Advances and Applications in Mathematical Sciences
Volume 21, Issue 6, April 2022, Pages 3161-3173

SOLUTION OF BURGER'S EQUATION ARISING IN LONGITUDINAL DISPERSION PHENOMENON USING TWO-DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD

ARUNA SHARMA and AMIT K. PARIKH

Research Scholar
Department of Mathematics
Mehsana Urban Institute of Sciences
Ganpat University, Kherva-384012
Mehsana, Gujarat, India

Principal, Mehsana Urban Institute of Sciences
Ganpat University, Kherva-384012
Mehsana, Gujarat, India
E-mail: amit.parikh.maths@gmail.com

Abstract

This study aims to find the solution of Burger's Equation, a non-linear partial differential equation that is obtained as a result of dispersion in the longitudinal direction when fluid flows through a porous medium. The governing equation is solved using the Two-Dimensional Differential Transform Method (TDDTM). The solution is obtained in the form of Infinite series. The numerical solution which is obtained by using MATLAB and represented graphically is discussed here. The current mathematical method TDDTM is dependable, efficient, and lowers computational work by simplifying all calculations. The accuracy of the results demonstrates that TDDTM is effective.

1. Introduction

The solution of the non-linear partial differential equation known as Burger's equation, which is obtained as a result of dispersion in the

2020 Mathematics Subject Classification: 35A25, 35A22, 35Q35.
Keywords: Longitudinal dispersion phenomena, Burger's equation, Two-dimensional differential transform method, Miscible flow, Porous medium.
*Corresponding author; E-mail: sharmaaruna961@gmail.com
Received September 18, 2021; Accepted January 1, 2022

longitudinal direction when fluid flows through a homogenous porous medium is obtained in this study.

When two miscible fluids flow in the same direction, i.e., along the longitudinal axis, the longitudinal dispersion phenomenon is said to occur.

If the molecules of one fluid freely mix with the molecules of the other fluid, the two fluids are said to be miscible, and hence there is no interface between these miscible fluids.

Chemical engineers, environmental engineers, petroleum engineers, hydrologists, mathematicians, and soil scientists have been interested in problems of dispersion or miscible displacement since the early nineteenth century. The phenomena of dispersion are most commonly observed in coastal areas, where seawater beds displace freshwater beds as a result of dispersion.

Longitudinal dispersion is also used to control the salinity of the soil where contaminated water is dispersed into the soil. Dispersion phenomena have a wide range of applications in many fields of engineering and science including oil recovery in petroleum engineering.

The Hope-Cole transformation approach was utilized by Patel and Mehta [1]. Backlund Transformations were used by Meher and Mehta [2]. To obtain the solution to Burger's Equation, Joshi et al. [3] adopted a theoretical approach. Kunjan and Twinkle [4] employed the New Integral Transform and Homotopy Perturbation Method. A numerical solution utilizing the finite difference approach and the Crank-Nicolson scheme was obtained by Ravi Borana, Vikas Pradhan and Manoj Mehta [5, 6].

Kajal Patel, Manoj Mehta and Twinkle Singh [7] solved the one-dimensional dispersion phenomenon by using Homotopy Analysis Method.

The mathematical formulation of the problem leads to a nonlinear differential equation known as Burger's equation, whose solution is obtained, in this paper by Two-Dimensional Differential Transform Method (TDDTM). Chen and Ho [11] discovered this mathematical method in 1999 to solve partial differential equations.

For solving nonlinear partial differential equations for which analytic solutions are sometimes difficult or impossible, this present method is found to be dependable and efficient.

2. Preliminaries

The primary goal here is to determine the concentration at a distance 'x' and time 't' when two fluids that are miscible flow through a homogeneous porous medium. The velocity of the fluid on either side of the mixed zone is represented by a single equation. Further, there is no effect of capillary forces between the miscible fluids, as the two phases are completely soluble in each other.

The longitudinal dispersion of impure or saline water with concentration $c(x, t)$ that occurs in homogeneous porous media along the x-axis is taken into account. The bulk coefficient of diffusion of one fluid into another is responsible for the change in concentration. It is assumed that there is no mass transfer between the solid and liquid phases.

Miscible flow occurs in both longitudinal and transverse directions. However, dispersion causes more spreading in the flow direction i.e. longitudinal direction than in the transverse direction. Our aim is to find the concentration $c(x, t)$ of impure water contaminant in terms of 'x' and time 't'.

