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ON FOUR TUPLE IN $\mathbb{N}^4$ WITH THE SUM OF SOME COORDINATES IS A FOURTH POWER OF AN INTEGER

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Abstract

In this paper, we have determined infinitely many families of (a, b, c, d) where $a, b, c, d \in \mathbb{N}$ such that sum of any three (and sum of any two) coordinates is a fourth power of an integer. We determined families of such four tuples where all coordinates are equal, exactly three of them are equal, exactly two of them are same, all of them are different.

1. Introduction

There are many fascinating conjectures in Number theory whose proofs

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have escaped from some of the most brilliant mathematicians.

For $a, b \in \mathbb{Z}, a \equiv b (\bmod n)$, where $n \in \mathbb{N}$, if $n \mid (a - b)$. For any $x \in \mathbb{Z}, x^2 \equiv 0$ or $1 (\bmod 3)$ and if $3 \nmid abc$, then $3 \mid (a^4 + b^4 + c^4), 3 \mid (a^2 + b^2 + c^2)$ for any $a, b, c \in \mathbb{Z}$.

In [5], Salunke and Ambulge found infinite families of four tuples of positive integers with sum some coordinates is a perfect square. In [2], with the help of taxicab numbers Salunke and Basude found $(a, b, c, d) \in \mathbb{N}^4$ with addition of any two coordinates is cubic number. Also Salunke, Basude and Jojar [7], found families of four tuples in $\mathbb{N}^4$ with addition of some coordinates to be cubic number.

2. Trivial four Tuples with Sum of any Three Coordinates is a Fourth Power of an Integer

Fourth power of positive integers are $1^4 = 1, 2^4 = 16, 3^4 = 81, 4^4 = 256, 5^4 = 625, \ldots$ and for any $n \in \mathbb{N}$, an integer k with $n^4 < k < (n+1)^4$ is not a fourth power of an integer. For any $n \in \mathbb{N}$, four tuples of positive integers $(27n^4, 27n^4, 27n^4, 27n^4)$ are such that when we add any three coordinates its addition is $(3n)^4$. Such four tuples are infinitely many, and here all coordinates are equal. For any $m, n \in \mathbb{N}$, with $m^4 > 54n^4$ and $m \neq 3n$, the four tuple $(m^4 - 54n^4, 27n^4, 27n^4, 27n^4)$ is such that any three coordinates' sum is m^4 or $(3n)^4$. Such four tuples are infinitely many and first coordinate is different from other coordinates. Here exactly three coordinates are equal.

Examples of such four tuples are:

$(27, 27, 27, 27), (432, 432, 432, 432), (2187, 2187, 2187, 2187), \ldots,$ and $(202, 27, 27, 27), (571, 27, 27, 27), \ldots$

3. Four Tuples of Distinct Naturals and Addition of Their any Three Coordinates is a Fourth Power of an Integer

For determination of distinct $a, b, c, d \in \mathbb{Z}_+$ such that addition of any three is fourth power of integer, we consider $p, q, r, s \in \mathbb{N}$ such that $p < q < r < s$ and $a + b + c = p^4$, $a + b + d = q^4$, $a + c + d = r^4$, $b + c + d = s^4$.

Above equations in matrix form is $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} p^4 \\ q^4 \\ r^4 \\ s^4 \end{bmatrix}$ and $a < b < c < d$ in $\mathbb{Z}$ Premultiplying by $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$, the inverse of the coefficient matrix $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$, we get

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} p^4 \\ q^4 \\ r^4 \\ s^4 \end{bmatrix}$$, which implies

$$a = \frac{1}{3}(p^4 + q^4 + r^4 - 2s^4)$$

$$b = \frac{1}{3}(p^4 + q^4 + 2r^4 + s^4) \qquad (1)$$

$$c = \frac{1}{3}(p^4 + 2q^4 + r^4 - s^4)$$

$$d = \frac{1}{3}(-2p^4 + q^4 + r^4 - s^4)$$

Clearly these values are in $\mathbb{Z} \Leftrightarrow 3 \mid (p^4 + q^4 + r^4 + s^4)$ and $a \in \mathbb{N}$ if $p^4 + q^4 + r^4 - 2s^4 > 0$ $[a \in \mathbb{Z}$ iff $3 \mid (p^4 + q^4 + r^4 + 2s^4)$ iff $3 \mid (p^4 + q^4 + r^4 + s^4)$ etc].

For $a, b, c, d \in \mathbb{Z}$, we have all p, q, r, s are divided by 3 or exactly one of them are divided by 3.

Now we consider cases with $\gcd(p, q, r, s) = 1$ and $s - p =$ minimum.

