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SOME FIXED POINT RESULTS FOR ρ-FUNDAMENTALLY NONEXPANSIVE MAPPINGS IN MODULAR FUNCTION SPACES

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Abstract

Our aim is to introduce ρ-fundamentally nonexpansive mapping in modular function spaces and prove some lemmas and fixed point existence results. Also, we provide an example to support our results.

1. Introduction

Modular function spaces are the generalization of some class of Banach spaces which attracts many analysts to work in this field. The concept of fixed point in modular function spaces was introduced by Khamsi, Kozlowski and Reich [6] in 1990. The existence and convergence theorems for fixed points for nonexpansive type mappings in modular function space have been given by many researchers ([2, 4, 5, 8]). In 2016, Moosaei [7] gave some basic properties and fixed point theorems for fundamentally nonexpansive mappings in Banach spaces. It can be easily shown that nonexpansiveness implies fundamentally nonexpansiveness but converse need not be true.

Example 1.1 [7]. Define a mapping T on [0, 4] as follows:

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$$Tx = \begin{cases} 1 & \text{if } x \neq 4 \\ 2.5 & \text{if } x = 4 \end{cases}$$

for all $x \in [0, 4]$. Then T is fundamentally nonexpansive but not nonexpansive. Motivated by him, we introduced the concept of fundamentally nonexpansive mappings in modular function spaces as ρ-fundamentally nonexpansive mappings. In this paper, we prove basic properties and some fixed point theorem for ρ-fundamentally nonexpansive mapping in context of function modular ρ.

2. Preliminaries

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a nontrivial δ-ring of subsets of Ω which means that $\mathcal{P}$ is closed under countable intersection, and finite union and differences. Suppose that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By ε we denote the linear space of all simple functions with support from $\mathcal{P}$. Also $\mathcal{M}_\infty$ denotes the space of all extended measurable functions, i.e., all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence

$$\{g_n\} \subset \varepsilon, |g_n| \leq |f| \text{ and } g_n(w) \to f(w) \text{ for all } w \in \Omega.$$

We define

$$\mathcal{M} = \{f \in \mathcal{M}_\infty : |f(w)| < \infty \, \rho - \text{a.e.}\}.$$

Definition 2.1 [9]**.** Let X be a vector space *(R or C)*. A functional ρ is called a modular if for arbitrary elements $f, g \in X$, the following hold:

(i) $\rho(f) = 0 \Leftrightarrow f = 0$

(ii) $\rho(\alpha f) = \rho(f)$ whenever $|\alpha| = 1$

(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ whenever $\alpha, \beta \geq 0, \alpha + \beta = 1$.

If we replace (iii) by

(iv) $\rho(\alpha f + \beta g) \le \alpha\rho(f) + \beta\rho(g)$ whenever $\alpha, \beta \ge 0$, $\alpha + \beta = 1$.

Then modular ρ is called convex.

Definition 2.2 [9]. If ρ is convex modular in X, then the set defined by

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}$$

is called modular function space. Generally, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. However, the modular function space L_ρ can be equipped with an F-norm defined by

$$\| f \|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le \alpha\right\}.$$

In the case ρ is convex modular

$$\| f \|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le 1\right\}.$$

defines a norm on modular function space L_ρ and it is called Luxemburge norm.

Definition 2.3 [9]. Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. Then ρ is a regular convex function pseudo modular if

1. $\rho(0) = 0$;

2. ρ is monotone, i.e., $|f(w)| \le |g(w)|$ for any $w \in \Omega$ implies $\rho(f) \le \rho(g)$, where $f, g \in \mathcal{M}_\infty$;

3. ρ is orthogonally sub-additive, i.e., $\rho(f\chi_{A\cup B}) \le \rho(f\chi_A + \rho(f\chi_B))$ for any $A, B \in \Sigma$ such that $A \cap B \ne \phi$, $f \in \mathcal{M}_\infty$;

4. ρ has Fatou property, i.e., $|f_n(w)| \uparrow |f(w)|$ for $w \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;

5. ρ is order continuous in ε, i.e., $g_n \in \varepsilon$ and $|g_n(w)| \downarrow 0$, then $\rho(g_n) \downarrow 0$.

