$

$$\leq \frac{(\delta + \lambda)V(y) - y - \lambda\int_0^{\rho(y,\, u)} V(y - z)dF_z(z)}{\alpha}$$

$$\leq \frac{(\delta + \lambda)V\left(\frac{y}{\delta} + \frac{\alpha}{\delta^2}\right) - y - \lambda\int_0^{\rho(y,\, u)} \frac{(y - z)}{\delta}dF_z(z)}{\alpha} \tag{21}$$

$$V'(y) \leq \frac{(\delta + \lambda)\frac{\alpha}{\delta^2} + \frac{\lambda\mu}{\delta} + H(y)}{\alpha}. \tag{22}$$

Where $H(y) = \frac{\lambda}{\delta} y(1 - F_z(y)) \geq 0$ also $H(y) = \frac{\lambda}{\delta} \int_y^{\infty} yF_z(z) \leq \frac{\lambda\mu}{\delta}$ equality (22) becomes now

$$\Rightarrow V'(y) \leq \frac{(\delta + \lambda)\frac{\alpha}{\delta^2} + \frac{2\lambda\mu}{\delta}}{\alpha}. \tag{23}$$

Now equation (23)

$$\alpha(u)V'(y) = (\delta + \lambda)V(y) - y - \lambda \int_0^{\rho(y,\, u)} V(y - r(z,\, u))dF_z(z)$$

$$= \delta V(y) - y + \lambda(V(y) - \int_0^{\rho(y,\, u)} V(y - r(z,\, u))dF_z(z))$$

$$\alpha(u)V'(y) \geq \frac{\alpha - \lambda\mu}{\delta + \lambda}$$

$$\alpha(u) \geq \frac{\alpha - \lambda\mu}{\delta + \lambda} \left(\frac{(\delta + \lambda)\frac{\alpha}{\delta^2} + \frac{2\lambda\mu}{\delta}}{\alpha} \right)^{-1} = L > 0.$$

Thus our compact set U is constrained to $\overline{U} = \{u \in U \mid \alpha(u) \geq L\}$.

7. Policy Iteration Method

Policy iteration process is a method of constructing a sequence of policies with monotonically increasing rewards. We solve the Hamilton Jacobi equation using Policy Iteration algorithm. It is an improved policy iteration method. A vector $V^n \in V$ has and the improving policy u_n. Hamilton Jacobi equation

$$\sup_{u \in U} \{y + \alpha(u)V'(y) - (\delta + \lambda)V(y) + \lambda \int_0^{\rho(y,\, u)} V(y - r(z,\, u))dF_z(z)\} = 0.$$

Finite difference discretization $V'(y) = \frac{v(y + h) - v(y)}{h}$ in each of intervals $[0,\, h],\, [h,\, 2h],\, \cdots [(n-1)h,\, nh],\, [nh,\, (n+1)h]$.

$$L_{u_n} V^n = r_{u_n} + bP_{u_n} V^n.$$

Where P_{u_n} is a stochastic matrix and the operator $L_{u_n} : V \to V$ is a contraction operator for given $b \in (0, 1)$ and I is a identity operator and L_{u_n} has a unique fixed point i.e. $\| I - bP_{u_n} \| < 1$ and the operator $I - bP_{u_n}$ is invertible. Substitute

$$r_{u_n} + bP_{u_n} V^n = y + \alpha(u_n) V'(y) - (\delta + \lambda) V(y) + \lambda \int_0^{\rho(y, u_n)} V(y - r(z, u_n)) dF_z(z)$$

Where $u_n \in \arg\max\ u \in U\{r_{u_n} + bP_{u_n} V^n\}$.

Algorithm:

Step (1). Set $n = 0$ and an initial strategy u^0

Step (2). (Policy Evaluation) obtain initial vector V^n By solving the following linear system of equations $(I - bP_{u_n}) V^n = r_{u_n}$

Step (3). (Policy improvement): Choose u_{n+1} so that $u_{n+1} = \arg\max u \in U\{r_{u_n} + bP_{u_n} V^n\}$ setting $u_{n+1} = u_n$ whenever possible

Step (4). If $u_{n+1} = u_n$ stop and set $u^* = u_n$ Otherwise, increase n by 1 and go to step 2.

Initial strategy: Let V^1 denotes initial value function. It can be calculated from when there is no reinsurance i.e., substitute $r(y, u) = 0$ and $\pi(y - r(y, u)) = 0$ in equation (10) obtained initial value function $V^1 = x + ct$ Then the initial strategy $u^1 = (u_t^1) t \geq 0$ can be calculated using HJB equation as follows

$$u^1(y) = \arg\sup_{u \in \overline{U}} \{y + \alpha(u) \frac{\partial V^1}{\partial y} - (\delta + \lambda) V^1(y) + \lambda \int_0^{\rho(y, u)} V^1(y - r(z, u)) dF_z(z)\}$$

8. Numerical Examples

Example (1). Proportional Reinsurance: Quota Share

Quota share is an example of proportional reinsurance where an insurance company use a quota share reinsurance as 25 percent on claims and claims has gamma distribution i.e.,

$$f(y) = \gamma^2 y e^{-\gamma y} \text{ with } \mu = \frac{2}{\gamma} \text{ and } \eta > 0 \text{ and } \theta > \eta.$$

Then $r(y, u) = 25\% = 0.25 = uy$ for all $u \in [0, 1]$. $\lambda\pi(z - r(z, u)) = \lambda\pi(z - 0.25)$ and insurer premium rate insurer $\alpha(u) = \alpha - \lambda\pi(z - 0.25)$. The insurer's premium rate is determined via the expected value principle and reads as $c = (1 + \eta)\gamma\mu$ are the input variables and function to apply the iteration process.