Mathematical Formulation

The equation of continuity for incompressible fluids is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) = 0 \tag{1}$$

where $\rho =$ density and $\bar{v} =$ pore seepage velocity, as per Darcy's Law

The diffusion equation is given by,

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\bar{v}) = \nabla \cdot \left[\rho \overline{D} \nabla \left(\frac{c}{\rho}\right)\right] \tag{2}$$

Here, the dispersing material does not increase or decrease when fluid flows through the homogenous porous medium.

In Equation (2), c is the concentration of fluid, D is the coefficient of dispersion with nine components 'D_{ij}'.

In the laminar flow of an incompressible fluid across a homogenous porous material, density ρ remains constant.

Therefore, equation (2) becomes

$$\frac{\partial c}{\partial t} + \bar{v} \cdot (\nabla c) = \nabla \cdot (\bar{D} \nabla (c)) \tag{3}$$

$\bar{v} = u(x, t)$ is the seepage velocity along direction of flow i.e. the x-axis $D_{11} \approx D_L = \gamma$ are the non-zero components of dispersion D, and other components are zero.

γ is called the coefficient of longitudinal dispersion.

Therefore, from Equation (3), we get

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = \gamma \frac{\partial^2 c}{\partial x^2} \tag{4}$$

Assuming $u = \frac{c(x, t)}{c_0}$, where $x > 0$, $c_0 \cong 1$

Equation (4) becomes $\frac{\partial c}{\partial t} + c \frac{\partial c}{\partial x} = \gamma \frac{\partial^2 c}{\partial x^2}$ (5)

Equation (5) is the nonlinear differential equation called Burger's equation due to dispersion in the longitudinal direction when two miscible fluids flow through porous media. The seepage flow velocity of impure water is assumed unsteady. Here the initial concentration of dispersion is the input concentration c_0 which is highest constant concentration of impurity at $x = 0$.

Equation (5) is solved using the Two-Dimensional Differential Transform Method, where

$\gamma =$ longitudinal dispersion coefficient,

$c =$ average cross-sectional concentration,

$u =$ seepage velocity along direction of flow i.e. the x-axis.

Solution of the problem

Consider Burger's equation

$$\frac{\partial c}{\partial t} + c\frac{\partial c}{\partial x} = \gamma\frac{\partial^2 c}{\partial x^2}$$

Taking $\gamma = 1$

We get $\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - c\frac{\partial c}{\partial x}$ (6)

Consider the initial and boundary conditions as

$$c(x, 0) = e^{-x}, \ x \geq 0$$

$$c(0, t) = 1, \ t > 0$$

because concentration decreases with increasing distance x, therefore $f(x)$ is considered as a decreasing function.

The nonlinear Partial differential equation (6) is solved by Two-Dimensional Differential Transform Method.

Two-Dimensional Differential Transform Method (TDDTM):

The Basic definition of TDDTM is given below

$$C(j, i) = \frac{1}{j!\,i!}\left[\frac{\partial^{j+1} c(x, t)}{\partial x^j \partial t^i}\right]_{(0,\,0)} \tag{1a}$$

where $c(x, t)$ in lower case letters represents the original function and $C(j, i)$ in upper case letters represent the t-dimensional spectrum of $c(x, t)$ also called as transformed T-function in short.

The differential inverse transform of $C(j, i)$ is defined as

$$c(x, t) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} C(j, i) x^j t^i \tag{1b}$$

and from equations (1a) and (1b) it can be concluded

$$c(x, t) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{1}{j!\,i!}\left[\frac{\partial^{j+i} c(x, t)}{\partial x^j \partial t^i}\right]_{(0,\,0)} x^j t^i$$

Theorems proved in [12] have been used and are as follows

Theorem 1. *If* $c(x, t) = p(x, t) \pm q(x, t)$, *then* $C(j, i) = P(j, i) \pm Q(j, i)$.