Definition 2.4. Let ρ be a regular convex pseudo modular. Then ρ is regular convex function modular if $\rho(f) = 0$ implies $f = 0$ a.e. The class of all nonzero regular convex function modular on Ω is denoted by $\mathfrak{R}$.

Proposition 2.5 [9]. *Let* $\rho \in \mathfrak{R}$.

(i) L_p *is* ρ*-complete.*

(ii) ρ*-balls* $B_\rho(f, r) = \{g \in L_\rho : (f - g) \leq r\}$ *are* ρ*-closed.*

(iii) *If* $\rho(\alpha f_n) \to 0$ *for* $\alpha > 0$ *then there exists a subsequence* $\{g_n\}$ *of* $\{f_n\}$ *such that* $g_n \to 0 \rho - a.e.$ *as* $n \to \infty$.

(iv) $\rho(f) \leq \liminf_{n\to\infty} \rho(f_n)$ *whenever* $f_n \to f \rho - a.e$ *as* $n \to \infty$. (*Note*: *this property is equivalent to the Fatou property*.)

(v) *Consider the set* $L_\rho^0 = \{f \in L_\rho : \rho(f, \cdot)$ *is order continuous*$\}$ *and* $E_\rho = \{f \in L_\rho : \lambda f \in L_\rho^0$ *for any* $\lambda > 0\}$.

Then we have $E_\rho \subset L_\rho^0 \subset L_\rho$.

Lemma 2.6 [2]. *Let* $\rho \in \mathfrak{R}$ *and satisfy* (*UUC1*). *Let* $\{t_n\} \subset (0, 1)$ *be bounded away from both* 0 *and* 1. *If there exists* $R > 0$ *such that*

$$\lim_{n\to\infty} \sup \rho(f_n) \leq R, \lim_{n\to\infty} \sup \rho(g_n) \leq R \text{ and}$$

$$\lim_{n\to\infty} \rho(t_n f_n + (1 - t_n) g_n) = R, \text{ then}$$

$$\lim_{n\to\infty} \rho(f_n - g_n) = 0.$$

The sequence $\{t_n\} \subset (0, 1)$ *is said to be bounded away from* 0 *if there exists* $a > 0$ *such that* $t_n \geq a$ *for all* $n \in \mathbb{N}$. *Similarly the sequence* $\{t_n\} \subset (0, 1)$ *is said to be bounded away from* 1 *if there exists* $b < 1$ *such that* $t_n \leq b$ *for all* $n \in \mathbb{N}$.

Lemma 2.7. *Let* $\{f_n\}$ *and* $\{g_n\}$ *be two bounded real sequences. Then*

1. $\lim_{n\to\infty} \sup \max \{f_n, g_n\} = \max \{\lim_{n\to\infty} \sup f_n, \lim_{n\to\infty} \sup g_n\}$

2. *let* $h_n = \alpha_n f_n + (1 - \alpha_n) g_n$ *with* $\alpha_n \in [0, 1]$ *convergent to a real number* $\alpha \in [0, 1]$. *Then* $\lim_{n\to\infty} \sup h_n \leq \alpha \lim_{n\to\infty} \sup f_n + (1 - \alpha) \lim_{n\to\infty} \sup g_n$.

Lemma 2.8 [1]**.** *Assume that* L_ρ *be modular function space. Let* ρ *satisfy the Fatou property. Let C be a nonempty* ρ*-closed convex subset of* L_ρ *and* $\{f_n\}$ *be a sequence in* L_ρ *with a finite asymptotic radius relative to C. If* ρ*-satisfies the* (*UUC1*)*-condition, then all the minimizing sequences of* τ *are modular-convergent, having the same* ρ*-limit.*

3. Main Results

In this section, firstly we define ρ-fundamentally nonexpansive mappings in modular function spaces and then prove some lemmas and fixed point existence results. Also, we provide an example in which the mapping T is ρ-fundamentally nonexpansive mapping but not nonexpansive.

