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ON GRACEFUL LABELING OF EXTENDED DOUBLE HUT GRAPHS

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Abstract

A labeled graph G which can be gracefully numbered is said to be graceful. Labeling the nodes of G with distinct nonnegative integers and then labeling the e edges of G with the absolute differences between node values, if the graph edge numbers run from 1 to e, the graph G is gracefully numbered. In this paper, we have discussed the gracefulness of some of the graphs formed from extended double hut graphs.

1. Introduction

Labeled graphs form useful models for a wide range of applications such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing and database management graceful labeling f of a graph G with q edges is an injective function from the vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct and nonzero. The concept above was put forward by Rosa in 1967.

Graphs consisting of any number of pairwise disjoint paths with common end vertices are called generalized theta graphs. Various labelings have been found for these graphs.

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In this paper, some new classes of graphs have been constructed by combining some subdivisions of theta graphs with the star graphs $St(n)$, $(n \geq 1)$. Only finite simple undirected graphs are considered here. Our notations and terminology are as in [1]. We refer to [2] for some basic concepts.

2. Results on the Gracefulness of Some Extended Double Hut Graphs

Definition 2.1. The theta graph $\theta(\alpha, \beta, \gamma)$ consists of three edge disjoint paths of lengths α, β, and γ having the same end points. Let the theta graph $\theta_1(2, 2, 3)$ have the paths $P_{1:v2, v1, v5;}P_{2:v2, v6, v5}$ and $P_{3:v2, v3, v4, v5.}$ Let the theta graph $\theta_2(2, 2, 3)$ have the paths $P_{1:x2, x1, x5;}P_{2:x2, x6, x5}$ and $P_{3:x2, x3, x4, x5.}$

Definition 2.2. Attach an edge $(v1, v6)$ to the theta graph $\theta_1(2, 2, 3)$ to form the hut graph $\theta_1'(2, 2, 3)$. Attach an edge $(x1, x6)$ to the theta graph $\theta_2(2, 2, 3)$ to form the hut graph $\theta_2'(2, 2, 3)$.

Definition 2.3. Let A be any graph and B be any tree graph. $A_i \circ B$ denotes the new graph formed by attaching a center vertex of B to a vertex V_i of A.

Definition 2.4. Let A be any graph. Let B and C be any tree graphs. $A_{ij} \circ (B, C)$ denotes the new graph formed by attaching a center vertex of B to a vertex v_i of A and a center vertex of C to a vertex v_j of A, where i and j are distinct.

Definition 2.5. Consider the theta graphs $\theta_1(2, 2, 3)$ and $\theta_2(2, 2, 3)$. Merge the vertices v_3 and x_4 and rename the single vertex as y_3. Merge the vertices v_4 and x_3 and rename the single vertex as y_4. Merge the edges v_3v_4 and x_4x_3 and rename the single edge as y_3y_4. The new graph formed is called a double theta graph Δ with 10 vertices and 13 edges. Consider the hut graphs $\theta_1'(2, 2, 3)$ and $\theta_2'(2, 2, 3)$. Merge the vertices v_3 and x_4 and rename the single vertex as y_3. Merge the vertices v_4 and x_3 and rename the single vertex as y_4. Merge the edges v_3v_4 and x_4x_3 and rename the single edge as y_3y_4. The new graph formed is called a double hut graph Δ' with 10 vertices and 15 edges.

Theorem 2.6. *The double theta graph* Δ *is graceful.*

Proof of Theorem 2.6. Consider the double theta graph Δ with 10 vertices $v1, v2, v5, v6, y3, y4, x1, x2, x5, x6$. Let $G = \Delta$. The vertex set $V(G) = \{vi, yj, xk / i = 1, 2, 5, 6; j = 3, 4; kk = 1, 2, 5, 6\}$. The edge set $E(G) = \{viv_{i+1}/i = 1, 5\} \cup \{viv_{i+4}/i = 1, 2\} \cup \{xix_{i+1}/i = 1, 5\} \cup \{xix_{i+4}/i = 1, 2\} \cup \{viy_{i+4}/i = 2\} \cup \{viy_{i-1}/i = 5\} \cup \{xiy_{i+2}/i = 2\} \cup \{xiy_{i-2}/i = 5\} \cup \{yiy_{i+1}/i = 3\}$.