Theorem 2. *If* $c(x, t) = \alpha p(x, t)$, *then* $C(j, i) = \alpha P(j, i)$ *where* α *is a constant.*

Theorem 3. *If* $c(x, t) = \frac{\partial p(x, t)}{\partial x}$. *Then* $C(j, i) = (j+1)P(j+1, i)$

Theorem 4. *If* $c(x, t) = \frac{\partial p(x, t)}{\partial \alpha}$. *Then* $C(j, i) = (i+1)P(j, i+1)$

Theorem 5. *If* $c(x, t) = \frac{\partial^{r+s} p(x, t)}{\partial x^r \partial t^s}$, *then* $C(j, i) = (j+1)(j+2)\ldots(j+r)$ $(i+1)(i+2)\ldots(i+s)P(j+r, i+s)$

Theorem 6. *If* $c(x, t) = p(x, t)q(x, t)$, *then*

$$C(j, i) = \sum_{r=0}^{j}\sum_{s=0}^{i} P(r, i-s)\, Q(j-r, s)$$

Theorem 7. *If* $c(x, t) = x^m t^n$, *then*

$$C(j, i) = \delta(j-m, i-n) = \delta(j-m)\,\delta(i-n)$$

Where $\delta(j-m)\,\delta(i-n) = 1,\ j = m,\ i = n$

$= 0$, *otherwise.*

Theorem 8. *If* $c(x, t) = \frac{\partial p(x, t)}{\partial x}\frac{\partial q(x, t)}{\partial x}$ *then*

$$C(j, i) = \sum_{r=0}^{j}\sum_{s=0}^{i} (r+1)(j-r+1)\, P(r+1, i-s)\, Q(j-r+1, s)$$

Theorem 9. *If* $c(x, t) = x^m e^{at}$ *then* $C(j, i) = \frac{a^i}{i!}\,\delta(j-m)$

Methodology

The differential transform $C(k, h)$ of the function $c(x, t)$ which denotes the concentration to be found is defined as

$$C(k, h) = \frac{1}{k!\,h!}\left[\frac{\partial^{k+h}c(x, t)}{\partial x^k \partial t^h}\right]_{(0, 0)} \tag{7}$$

The inverse differential transform of $C(k, h)$ is defined as

$$c(x, t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} C(k, h)\, x^k t^h \tag{8}$$

Applying Two Dimensional Differential Transform Method and theorems given above on Burger's equation (6)

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - c\,\frac{\partial c}{\partial x}$$

From Theorem (4) $\frac{\partial c}{\partial t} = (h+1)C(k, h+1)$

From Theorem (5) $\frac{\partial^2 c}{\partial x^2} = (k+1)(k+2)C(k+2, h)$

From Theorem (6) $c\,\frac{\partial c}{\partial x} = \sum_{r=0}^{k}\sum_{s=0}^{h} C(r, h-s)(k-r+1)C(k-r+1, s)$

Therefore, Burger's Equation transforms to

$$(h+1)C(k, h+1) = (k+1)(k+2)C(k+2, h)$$

$$-\sum_{r=0}^{k}\sum_{s=0}^{h} C(r, h-s)(k-r+1)C(k-r+1, s) \tag{9}$$

The initial conditions $c(x, 0) = e^{-x},\ x \geq 0$ are transformed to

$$C(k, 0) = \frac{1}{k!}\left[\frac{\partial^k c(x, 0)}{\partial x^k}\right]_{(0, 0)}$$

$$C(k, 0) = \frac{(-1)^k}{k!}\ k = 0, 1, 2 \ldots \tag{10}$$

Therefore, we get

$$C(0,\ 0) = 1,\ C(1,\ 0) = -1,\ C(2,\ 0) = \frac{1}{2},\ C(3,\ 0) = \frac{-1}{6}$$

Now we calculate the values of $C(k,\ h)$ by giving different values to h and k in equation (9)

For $k = 1,\ h = 0,\ C(1,\ 1) = -3$

For $k = 2,\ h = 0,\ C(2,\ 1) = \frac{5}{2}$

The calculated coefficients from the recursive relation (12) are shown in the Table 1

Table 1. Coefficients $C(h,\ k)$ for $h = 0, 1, 2, 3$ and $k = 0, 1, 2, 3, 4$

k	h			
	0	1	2	3
0	$C(0, 0) = 1$	$C(0, 1) = 2$	$C(0, 2) = 5$	$C(0, 3) = 16$
1	$C(1, 0) = -1$	$C(1, 1) = -3$	$C(1, 2) = -11$	
2	$C(2, 0) = \frac{1}{2}$	$C(2, 1) = \frac{5}{2}$	$C(2, 2) = 13$	
3	$C(3, 0) = \frac{-1}{6}$	$C(3, 1) = \frac{-3}{2}$	$C(3, 2) = \frac{-32}{3}$	
4	$C(4, 0) = \frac{1}{24}$	$C(4, 1) = \frac{17}{24}$		