3.1 $s - p = 3$ (and p, q, r, s as consecutive natural numbers)

Case 1. Let $p = 3m - 1, q = 3m, r = 3m + 1, s = 3m + 2, (m \in \mathbb{N})$ So by equations (1), we have

$$a = \frac{1}{3}[(3m-1)^4 + (3m)^4 + (3m+1)^4 - 2(3m+2)^4],$$

$$b = \frac{1}{3}[(3m-1)^4 + (3m)^4 + 2(3m+1)^4 - (3m+2)^4],$$

$$c = \frac{1}{3}[(3m-1)^4 + 2(3m)^4 + (3m+1)^4 + 2(3m+2)^4],$$

$$d = \frac{1}{3}[-2(3m-1)^4 + (3m)^4 + (3m+1)^4 + (3m+2)^4]$$

and $a \in \mathbb{N}$ iff $(3m-1)^4 + (3m)^4 + (3m+1)^4 - 2(3m+2)^4 > 0$ i.e. $m \geq 7$ $(m = 7, 8, 9, \ldots)$.

For $m = 7$, we get $(a, b, c, d) = (9685, 55270, 95045, 129526)$ as per our required condition: $9685 + 55270 + 95045 = 160000 = (20)^4$; $9685 + 55270 + 129526 = 194481 = (21)^4$ $9685 + 95045 + 129526 = 234256 = (22)^4$; $55270 + 95945 + 129526 = 279841 = (23)^4$. Taking $m = 8, 9, 10, 11, \ldots$ we get infinitely many four tuples of positive integers which are distinct with our condition.

Case 2. Similarly, for $p = 3m - 2, q = 3m - 1, r = 3m - 1, r = 3m, s = 3m + 1 (m \in \mathbb{N}, m = 7, 8, 9, \ldots)$, using in equations (1), we get infinitely many four tuples with the required condition.

3.2 $s - p = 4$. **(either *p* of *s* is divisible by 3)**

Let $p = 3m, q = 3m + 1, r = 3m + 2, s = 3m + 4 (m \in \mathbb{N})$.

Then by equations (1), we have

$$a = \frac{1}{3}[(3m)^4 + (3m+1)^4 + (3m+2)^4 - 2(3m+4)^4],$$

$$b = \frac{1}{3}[(3m)^4 + (3m+1)^4 - 2(3m+2)^4 + 2(3m+4)^4],$$

$$c = \frac{1}{3}[(3m)^4 - 2(3m+1)^4 + (3m+2)^4 + 2(3m+4)^4],$$

$$d = \frac{1}{3}[-2(3m)^4 + (3m+1)^4 + (3m+2)^4 + (3m+4)^4]$$

and $a > 0$ iff $(3m)^4 + (3m+1)^4 + (3m+2)^4 - 2(3m+4)^4 > 0$ i.e. $m \geq 9 (m = 9, 10, 11, \ldots)$. For $m = 9$, we get a four tuple $(a, b, c, d) = (2112, 218352, 310977, 394192)$ as follows: $2112 + 218352 + 310977 = 531441 = (27)^4; 212 + 218352 + 394192 = 614656 = (28)^4$ $2112 + 310977 + 394192 = 707281 = (29)^4; 218352 + 310977 + 394192 = 923521 = (31)^4$. Taking $m = 10, 11, 12, 13, \ldots$ we get infinitely many four tuples of distinct positive integers with required condition.

4. Four Tuples of Positive Integers with two Coordinates Equal and Addition of any Three Coordinates is Fourth Power

4.1 $s - p = 2$ **and equal coordinates not divisible by 3**

For $p = 3m - 1 = q, r = 3m, s = 3m + 1, (m \in \mathbb{N})$, by equations (1), we have

$$a = \frac{1}{3}[2(3m-1)^4 + (3m)^4 - 2(3m+1)^4],$$

$$b = \frac{1}{3}[2(3m-1)^4 - 2(3m)^4 + (3m+1)^4],$$

$$c = d = \frac{1}{3}[-(3m-1)^4 + (3m)^4 + (3m+1)^4],$$

and $a > 0$ iff $2(3m-1)^4 + (3m)^4 - 2(3m+1)^4 > 0$ iff $m \geq 6 (m = 6, 7, 8, \ldots)$. For $m = 6$, we get $(a, b, c, d) = (3792, 29137, 50592, 50592)$ as a four tuple as follows: $3792 + 29137 + 50592 = 83521 = (17)^4; 3792 + 50592 + 50592 = 104976$ $29137 + 50592 + 50592 = 130321 = (19)^4$.

Taking $n = 7, 8, 9, 10, \ldots$ we get infinitely many four tuples of positive integers where in each tuple exactly two coordinates are equal and satisfy the required condition.