Let f be the labeling on the set of vertices of G and g be the induced labeling on the set of edges of G. The vertex label set of G can be written as $V_1 \cup V_2 \cup V_3$ where $V_1 = \{f(v_i)/i = 1, 2, 5, 6\}, V_2 = \{f(y_i)/j = 3, 4\}, V_3 = \{f(x_k)/k = 1, 2, 5, 6\}$. The edge label set of G can be written as $E_1 \cup E_2 \cup E_4 \cup E_5 \cup E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11}$ where $E_1 = \{g(viv_{i+1})/i = 1, 5\}, E_2 = \{g(viv_{i+4})/i = 1, 2\}$, $E_4 = \{g(xix_{i+1})/i = 1, 5\}$, $E_5 = \{g(xix_{i+4})/i = 1, 2\}$, $E_7 = \{g(viy_{i+1})/i = 2\}$, $E_8 = \{g(viy_{i|-1})/i = 5\}$, $E_9 = \{g(xiy_{i+2})/i = 2\}$, $E_{10} = \{g(xiy_{i-2})/i = 5\}$, $E_{11} = \{g(yiy_{i+1})/i = 3\}$.

Let the labeling f on the vertices of G be defined by

$f(v_i) = (2)$ for $i = 1$, $f(v_i) = (6)$ for $i = 2$,

$f(v_i) = (10)$ for $i = 5$, $f(v_i) = (5)$ for $i = 6$,

$f(x_i) = (13)$ for $i = 1$, $f(x_i) = (1)$ for $i = 2$,

$f(x_i) = (0)$ for $i = 5$, $f(x_i) = (11)$ for $i = 6$,

$f(y_i) = (9)$ for $i = 3$, $f(y_1) = (3)$ for $i = 4$.

The induced labeling g on the edges of G is defined by

$g(vi \quad v_{i+1}) = (4)$ for $i = 1$, $g(vi \quad v_{i+1}) = (5)$ for $i = 5$,

$g(vi \quad v_{i+4}) = (8)$ for $i = 1$, $g(vi \quad v_{i+4}) = (1)$ for $i = 2$,

$g(xi \quad x_{i+1}) = (12)$ for $i = 1$, $g(xi \quad x_{i+1}) = (11)$ for $i = 5$,

$g(xi \quad x_{i+4}) = (13)$ for $i = 1$, $g(xi \quad x_{i+4}) = (10)$ for $i = 2$,

$g(vi \quad y_{i+1}) = (3)$ for $i = 2$, $g(vi \quad y_{i-1}) = (7)$ for $i = 5$,

$g(xi \quad y_{i+2}) = (2)$ for $i = 2$, $g(xi \quad y_{i-2}) = (9)$ for $i = 5$,

$g(yi \quad y_{i+1}) = (6)$ for $i = 3$.

The vertex labels of G can be arranged in the following order. $V_1 = \{2, 5, 6, 10\}$, $V_2 = \{3, 9\}$, $V_3 = \{0, 1, 11, 13\}$. The set of vertex labels of G is $V_1 \cup V_2 \cup V_3 = \{0, 1, 2, 3, 5, 6, 9, 10, 11, 13\}$.

The edge labels of G can be arranged in the following order. $E_1 = \{4, 5\}$, $E_2 = \{1, 8\}$, $E_4 = \{11, 12\}$, $E_5 = \{10, 13\}$, $E_7 = \{3\}$, $E_8 = \{7\}$, $E_9 = \{2\}$, $E_{10} = \{9\}$, $E_{11} = \{6\}$. The set of edge labels of G is $E_1 \cup E_2 \cup E_4 \cup E_5 \cup E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11} = \{1, 2, 3, \ldots, 13\}$.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence $G = \Delta$ is a graceful graph.