The parameter set for quota share was given in the table 1 for gamma distribution of claims. Here the convergence or optimization of strategy occurs at the value of $y = 50$.

Table 1. Set of parameters for Quota share reinsurance.

$\gamma = 0.2$	$\eta = 0.1$	$\theta = 0.11$	$\lambda = 1$	$\delta = 0.01$

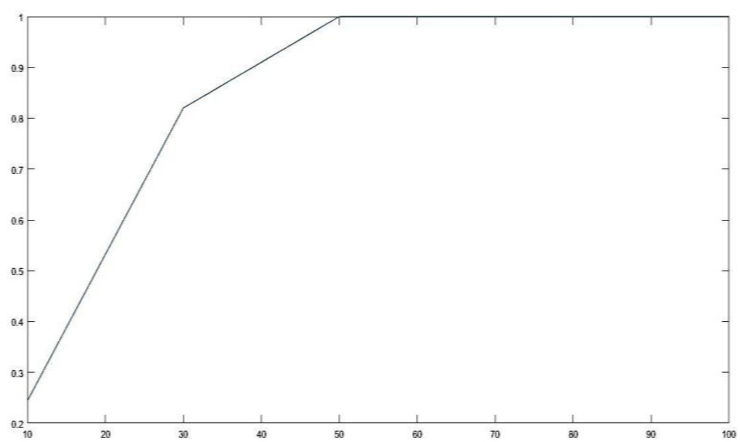

Figure 1. Illustration of optimal strategy in quota share.

Example (2). Non proportional Reinsurance: Excess of Loss reinsurance (XL-type) Excess of Loss is a non proportional reinsurance type. Consider dynamic XL-reinsurance whose claim amounts distribution is $Exp(v)$ i.e distribution function is given by

$$f(y, v) = \begin{cases} v\exp(-vy) & if\ y \geq 0 \\ 0 & if\ y < 0. \end{cases}$$

Derivative of cumulative distribution function is probability density function $\frac{dF_y}{dy} = f(y, v) \Rightarrow dF_y = f(y, v)dy.$ Retention function $r(y, u) = \min(y, u)$ where $u \in [0, \infty]$ if $r = \infty$ then there no reinsurance. $\lambda\pi(z - r(z, u))$ where $\pi(z - r(z, u)) = E[z - r(z, u)] + \beta Var[z - r(z, u)]$ and the premium income of insurer $\alpha(u) = \alpha - \lambda\pi(z - r(Z, u))$. Expected value premium principle $\alpha(u) = \lambda\mu(u(1 + \theta) - (\theta - \eta))$ are the input variables and functions in the policy iteration procedure. When considering the exponential form of claims, the optimal strategy for the non-proportional reinsurance in table 2, the optimum strategy for the XL reinsurance and stop loss reinsurance as shown in figure 2 and 3.

Table (2). Set of Parameters for Excess of Loss XL-Type and stop loss reinsurance.

$v = 1$	$\eta = 0.5$	$\theta = 0.65$	$\lambda = 1$	$\delta = 0.01$

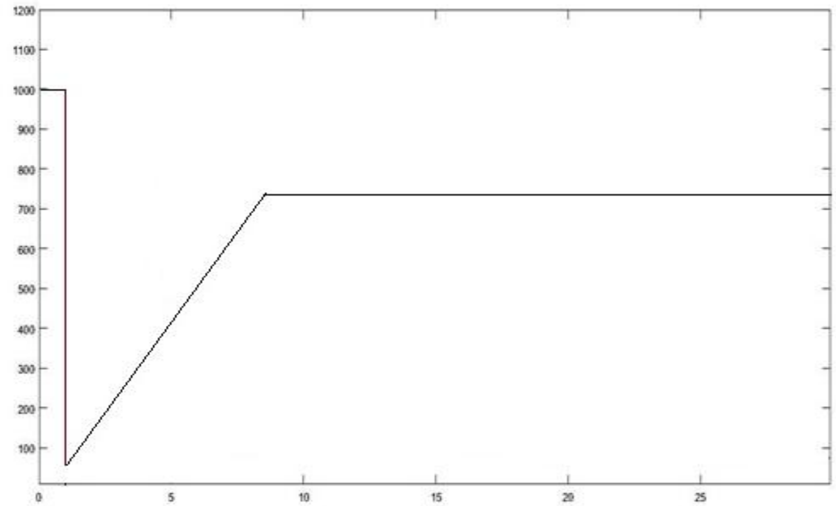

Figure 2. Illustration of optimal strategy in XL reinsurance.

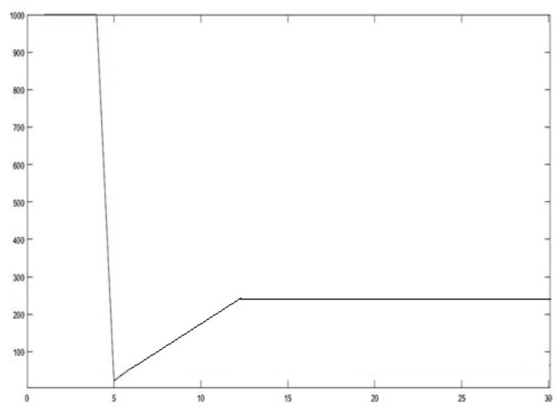

Figure 3. Illustration of optimal strategy in stop loss reinsurance.

9. Conclusions

In this model, we used the iteration algorithm to optimize the value function that is a solution of Hamilton Jacobi Bellman equation. The model development is based on dynamical methods in risk theory. We also applied the model quota share and Excess of Loss. Policy method provides better return functions. Model suggests that reinsurance can accelerate the process of building up a surplus and that the use of reinsurance is beneficial in economic firms.

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