By Equation (8). we get

$$\begin{aligned} c(x,\ t) = {} & C(0,\ 0)x^0t^0 + C(1,\ 0)x^1t^0 + C(2,\ 0)x^2t^0 + C(3,\ 0)x^3t^0 \\ & + C(4,\ 0)x^4t^0 + C(5,\ 0)x^5t^0 + C(6,\ 0)x^6t^0 + C(0,\ 1)x^0t^1 \\ & + C(0,\ 2)x^0t^2 + C(0,\ 3)x^0t^3 + C(1,\ 1)x^1t^1 + C(2,\ 1)x^2t^1 \\ & + C(3,\ 1)x^3t^1 + C(4,\ 1)x^4t^1 + C(5,\ 1)x^5t^1 + C(1,\ 2)x^1t^2 \\ & + C(2,\ 2)x^2t^2 + C(3,\ 2)x^3t^2 \ldots \end{aligned} \tag{11}$$

Substituting the values of coefficients $C(k, h)$ from above table we have

$$c(x, t) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{720}x^6 + 2t + 5t^2 + 16t^3$$

$$- 3xt + \frac{5}{2}x^2t - \frac{3}{2}x^3t + \frac{17}{24}x^4t - \frac{33}{120}x^5t - 11xt^2 + 13x^2t^2 - \frac{32}{3}x^3t^2 \dots \quad (12)$$

which is the solution of Burger's equation in the form of an infinite series.

3. Results and Discussion

Numerical representation of the solution

The numerical values of the concentration obtained from equation (12) for various distances x at fixed time $t = 0.001, 0.002, 0.003, 0.004, 0.005, 0.006,$ $0.007, 0.008, 0.009, 0.01$ are obtained by using MATLAB and presented in Table 2 below.

Table 2. Concentration Vs. Distance x at fixed time t.

t	x										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.001	1	0.906565	0.820223	0.74211	0.671441	0.607505	0.549659	0.49732	0.44996	0.407102	0.368312
0.002	1	0.908301	0.821722	0.743408	0.672567	0.608483	0.550508	0.498054	0.45059	0.40763	0.368734
0.003	1	0.910045	0.823227	0.744711	0.673696	0.609463	0.551358	0.498789	0.451217	0.408153	0.369149
0.004	1	0.911797	0.82474	0.746019	0.67483	0.610447	0.552211	0.499524	0.451843	0.408673	0.369558
0.005	1	0.913558	0.826259	0.747333	0.675969	0.611434	0.553065	0.500259	0.452468	0.409188	0.369959
0.006	1	0.915327	0.827785	0.748652	0.677111	0.612424	0.553922	0.500995	0.453091	0.4097	0.370353
0.007	1	0.917105	0.829318	0.749977	0.678259	0.613418	0.55478	0.501732	0.453713	0.410208	0.370741
0.008	1	0.918891	0.830858	0.751308	0.679411	0.614415	0.555641	0.502469	0.454333	0.410712	0.371122
0.009	1	0.920687	0.832406	0.752645	0.680567	0.615416	0.556504	0.503207	0.454953	0.411213	0.371497
0.01	1	0.922491	0.833961	0.753988	0.681729	0.61642	0.557369	0.503946	0.455571	0.411711	0.371865

From the table. we observe that the concentration $c(x, t)$ of impure water is decreasing when distance x is increasing for fixed time t .

Graphical Representation of the solution:

The graphical interpretation of the solution (12) is given in figures 1-3. The graphs have been plotted using MATLAB

Figure 1 shows a graph of concentration 'c' versus distance 'x' for fixed times $t = 0.001, 0.002, 0.003, 0.004, 0.005, 0.006, 0.007, 0.008, 0.009, 0.01$

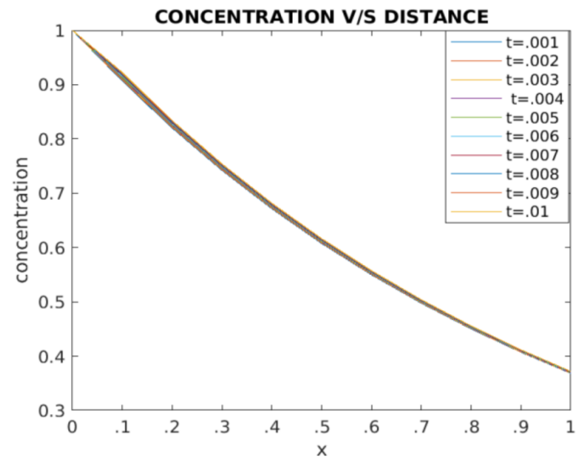

Figure 1. Concentration Vs. Distance for fixed time *t*.