Theorem 2.7. *The graph $\Delta_a \circ (St(n))$ is graceful, for $a = 1$ and $n \geq 1$.*

Proof of Theorem 2.7. Let $G = \Delta_a \circ (St(n))$ where Δ is the double theta graph with 10 vertices $v1, v2, v5, v6, y3, y4, x1, x2, x5, x6$ and 13 edges and $St(n)$ is the star graph on n vertices $u, u_1, u_2, \ldots, u_n.(n \geq 1)$ and $(n-1)$ edges. Here u is the center vertex and the other vertices are pendant vertices. To form the graph G, attach the center vertex u of $St(n)$ to the vertex v_α of the double hut graph Δ and name it as v_α. Let $a = 1. G$ has $(n+10)$ vertices and $(n+13)$ edges.

The vertex set $V(G) = \{u_1, v_i, y_j, x_k / 1 = 1, 2, \ldots, n; i = 1, 2, 5, 6; j = 3, 4; k = 1, 2, 5, 6\}$. The edge set $E(G) = \{viv_{i+1}/i = 1, 5\} \cup \{viv_{i+4}/i = 1, 2\} \cup \{xix_{i+1}/i = 1, 5\} \cup \{xix_{i+4}/i = 1, 2\} \cup \{viy_{i+4}/i = 2\} \cup \{viy_{i-1}/i = 5\} \cup \{xiy_{i+2}/i = 2\} \cup \{xiy_{i-2}/i = 5\} \cup \{yiy_{i+1}/i = 3\} \cup \{viu_1/i = 1, 1 = 1, 2, \ldots, n\}$.

Let f be the labeling on the set of vertices of G and g be the induced labeling on the set of edges of G.

Let f be the labeling on the set of vertices of G and g be the induced labeling on the set of edges of G. The vertex label set of G can be written as $V_1 \cup V_2 \cup V_3 \cup V_4$ where $V_1 = \{f(v_i)/i = 1, 2, 5, 6\}$, $V_2 = \{f(y_i)/j = 3, 4\}$, $V_3 = \{f(x_k)/k = 1, 2, 5, 6\}$ and $V_4 = \{f(u_i)/1 = 1, 2, \ldots, n\}$. The edge label set of G can be written as $E_1 \cup E_2 \cup E_4 \cup E_5 \cup E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11} \cup E_{12}$ where $E_1 = \{g(viv_{i+1})/i = 1, 5\}, E_2 = \{g(viv_{i+4})/i = 1, 2\}, E_4 = \{g(xix_{i+1})/i = 1, 5\}$, $E_5 = \{g(xix_{i+4})/i = 1, 2\}, E_7 = \{g(viy_{i+1})/i = 2\}, E_8 = \{g(viy_{i|-1})/i = 5\}$, $E_9 = \{g(xiy_{i+2})/i = 2\}, E_{10} = \{g(xiy_{i-2})/i = 5\}, E_{11} = \{g(yiy_{i+1})/i = 3\}$, $E_{12} = \{g(viu_1)/i = 1, 1 = 1, 2, \ldots, n\}$.

Let the labeling f on the vertices of G be defined by

$f(v_i) = (4)$ for $i = 1$, $f(v_i) = (0)$ for $i = 2$,

$f(v_i) = (1)$ for $i = 5$, $f(v_i) = (n+13)$ for $i = 6$,

$f(x_i) = (n+12)$ for $i = 1$, $f(x_i) = (2)$ for $i = 2$,

$f(x_i) = (5)$ for $i = 5$, $f(x_i) = (7)$ for $i = 6$,

$f(y_i) = (n+11)$ for $i = 3$, $f(y_1) = (n+10)$ for $i = 4$,

$f(u_i) = (i+9)$ for $1 \leq i \leq n$

The induced labeling g on the edges of G is defined by

$g(vi \quad v_{i+1}) = (4)$ for $i = 1$, $g(vi \quad v_{i+1}) = (n+12)$ for $i = 5$,

$g(vi \quad v_{i+4}) = (3)$ for $i = 1$, $g(vi \quad v_{i+4}) = (n+13)$ for $i = 2$,