Definition 3.1. Let $\rho \in \Re$ and D_ρ be a nonempty ρ-bounded, ρ-closed, ρ-convex subset of L_p. Then $T : D_\rho \to D_\rho$ is said to be fundamentally nonexpansive mapping if for each $f, g \in D_\rho$

$$\rho(T^2 f - Tg) \leq \rho(Tf - g).$$

Lemma 3.2. *Let* $\rho \in \Re$ *and* D_ρ *be a nonempty subset of* L_ρ. *Let* $T : D_\rho \to D_\rho$ *be a mapping. Then the following hold*:

(i) *If T is nonexpansive*, *then it is fundamentally nonexpansive*.

(ii) *If T is fundamentally nonexpansive with* $F_\rho(T) \neq \phi$, *then T is quasi-nonexpansive*.

Proof. (i) If T is nonexpansive, then

$$\rho(T^2 f - Tg) \leq \rho(Tf - g),$$

which shows that T is fundamentally nonexpansive.

(ii) If T is fundamentally nonexpansive with $F_\rho(T) \neq \phi$, then let $p \in F_\rho(T)$

$$\rho(Tf - p) = \rho(Tf - T^2 p) = \rho(T^2 p - Tf) \leq \rho(Tp - f) = \rho(p - f),$$

hence $\rho(Tf - p) \leq \rho(f - p)$. This proves that T is quasi-nonexpansive.

Lemma 3.3. *Let $\rho \in \Re$ and D_ρ be a nonempty ρ-bounded, ρ-closed, ρ-convex subset of L_ρ. Let $T : D_\rho \to D_\rho$ be fundamentally nonexpansive mapping and $F_\rho(T) \neq \phi$, then $F_\rho(T)$ is closed and convex.*

Proof. Suppose that $\{f_n\}$ is a sequence in $F_\rho(T)$ which ρ-converges to some $f \in D_\rho$.

$$\rho(f_n - Tf) = \rho(T^2 f_n - Tf) \leq \rho(Tf_n - f) = \rho(f_n - f)$$

$$\lim_{n\to\infty} \sup \rho(f_n - Tf) \leq \lim_{n\to\infty} \sup \rho(f_n - f).$$

By uniqueness of asymptotic center, $Tf = f$ and hence $f \in F_\rho(T)$ which shows that $F_\rho(T)$ is closed. Now, it remains to show that $F_\rho(T)$ is convex. Let $f, g \in F_\rho(T)$ and $h = \frac{f+g}{2}$.

$$\rho(f - Th) = \rho(T^2 f - Th) \leq \rho(Tf - h) = \rho(f - h) = \rho\left(\frac{f-g}{2}\right) \tag{3.1}$$

$$\rho(g - Th) = \rho(T^2 g - Th) \leq \rho(Tg - h) = \rho(g - h) = \rho\left(\frac{f-g}{2}\right). \tag{3.2}$$

Also,

$$\rho(f - h) = \rho\left(\frac{f-g}{2}\right), \ \rho(g - h) = \rho\left(\frac{f-g}{2}\right). \tag{3.3}$$

$$\rho\left(f - \frac{h + Th}{2}\right) = \rho\left(\frac{1}{2}\rho(f - h) + \frac{1}{2}\rho(f - Th) \leq \frac{1}{2}\rho(f - h)\right) + \frac{1}{2}\rho(f - Th)$$

$$\leq \frac{1}{2}\rho\left(\frac{f-g}{2}\right) + \frac{1}{2}\rho\left(\frac{f-g}{2}\right) = \rho\left(\frac{f-g}{2}\right).$$

$$\rho\left(g - \frac{h + Th}{2}\right) = \rho\left(\frac{1}{2}\rho(g - h) + \frac{1}{2}\rho(g - Th)\right) \leq \frac{1}{2}\rho(g - h) + \frac{1}{2}\rho(g - Th)$$

$$\leq \frac{1}{2}\rho\left(\frac{f - g}{2}\right) + \frac{1}{2}\rho\left(\frac{f - g}{2}\right) = \rho\left(\frac{f - g}{2}\right).$$