From Figure 1, it can be concluded that for any fixed time t, as the distance x increases the concentration $c(x, t)$ of impure water decreases.

Figure 2 shows a graph of concentration 'c' versus time 't' for fixed distances $x = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$

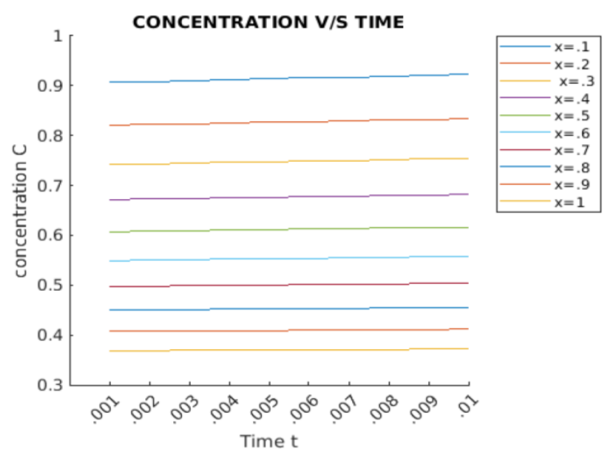

Figure 2. Concentration Vs. Time for fixed distance *x*.

From the graph of Figure 2 of concentration 'c' versus time 't' for given distance 'x', it can be observed that the concentration of impure water is increasing for a small time 't' and then it becomes steady and constant as time 't' increases.

Figure 3 shows a graph of concentration 'c' versus time 't' and distance 'x'

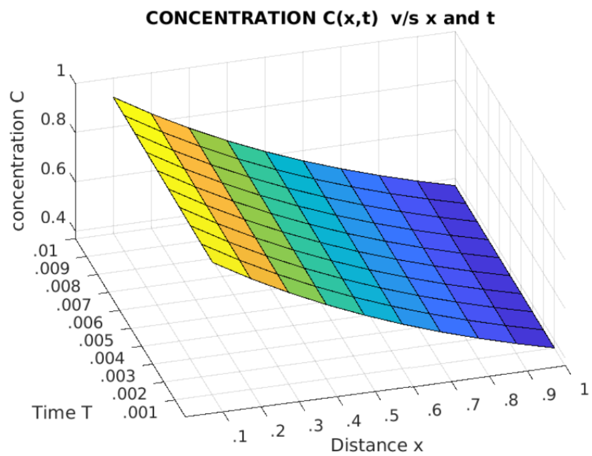

Figure 3. Concentration $c(x, t)\ v/s$ 'x' and 't'.

Figure 3, is a 3D Plot of concentration $c(x, t)$ versus distance 'x' and time 't', which clearly shows how for fixed time 't' the concentration 'c' is decreasing as the distance 'x' is increasing. It also shows that for fixed distance 'x' the concentration 'c' is increasing as time 't' increases.

4. Conclusion

Numerical solution and graphs for obtaining the probable concentration of impure water in unstable unidirectional seepage flow with regard to distance and time are obtained using MATLAB.

For a fixed time 't', the initial concentration of impure water is highest at $x = 0$ and is observed to decrease with increasing distance 'x', which in agreement with the physical situation wherein the concentration of impure water is always highest at the source and gradually decreases and disperses from the source.

It is clear, that the concentration of impure water is slightly increasing for a small-time t, for fixed distance 'x', and then it becomes constant through the time for given distance 'x'.

The significance of this research is it can be utilized to predict salt intrusion in groundwater and also can predict whether there is a possibility of groundwater supplies being contaminated due to groundwater flowing through pollutants buried in the ground.

When compared to other approaches, the Two-dimensional Differential Transform Method reduces computational difficulties, and all calculations can be performed easily and accurately.

The solution expressed by equation (12) demonstrates that for $t > 0$, we can obtain the approximate value of concentration 'c' at any 'x'. The series expression obtained in equation (12) has been used to obtain numerical values emphasizing the ability to obtain analytic and numerical solutions by Two-Dimensional Differential Transform Method.