$g(xi \quad x_{i+1}) = (n+10)$ for $i = 1$, $g(xi \quad x_{i+1}) = (2)$ for $i = 5$,

$g(xi \quad x_{i+4}) = (n+7)$ for $i = 1$, $g(xi \quad x_{i+4}) = (5)$ for $i = 2$,

$g(vi \quad y_{i+1}) = (n+11)$ for $i = 2$, $g(vi \quad y_{i-1}) = (n+9)$ for $i = 5$,

$g(xi \quad y_{i+2}) = (n+8)$ for $i = 2$, $g(xi \quad y_{i-2}) = (n+6)$ for $i = 5$,

$g(yi \quad y_{i+1}) = (1)$ for $i = 3$. $g(v_i u_1) = (1+5)$ for $i = 1, 1 \leq 1 \leq n$.

The vertex labels of G can be arranged in the following order. $V_1 = \{0, 1, 4, (n+13)\}$, $V_2 = \{(n+10), (n+11)\}$, $V_3 = \{2, 5, 7, (n+12)\}$,

$V_4\{10, 11, \ldots, (n+9)\}$. The set of vertex labels of G is $V_1 \cup V_2 \cup V_3 \cup V_4$ $= \{0, 1, 2, 4, 5, 7, 10, 11, 12, \ldots, (n+13)\}$.

The edge labels of G can be arranged in the following order. $E_1 = \{4, (n+12)\}$, $E_2 = \{(3, (n+13))\}$, $E_4 = \{2, (n+10)\}$, $E_5 = \{5, (n+7)\}$, $E_7 = \{(n+11)\}$, $E_8 = \{(n+9)\}$, $E_9 = \{(n+8)\}$, $E_{10} = \{(n+6)\}$, $E_{11} = \{1\}$, E_{12} $= \{6, 7, \ldots, (n+5)\}$. The set of edge labels of G is $E_1 \cup E_2 \cup E_4 \cup E_5 \cup E_7$ $\cup E_8 \cup E_9 \cup E_{10} \cup E_{11} \cup E_{12} = \{1, 2, 3, \ldots, (n+13)\}$.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence $G = \Delta_\alpha \circ (St(n))$ for is a graceful graph $\alpha = 1$, $n \geq 1$.

Corollary 2.8 *The graph* $\Delta_{\alpha\beta} \circ (St(n), St(m))$ *is graceful, for* $\alpha = 1, \beta = 2$ *and* $m, n \geq 1$.

Proof of Corollary 2.8. To form the graph $G = \Delta_{\alpha\beta} \circ (St(n), St(m))$, attach the center vertex u of $St(n)$ to the vertex of the double theta graph Δ and name it as v_α and attach the center vertex w of $St(m)$ to the vertex v_β of the double hut graph Δ and name it as v_β, for $\alpha = 1, \beta = 2$ and $m, n \geq 1$, in the above theorem. $G = \Delta_{\alpha\beta} \circ (St(n), St(m))$, is a graceful graph.

Theorem 2.9. *The double hut graph* Δ' *is graceful.*

Proof of Theorem 2.9. Consider the double hut graph $G = \Delta'$ with 10 vertices $v1, v2, v5, v6, y3, y4, x1, x2, x5, x6$.

The vertex set $V(G) = \{v_i, y_j, x_k / i = 1, 2, 5, 6; j = 3, 4; k = 1, 2, 5, 6\}$. The edge set $E(G) = \{viv_{i+1}/i = 1, 5\} \cup \{viv_{i+4}/i = 1, 2\} \cup \{xix_{i+5}/i = 1\} \cup$ $\{xix_{i+1}/i = 1, 5\} \cup \{viy_{i+4}/i = 2\} \cup \{xix_{i+5}/i = 1\} \cup \{viy_{i+1}/i = 2\} \cup \{viy_{i-1}/i = 5\} \cup$ $\{xiy_{i+2}/i = 2\} \cup \{xiy_{i-2}/i = 5\} \cup \{yiy_{i+1}/i = 3\}$.