Therefore,

$$\rho\left(\frac{f - g}{2}\right) \leq \frac{1}{2}\rho\left(f - \frac{h + Th}{2}\right) + \frac{1}{2}\rho\left(\frac{h + Th}{2} - g\right) \text{ or } \rho\left(\frac{f - g}{2}\right) \leq \rho\left(f - \frac{h + Th}{2}\right)$$

We conclude that

$$\rho\left(f - \frac{h + Th}{2}\right) = \rho\left(\frac{1}{2}\rho(f - Th) + \frac{1}{2}\rho(f - h)\right) = \rho\left(\frac{f - g}{2}\right). \tag{3.4}$$

Using (3.1), (3.3), (3.4) and Lemma 2.6, $\rho(h - Th) = 0$ and hence $h \in F_\rho(T)$ which implies that $F_\rho(T)$ is convex.

Lemma 3.4. *Let* $\rho \in \Re$ *and* D_ρ *be a nonempty* ρ*-bounded*, ρ*-closed,* ρ*-convex subset of* L_ρ. *Let* $T : D_\rho \to D_\rho$ *be fundamentally nonexpansive mapping, then* $F_\rho(T)$ *is nonempty.*

Proof. By Lemma 2.8, the asymptotic center of a sequence in D_ρ, particularly, the approximate fixed point for T is in D_ρ. Let $A(\{f_n\}) = f$. We show that $f \in F_\rho(T)$.

$$\rho(f_n - Tf) = \rho(T^2 f_n - Tf) \leq \rho(Tf_n - f) = \rho(f_n - f)$$

$$\lim_{n\to\infty} \sup \rho(f_n - Tf) \leq \lim_{n\to\infty} \sup \rho(f_n - f).$$

As asymptotic center is unique, $Tf = f$ and hence $f \in F_\rho(T)$. This shows that $F_\rho(T)$ is nonempty. □

Theorem 3.5. *Let* $\rho \in \Re$ *and* D_ρ *be a nonempty* ρ*-bounded,* ρ*-closed,* ρ*-convex subset of* L_p. *Let* $T : D_\rho \to D_\rho$ *be fundamentally nonexpansive mapping and* $T(D_\rho)$ *be* ρ*-bounded,* ρ*-convex. Define a sequence* $\{Tf_n\}$ *in* $T(D_\rho)$ *by* $f_1 \in D_\rho$ *and*

$$Tf_{n+1} = \lambda T^2 f_n + (1 - \lambda) Tf_n \text{ for all } n \in \mathbb{N},$$

where $\lambda \in (0, 1)$. *Then* $F_\rho(T) \neq \phi$ *if and only if* $\lim_{n\to\infty} \rho(T^2 f_n - Tf_n) = 0$.

Proof. Let $p \in F_\rho(T)$. Then, using the convexity of ρ, we have

$$\begin{aligned} \rho(Tf_{n+1} - p) &= \rho(\lambda T^2 f_n + (1 - \lambda) Tf_n - p) \\ &= \rho(\lambda(T^2 f_n - p + (1 - \lambda)(Tf_n - p)) \\ &\leq \lambda\rho(T^2 f_n - p) + (1 - \lambda)\rho(Tf_n - p) \\ &\leq \lambda\rho(Tf_n - p) + (1 - \lambda)\rho(Tf_n - p) \\ &= \rho(Tf_n - p). \end{aligned}$$