5. Four Tuples of Naturals where Addition of any two Coordinates is Fourth Power

For any $n \in \mathbb{N}, (8n^4, 8n^4, 8n^4, 8n^4)$ is a four tuple of positive integers, where all coordinates are equal, is such that addition of any two coordinates is fourth power , i.e. $(2n)^4$. Such four tuples are infinitely many which are $(8, 8, 8, 8), (128, 128, 128, 128), (648, 648, 648, 648), (2048, 2048, 2048, 2048), \ldots$

For any $m, n \in \mathbb{N}, m^4 > 8n^4$ and $m \neq 2n$, the four tuple $(m^4 - 8n^4, 8n^4, 8n^4, 8n^4)$ is of positive integers such that addition of its two coordinates is fourth power m^4 or $(2n)^4$. Such four tuples are infinitely many, where first coordinate is different from other coordinates and exactly three coordinates are equal. For example $(73, 8, 8, 8), (248, 8, 8, 8),$ $(617, 8, 8, 8), (497, 128, 128, 128), (1168, 128, 128, 128), (2273, 128, 128, 128), \ldots$.

5.1. Determination of four tuples of positive integers with exactly two coordinates are equal and addition of two coordinates is fourth power

Suppose we have (a, b, c) of three distinct naturals with $a + b$, $a + c$, $b + c$ and $2a$ as fourth power of an integer [6]. Then (a, a, b, c)

is a four tuple having exactly two coordinates equal and addition of two coordinates as a fourth power.

Using a generalized taxicab number [1], $m \in \mathbb{N}$ which is expressed as $m = x^4 + y^4 = z^4 + t^4$ where $x, y, z, t \in \mathbb{N}$ and $\{x, y\} \neq \{z, t\}$, we get required four tuple. For such a number m, let $t = \min\{x, y, z, t\}$, i.e. $z = \max\{x, y, z, t\}$. Consider z, t as even numbers (otherwise consider the number $16m = (2x)^4 + (2y)^4 = (2z)^4 + (2t)^4$, where $2z, 2t$ are even numbers etc). Then both x, y are either even or odd. Then

$a = \frac{1}{2}(x^2 + y^4 - z^4), b = \frac{1}{2}(x^2 + y^4 - z^4), c = \frac{1}{2}(-x^2 + y^4 - z^4)$ are distinct naturals such that $a + b = x^4, a + c = y^4, b + c = z^4$ are fourth power of integers and $2a = (a + c) + (a + b) - (b + c) = x^4 + y^4 - z^4 = t^4$ is fourth power of an integer. Now we get required tuple (a, a, b, c) where addition of two coordinates is fourth power.

Illustration: We have [due to Euler]

$635318657 = (59)^4 + (158)^4 = (133)^4 + (134)^4$. Multiplying by 16, we get

$$1270637314 = (118)^4 + (316)^4 = (266)^4 + (268)^4$$

Take $p = 268, q = 266, r = 316, s = 166$. Now

$$a = \frac{1}{2}[p^4 + q^4 - r^4] = \frac{1}{2}[5158686976 + 5006411536 - 9971220736] = 96938888$$

$$b = \frac{1}{2}[p^4 + q^4 - r^4] = \frac{1}{2}[5158686976 + 5006411536 - 9971220736] = 5061748088$$

$$c = \frac{1}{2}[p^4 + q^4 - r^4] = \frac{1}{2}[-5158686976 + 5006411536 + 9971220736] = 4909472648$$

are three distinct positive integers.

$(a, a, b, c) = (96938888, 96938888, 5061748088, 4909472648)$ is a four tuple of positive integers where two coordinates are equal and addition of two coordinates is fourth power : $96938888 + 96938888 = 19387776 = (118)^4$,

$96938888 + 5061748088 = 5158686976 = (268)^4$ $96938888 + 4909472648 = 5006411536 = (266)^4$, $5061748088 + 4909472648 = 9971220736 = (316)^4$

From above four tuple we get infinitely many four tuples $(n^4 a, n^4 a, n^4 b, n^4 c)$, for $n \in \mathbb{N}$, with addition of two coordinates of is a fourth power of natural.

6. Conclusions

There are infinite families of four tuples of distinct naturals different from obtained in sections 3.1 and 3.2 with required condition. For example, by taking $p = 3m - 1, q = 3m, r = 3m + 1, s = 3m + 4, m \in \mathbb{N}$ etc, we obtain such four tuples. There are infinite families of four tuples other than obtained in section 4, of positive integers where exactly two coordinates are same and in each of such tuple, addition of three coordinates is fourth power of a natural. For example taking $p = 3m, q = 3m + 2 = r, s = 3m + 4, m \in \mathbb{N}$ etc. we obtain such four tuples.

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