Let f be the labeling on the set of vertices of G and g be the induced labeling on the set of edges of G. The vertex label set of G can be written as $V_1 \cup V_2 \cup V_3$ where $V_1 = \{f(v_i)/i = 1, 2, 5, 6\}, V_2 = \{f(y_i)/j = 3, 4\}$, $V_3 = \{f(x_k)/k = 1,$ $2, 5, 6\}$. The edge label set of G can be written as $E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$

$\cup E_6 \cup E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11}$ where

$E_1 = \{g(viv_{i+1})/i = 1, 5\}, E_2 = \{g(viv_{i+4})/i = 1, 2\}, E_3 = \{g(viv_{i+5})/i = 1\},$

$E_4 = \{g(xix_{i+1})/i = 1, 5\}, E_5 = \{g(xix_{i+4})/i = 1, 2\}, E_6 = \{g(xix_{i+5})/i = 1\},$

$E_7 = \{g(viy_{i+1})/i = 2\}, E_8 = \{g(viy_{i|-1})/i = 5\}, E_9 = \{g(xiy_{i+2})/i = 2\},$

$E_{10} = \{g(xiy_{i-2})/i = 5\}, E_{11} = \{g(yiy_{i+1})/i = 3\}.$

Let the labeling f on the vertices of G be defined by

$f(v_i) = (1)$ for $i = 1$, $f(v_i) = (12)$ for $i = 2$,

$f(v_i) = (14)$ for $i = 5$, $f(v_i) = (7)$ for $i = 6$,

$f(x_i) = (13)$ for $i = 1$, $f(x_i) = (4)$ for $i = 2$,

$f(x_i) = (3)$ for $i = 5$, $f(x_i) = (5)$ for $i = 6$,

$f(y_i) = (15)$ for $i = 3$, $f(y_1) = (0)$ for $i = 4$.

The induced labeling g on the edges of G is defined by

$g(v_i \quad v_{i+1}) = (11)$ for $i = 1$, $g(vi \quad v_{i+1}) = (7)$ for $i = 5$,

$g(v_i \quad v_{i+4}) = (13)$ for $i = 1$, $g(vi \quad v_{i+4}) = (5)$ for $i = 2$,

$g(v_i \quad v_{i+5}) = (6)$ for $i = 1$, $g(xi \quad x_{i+1}) = (9)$ for $i = 1$,

$g(x_i \quad x_{i+1}) = (2)$ for $i = 5$, $g(xi \quad x_{i+4}) = (10)$ for $i = 1$,

$g(x_i \quad x_{i+4}) = (1)$ for $i = 2$, $g(xi \quad x_{i+5}) = (8)$ for $i = 1$,

$g(v_i \quad y_{i+1}) = (3)$ for $i = 2$, $g(vi \quad y_{i-1}) = (14)$ for $i = 5$,

$g(x_i \quad y_{i+2}) = (4)$ for $i = 2$, $g(xi \quad y_{i-2}) = (12)$ for $i = 5$,

$g(y_i \quad y_{i+1}) = (15)$ for $i = 3$.

The vertex labels of G can be arranged in the following order. $V_1 = \{1, 7, 12, 14\}$, $V_2 = \{0, 15\}$, $V_3 = \{3, 4, 5, 13\}$. The set of vertex labels of G is $V_1 \cup V_2 \cup V_3 = \{0, 1, 3, 4, 5, 7, 12, 13, 14, 15\}$.

The edge labels of G can be arranged in the following order. $E_1 = \{7, 11\}$, $E_2 = \{5, 13\}$, $E_3 = \{6\}$, $E_4 = \{2, 9\}$, $E_5 = \{1, 10\}$, $E_6 = \{8\}$, $E_7 = \{3\}$, $E_8 = \{14\}$, $E_9 = \{4\}, E_{10} = \{12\}$, $E_{11} = \{15\}$. The set of edge labels of G is $E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11} = \{1, 2, 3, \ldots, 15\}$. Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence $G = \Delta'$ is a graceful graph.