This implies that $\{\rho(Tf_n - p)\}$ is non-negative decreasing sequence. Since $T(D_\rho)$ is ρ-bounded and ρ-convex, therefore $\{Tf_n - p\}$ lies in $T(D_\rho)$ and hence $\rho(Tf_n - p) < \infty$. Therefore, $\{\rho(Tf_n - p)\}$ is convergent. Let

$$\lim_{n\to\infty} \rho(Tf_n - p) = R \tag{3.5}$$

And

$$\rho(T^2 f_n - p) = \rho(T^2 f_n - Tp) \leq \rho(Tf_n - p)$$

$$\lim_{n\to\infty} \rho(T^2 f_n - p) = R \tag{3.6}$$

$$\lim_{n\to\infty} \rho(Tf_{n+1} - p) = \lim_{n\to\infty} \rho(\lambda(T^2 f_n - p) + (1 - \lambda)(Tf_n - p)) = R. \tag{3.7}$$

Using (3.5), (3.6), (3.7) and Lemma 2.6, $\lim_{n\to\infty} \rho(Tf_n - T^2 f_n) = 0$. Now, we prove the converse part. Let $\tau, \bar{\tau} : D_\rho \to [0, \infty)$ be ρ-type functions corresponding to sequence $\{Tf_n\}$ and $\{T^2 f_n\}$. We prove that for each $p \in D_\rho \bar{\tau}(Tp) \leq \tau(p)$.

$$\rho(T^2 f_n - Tp) \leq \rho(Tf_n - p) \tag{3.8}$$

Now,

$$\rho(T^2 f_n - Tp) \le w(2)[\rho(T^2 f_n - Tf_n) + \rho(Tf_n - Tp)]$$

$$\le w(2)(\rho(T^2 f_n - Tf_n) + w(2)[\rho(Tf_n - T^2 f_n) + \rho(T^2 f_n - Tp)])$$

$$\le w(2)[(1 + w(2))\rho(Tf_n - T^2 f_n) + w(2)\rho(T^2 f_n - Tp)]$$

$$(1 - (w(2)^2)\rho(T^2 f_n - Tp) \le w(2)(1 + w(2))\rho(Tf_n - T^2 f_n)$$

$$\rho(T^2 f_n - Tp) \le \frac{w(2)(1 + w(2))}{(1 - (w(2))^2)}\rho(Tf_n - T^2 f_n)$$

$$\rho(T^2 f_n - Tp) \le \frac{\omega(2)}{(1 - \omega(2))}\rho(Tf_n - T^2 f_n). \tag{3.9}$$

From inequality (3.8) and (3.9),

$$\rho(T^2 f_n - Tp) \le \max\left\{\rho(Tf_n - p), \frac{\omega(2)}{(1 - \omega(2))}\rho(Tf_n - T^2 f_n)\right\}$$

Applying $\lim_{n\to\infty}\sup$ and Lemma 2.7,

$$\bar{\tau}(Tp) \le \max\{\tau(p), 0\} \tag{3.10}$$

Also,

$$\rho(Tf_{n+1} - p) = \rho(\lambda(T^2 f_n - p) + (1 - \lambda)(Tf_n - p))$$

$$\le \lambda\rho(T^2 f_n - p) + (1 - \lambda)\rho(Tf_n - p)$$

Again from Lemma 2.6

$$\tau(p) \le \lambda\bar{\tau}(p) + (1 - \lambda)\tau(p)$$

$$\tau(p) \le \bar{\tau}(p). \tag{3.11}$$

Combining inequality (3.10) and (3.11),

$$\tau(Tp) \le \bar{\tau}(Tp) \le \tau(p) \le \bar{\tau}(p).$$

Let $\{g_n\}$ be as minimizing sequence of τ. Then, $\lim_{n\to\infty}\tau(g_n) = r(D_\rho)$. According to inequality (3.11), $\tau(Tg_n) \le \tau(g_n)$, which shows that $\{Tg_n\}$ is minimizing sequence of τ. Also, $\{T^2 g_n\}$ is a minimizing sequence.

By Lemma 2.8, all the minimizing sequences are ρ-convergent to same limit g. Therefore,

$$\lim_{n\to\infty} \rho(g_n - g) = \lim_{n\to\infty} \rho(Tg_n - g) = \lim_{n\to\infty} \rho(T^2 g_n - g) = 0.$$

As $\rho(T^2 g_n - Tg) \le \rho(Tg_n - g)$, so, $\{T^2 g_n\}$ is also a minimizing sequence. Since ρ-limit unique, therefore, $Tg = g$. Hence $g \in F_\rho(T)$.

