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# UNITAL BANACH ALGEBRA

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## Abstract

The theory of real Banach algebras can be very different from the theory of complex Banach algebras. For example, the spectrum of an element of a nontrivial complex Banach algebra can be empty, where as in a real Banach algebra it could be empty for some elements. The aim of this notes is to provide basic information about Banach algebras. The final goal is to show that a Banach algebra can be embedded into a unital Banach algebra $A1$ as an ideal of codimension one. The object of this paper is to show that if $A$ is a Banach algebra and Every maximal ideal in a unital Banach is closed.

## Introduction

A Banach algebra is an algebra (a vector space with multiplication, satisfying the usual algebraic rules) A Banach algebra, is an associative algebra $A$ over the real or complex numbers that at the time also a Banach space, i.e. a normed space and complete in the metric induced by the norm. A Banach algebra is called unital if it has an identity element for the multiplication whose norm is 1. The set of invertible elements in any unital Banach Algebra is an open set.

**Theorem.** *If $X$ is a unital Banach algebra, with identity $e$, then $\| e - x \| < 1$ implies that $x$ is invertible. Also,*

$$x^{-1} = e + \sum_{k=0}^{\infty} (e - x)^n$$

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*and* $\| x \|^{-1} \leq \dfrac{1}{1 - \| e - x \|}$.

**Proof.** We have

$$\| (e-x)^2 \| = \| (e-x)(e-x) \| \leq \| e - x \|^2$$

And so for any positive integer

$$\| (e-x)^n \| \leq \| e - x \|^2 \tag{1}$$

Since $\| e - x \| \leq 1$, it follows that

$$\| e - x \|^n \to 0 \text{ as } n \to \infty. \tag{2}$$

Consequently

$$e - x^n \to 0 \text{ as } n \to \infty.$$

Consider the infinite series

$$e + (e-x) + (e-x)^2 + \ldots$$

Let $Sn = e + (e-x) + \ldots + (e-x)^n$ for $n = 1, 2, \ldots$

But then. by using (1) and (2) for $m > n$, we have

$$\begin{aligned}\| Sm - Sn \| &= \| (e-x)^{n+1} + \ldots + (e-x)^m \| \\ &\leq \| (e-x)^{n+1} + \ldots + (e-x)^m \| \\ &\leq \| e - x \|^{n+1} + \ldots + \| e - x \|^m \to 0 \text{ as } n \to \infty.\end{aligned}$$

Therefore $(Sn)$ is a Cauchy sequence in the complete space $X$.

$$y = e + \sum_{n=1}^{\infty} (e-x)^n$$

Now

$$[\, e - (e-x)] \, [e + (e-x)] + \ldots + (e-x)^n = e - (e-x)^{n+1}$$

Letting $n \to \infty$ and using (3),

We get $xy = e$.

Similarly, $yx = e$.

That is,

$$x^{-1} = y = e + \sum_{n=1}^{\infty} (e - x)$$

Hence $x$ is invertible and is the inverse of.

Also

$$\| x - 1 \| \lim_{n \to \infty} \sum_{n=1}^{\infty} (e - x)^n$$

$$\leq \lim_{n \to \infty} \sum_{k=0}^{n} \| e - x \|^k$$

$$\leq \frac{1}{1 - \| e - x \|}$$

Hence proved.

**Theorem.** *A Banach algebra A without a unit can be embedded into A unital Banach algebra* $A_1$ *as an ideal of codimension one*.

**Proof.** Let $A_1 = A \oplus \mathbb{C}$ as a linear space, and define a Multiplication in $A_1$ by

$$(x, \lambda)(y, \mu) = (xy + \mu x + \lambda y, \lambda\mu).$$

It is easily checked that this is associative and distributive.

Moreover, the element (0, 1) is a unit for this multiplication.

$$(x, \lambda)(0, 1) = (x0 + x + \lambda 0, \lambda 1) = (x, \lambda) = (0, 1)(x, \lambda).$$

Put $\| (x) \| = \| x \| + \| \lambda \|$.

Then $A_1$ is a Banach space when equipped with this norm. Furthermore,

$$\| (x, \lambda)(y, \mu) \| = \| (xy + \mu x + \lambda y, \lambda\mu) \|$$

$$= \| xy + \mu x + \lambda y \| + | \lambda\mu |$$

$$\leq \| x \| \| y \| + | \mu | \| x \| + | \lambda | + | \lambda | | \mu |$$

$$= (\| x \| + | \lambda |)(\| y \| + | \mu |)$$

$$= \| (x, \lambda) \| \| (y, \mu) \|.$$

Hence $A_1$ is a Banach algebra with unit. We may identify $A$ with the ideal $\{(x, 0) : x \in A\}$ in $A_1$ via the isometric isomorphism $x \mapsto (x, 0)$.

Hence proved.

**Theorem.** *Every maximal ideal in a unital Banach is closed.*

**Proof.**

Let $J$ be a maximal ideal in the unital Banach algebra.

Then $J$ cannot contain any invertible elements, otherwise we would have $J = A$.

Hence $J \subseteq \setminus(A)$.

Now, $(A)$ is open and so $A \setminus \mathcal{G}(A)$ is closed,

In particular, $\overline{J} \neq A$.

But $\overline{J}$ is an ideal containing $J$, and so $\overline{J} = J$ since $J$ is a maximal ideal. That is, $J$ is closed.

Hence proved.

**Theorem.** *Let X be a unital Banach algebra with ide-ntity e. The set all invertible elements of X is an open set X.*

**Proof.** Consider the open sphere

$$B = \left\{ y \in X : \| x - y \| < \frac{1}{\| X^{-1} \|} \right\}.$$

If $y \in B$, then

$$\| x - y \| \| x^{-1} \| < 1.$$

But $\| (x - y) x^{-1} \| \leq \| x - y \| \| x^{-1} \|$

Which is same as

$$\| e - y\, x^{-1} \| \leq \| x - y \| \| x^{-1} \| \leq 1,$$

By using (1)

$(yx^{-1})^{-1}$ exists, so that

$z(x^{-1}) = (yx^{-1})\, z = e$

Where $z = (yx^{-1})^{-1}$. But then $p = x^{-1}z$.

So $y$ is invertible.

That is, $y \in U$. We have proved that $x \in B \subset U$.

Therefore, $U$ is open.

Hence proved.

## Conclusion

We seen some basic theorems that are based on Banach algebra. A Banach algebra $A$ without a unit can be embedded into a unital Banach algebra $A_1$ as an ideal of codimension one. Then we discussed about Every maximal ideal in a unital Banach is closed.

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