Even, the results are found to be reliable and accurate and the series has been found to converge to exact solutions.

As a result, we can use this method to solve a wide range of difficult partial differential equations both linear and non-linear without the need for linearization, discretization, or perturbation.

References

[1] T. Patel and M. N. Mehta, A solution of Burger's equation for longitudinal dispersion of miscible fluid flow through porous media, Indian Journal of Petroleum Geology 14(2) (2005), 49-54.

[2] R. Meher and M. N. Mehta, A new approach to Backlund Transformations of Burger's equation arising in longitudinal dispersion of miscible fluid flow through porous media, International Journal of Applied Mathematics and Computation 2 (2010), 17-24.

[3] M. S. Joshi, B. D. Narendrasinh and N. M. Monika, Solution of Burger's equation for longitudinal dispersion phenomena occurring in miscible phase flow through porous media, ITB Journal of Engineering and Technological Sciences 44(1) (2012), 61-76.

[4] S. Kunjan and S. Twinkle, A solution of Burger's equation arising in the longitudinal dispersion phenomena in fluid flow through porous media by the mixture of new integral transform and homotopy perturbation method, Journal of Geoscience and Environment Protection 3 (2015), 24-30.

[5] Borana, N. Ravi et al., Numerical solution of Burger's equation in a one-dimensional groundwater recharge by spreading using finite difference Method, International Journal of Advance Research in Science and Engineering 2(11) (2013), 121-130.

[6] Borana, N. Ravi et al., Numerical solution of Burger's equation arising in longitudinal

dispersion phenomenon in fluid flow through porous media by crank-nicolson scheme, 5th International Conference on porous media and their applications in science, engineering industry, Eds., ECI symposium series, (2014).

[7] Kajal Patel et al., A solution of one-dimensional dispersion phenomenon by homotopy analysis method, International Journal of Modern Engineering Research 3(6) (2013), 3626-3631.

[8] Ramakanta Meher et al., Adomain decomposition method for dispersion phenomenon arising in longitudinal dispersion of miscible fluid flow through porous media, Advances in Theoretical and Applied Mechanics 3(5) (2010), 211-220.

[9] K. A. Patel, P. H. Bhathawala and S. Priyanka Patel, Numerical solutions of instabilities in displacement through porous media using differential transform method, Global Journal for Research Analysis 4(7) (2015), 2277-8160.

[10] K. A. Patel, P. H. Bhathawala and S. Priyanka Patel, Numerical solutions of imbibition in double phase flow through porous medium with capillary pressure using differential transform method, International Journal of Scientific and Innovative Mathematical Research 3(8) (2015), 43-48.

[11] C. K. Chen and S. H. Ho, Solving partial differential equations by two-dimensional differential transform, Applied Mathematics and Computation 106(2-3) (1999), 171-179.

[12] F. Ayaz, On the two-dimensional differential transform method, Applied Mathematics and Computation 143(2-3) (2003), 361-374.

[13] Mahendra A. Patel and N. B. Desai, An approximate analytical solution of the Burger's equation for longitudinal dispersion phenomenon arising in fluid flow through porous medium, International Journal on Recent and Innovation Trends in Computing and Communication 5(5) (2017), 1103-1107.

[14] Mritunjay Kumar Singh, Sohini Rajput and Rakesh Kumar Singh, Study of 2D contaminant transport with depth varying input source in a groundwater reservoir, Water Supply 4 (2021), 1464-1480.

[15] M. A. Patel and N. B. Desai, An approximate analytical solution of one-dimensional groundwater recharge by spreading, TWMS Journal of Applied and Engineering Mathematics 9(4) (2019), 838-850.

[16] Abdallah Habila Ali, Applications of differential transform method to initial value problems, American Journal of Engineering Research 6(12) (2017), 365-371.

[17] C. Huang, J. Li and F. Lin, A new algorithm based on differential transform method for solving partial differential equation system with initial and boundary conditions, Advances in Pure Math 10(5) (2020), 337-349. https://doi.org/ 10.4236/apm.2020.105020.

[18] P. Ya Polubarinova-Kochina, Theory of Groundwater Movement, Princeton Uni. Press (1962).

[19] J. Bear, Dynamics of fluids in porous media, Dover Publication, New York, (1972).