Theorem 2.10. *The graph* $\Delta'_\alpha \circ (St(n))$ *is graceful, for* $\alpha = 1$ *and* $n \geq 1$.

Proof of Theorem 2.10. Let $G = \Delta'_\alpha \circ (St(n))$ where Δ' is the double hut graph with 10 vertices $v1, v2, v5, v6, y3, y4, x1, x2, x5, x6$ and 15 edges and $St(n)$ is the star graph on n vertices $u, u_1, u_2, \ldots, uu_n.(n \geq 1)$ and $(n-1)$ edges. Here u is the center vertex and the other vertices are pendant vertices. To form the graph G, attach the center vertex u of $St(n)$ to the vertex v_α of the double hut graph Δ' and name it as v_α. Let $\alpha = 1$. G has $(n+10)$ vertices and $(n+10)$ edges.

The vertex set $V(G) = \{vi, vi, yj, xk/i = 1, 2, \ldots, n; i = 1, 2, 5, 6; j = 3, 4; k = 1, 2, 5, 6\}$. The edge set $E(G) = \{viv_{i+1}/i = 1, 5\} \cup \{viv_{i+4}/i = 1, 2\} \cup \{xix_{i+5}/i = 1\} \cup \{xix_{i+1}/i = 1, 5\} \cup \{viy_{i+4}/i = 1, 2\} \cup \{xix_{i+5}/i = 1\} \cup \{viy_{i+1}/i = 2\} \cup \{viy_{i-1}/i = 5\} \cup \{xiy_{i+2}/i = 2\} \cup \{xiy_{i-2}/i = 5\} \cup \{yiy_{i+1}/i = 3\} \cup \{viu_1/i = 1, 2, \ldots, n\}$.

Let f be the labeling on the set of vertices of G and g be the induced labeling on the set of edges of G.

Let f be the labeling on the set of vertices of G and g be the induced labeling on the set of edges of G. The vertex label set of G can be written as $V_1 \cup V_2 \cup V_3 \cup V_4$ where $V_1 = \{f(v_i)/i = 1, 2, 5, 6\}, V_2 = \{f(y_i)/j = 3, 4\}$, $V_3 = \{f(x_k)/k = 1, 2, 5, 6\}$ and $V_4 = \{f(u_1)/1 = 1, 2, \ldots, n\}$. The edge label set of G can be written as $E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11} E_{12}$ where

$$E_1 = \{g(viv_{i+1})/i = 1, 5\}, E_2 = \{g(viv_{i+4})/i = 1, 2\}, E_3 = \{g(viv_{i+5})/i = 1\},$$

$$E_4 = \{g(xix_{i+1})/i = 1, 5\}, E_5 = \{g(xix_{i+4})/i = 1, 2\}, E_6 = \{g(xix_{i+5})/i = 1\},$$

$E_7 = \{g(viy_{i+1})/i = 2\}, E_8 = \{g(viy_{i|-1})/i = 5\}, E_9 = \{g(xiy_{i+2})/i = 2\},$

$E_{10} = \{g(xiy_{i-2})/i = 5\}, E_{11} = \{g(yiy_{i+1})/i = 3\}, E_{12} = \{g(viu_1)/i = 1, 2, \ldots, n\}.$

Let the labeling f on the vertices of G be defined by

$f(v_i) = (1)$ for $i = 1$, $f(v_i) = (11)$ for $i = 2$,

$f(v_i) = (n+12)$ for $i = 5$, $f(v_i) = (4)$ for $i = 6$,

$f(x_i) = (0)$ for $i = 1$, $f(x_i) = (13)$ for $i = 2$,

$f(x_i) = (n+14)$ for $i = 5$, $f(x_i) = (n+15)$ for $i = 6$,

$f(y_i) = (2)$ for $i = 3$, $f(y_1) = (n+8)$ for $i = 4$.