Example 3.6. Let $T : [0, 1] \to [0, 1]$ be a mapping defined as:

$$Tf = \begin{cases} 1 - f & \text{if } 0 \le f < \frac{1}{5} \\ \frac{f + 4}{5} & \text{if } \frac{1}{5} \le f \le 1. \end{cases}$$

Let $f = \frac{195}{1000}$ and $g = \frac{1.05}{5} = 0.21$. We have the following calculations:

$$\rho(Tf - Tg) = | Tf - Tg | = | 1 - \frac{195}{1000} - \frac{0.21 + 4}{5} | = \frac{37}{1000}$$

$$\rho(f - g) = | f - g | = | \frac{195}{1000} - \frac{1.05}{5} | = \frac{15}{1000}$$

which shows that $\rho(Tf - Tg) > \rho(f - g)$. Therefore, T is not nonexpansive mapping. Now, we prove that T is ρ-fundamentally nonexpansive mapping.

Case 1. If $0 \le f, g < \frac{1}{5}$, then $\frac{1}{5} \le 1 - f \le 1$.

$$\rho(T^2 f - Tg) = | T^2 f - Tg | = | T(1 - f) - (1 - g) |$$

$$= | \frac{1 - f + 4}{5} - (1 - g) | = | \frac{-f + 5g}{5} | < \frac{1}{25}$$

$$\rho(Tf - g) = | Tf - g | = | 1 - f - g | > \frac{3}{5}.$$

Therefore,

$$\rho(T^2 f - Tg) < \rho(Tf - g).$$

Case 2. If $\frac{1}{5} \leq f,\ g \leq 1$

$$\rho(T^2 f - Tg) = |\, T^2 f - Tg \,| = |\, T\left(\frac{f+4}{5}\right) - \left(\frac{g+4}{5}\right) |$$

$$= \left|\ \frac{f+24}{25} - \left(\frac{g+4}{5}\right) \right| = \left|\ \frac{f-5g+4}{25}\ \right|$$

$$\rho(Tf - g) = |\, Tf - g \,| = \left|\ \frac{f+4}{5} - g \ \right| = \left|\ \frac{f-5g+4}{5}\ \right|.$$

Therefore,

$$\rho(T^2 f - Tg) < \rho(Tf - g).$$

Case 3. If $0 \leq f < \frac{1}{5}$ and $\frac{1}{5} \leq g \leq 1$

$$\rho(T^2 f - Tg) = |\, T^2 f - Tg \,| = \left|\, T(1-f) - \left(\frac{g+4}{5}\right) \right|$$

$$= \left|\ \frac{1-f+4}{5} - \left(\frac{g+4}{5}\right) \right| = \left|\ \frac{1-f-g}{5}\ \right|$$

$$\rho(Tf - g) = |\, Tf - g \,| = |\, 1 - f - g \,|.$$

Therefore,

$$\rho(T^2 f - Tg) < \rho(Tf - g).$$

Case 4. If $\frac{1}{5} \leq f \leq 1$ and $0 \leq g < \frac{1}{5}$

$$\rho(T^2 f - Tg) = |\, T^2 f - Tg \,| = \left|\, T\left(\frac{f+4}{5}\right) - (1-g) \right|$$

$$= \left|\ \frac{f+24}{25} - (1-g) \right| = \left|\ \frac{f+25g-1}{25}\ \right| < \frac{1}{5}$$

$$\rho(Tf - g) = |\, Tf - g \,| = \left|\ \frac{f+4}{5} - g \ \right| = \left|\ \frac{f-5g+4}{5}\ \right| > \frac{4}{5}.$$

Therefore,

$$\rho(T^2 f - Tg) < \rho(Tf - g).$$

From all cases, we observed that T is a ρ-fundamentally nonexpansive mapping.

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