$f(u_i) = (i+6)$ for $1 \leq i \leq n$

The induced labeling g on the edges of G is defined by

$g(vi \quad v_{i+1}) = (n+10)$ for $i = 1$, $g(vi \quad v_{i+1}) = (n+8)$ for $i = 5$,

$g(vi \quad v_{i+4}) = (11)$ for $i = 1$, $g(vi \quad v_{i+4}) = (n+7)$ for $i = 2$,

$g(vi \quad v_{i+5}) = (3)$ for $i = 1$, $g(xi \quad x_{i+1}) = (n+13)$ for $i = 1$,

$g(xi \quad x_{i+1}) = (1)$ for $i = 5$, $g(xi \quad x_{i+4}) = (n+14)$ for $i = 1$,

$g(xi \quad x_{i+4}) = (2)$ for $i = 2$, $g(xi \quad x_{i+5}) = (n+15)$ for $i = 1$,

$g(vi \quad y_{i+1}) = (n+9)$ for $i = 2$, $g(vi \quad y_{i-1}) = (4)$ for $i = 5$,

$g(xi \quad y_{i+2}) = (5)$ for $i = 2$, $g(xi \quad y_{i-2}) = (n+12)$ for $i = 5$,

$g(yi \quad y_{i+1}) = (n+6)$ for $i = 3$, $g(v_i u_1) = (1+5)$ for $1 \leq 1 \leq n$.

The vertex labels of G can be arranged in the following order. $V_1 = \{1, 4, (n+11), (n+12)\}$, $V_2 = \{(n+8)\}$, $V_3 = \{0, (n+13), (n+14), (n+15)\}$, $V_4\{7, 8, \ldots, (n+6)\}$ The set of vertex labels of G is $V_1 \cup V_2 \cup V_3 \cup V_4$ $= \{0, 1, 2, 4, \ldots, (n+15)\}$.

The edge labels of G can be arranged in the following order. $E_1 = \{(n+8), (n+10)\}$, $E_2 = \{((n+7)), (n+11)\}$, $E_3 = \{1, (n+13)\}$, $E_5 =$

$\{2, (n+14)\}$, $E_6\{(n+15)\}$, $E_7 = \{(n+9)\}$, $E_8 = \{(4)\}$, $E_9 = \{(5)\}$, $E_{10} = \{(n+12)\}$, $E_{11} = \{(n+6)\}$, $E_{12} = \{6, 7, \ldots, (n+5)\}$. The set of edge labels of G is $E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11} E_{12} = \{1, 2, 3, \ldots, (n+15)\}$. Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence $G = \Delta'_\alpha \circ (St(n))$ for is a graceful graph $a = 1$, $n \geq 1$.

Corollary 2.11. *The graph* $\Delta'_{\alpha\beta} \circ (St(n), St(m))$ *is graceful*, *for* $\alpha = 1$, $\beta = 1$ *and* m, $n \geq 1$.

Proof of Corollary 2.11. To form the graph $G = \Delta'_{\alpha\beta} \circ (St(n), St(m))$, attach the center vertex u of $St(n)$ to the vertex v_α of the double hut graph Δ' and name it as v_α and attach the center vertex w of $St(m)$ to the vertex x_β of the double hut graph Δ' and name it as x_β, for $\alpha = 1, \beta = 1$ and m, $n \geq 1$, in the above theorem. $G = \Delta'_{\alpha\beta} \circ (St(n), St(m))$, is a graceful graph.

References

[1] F. Harary, Graph Theory, Addision Wesley, New Delhi, 1972.

[2] J. A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics 24 (2021), # DS6.

[3] R. Raziya Begam, M. Palanivelrajan, K. Gunasekaran and A. R. Shahul Hameed, Graceful labeling of some theta related graphs, International Journal of Fuzzy Mathematical Archive 2 (2013), 78-84.

[4] A. R. Shahul Hameed, M. Palanivelrajan, K. Gunasekaran and R. Raziya Begam, On Graceful labeling of some Bicyclic graphs, International Journal of Fuzzy Mathematical Archive 3 (2013), 1-8.

[5] A. R. Shahul Hameed, M. Palanivelrajan, K. Gunasekaran and R. Raziya Begam, Graceful labeling of some combinations of hut graphs, Archimedes J. Math. 4(1) (2014), 27-35.