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GENERALIZED STATISTICAL CONVERGENCE OF ORDER α IN RANDOM n-NORMED SPACE

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Abstract

The concept of generalized statistical convergence was presented by Mursaleen [17]. In present paper we describe the idea of generalized statistical convergence of order alpha in random n-normed space and prove some inclusion theorems.

I. Introduction

The statistical convergence is an attractive theory not in the field of pure Mathematics but also in the area of science and technology. The term statistical convergence was presented by Fast [6] and Steinhauss [26] autonomously. It helps to understand the behaviour of sequence beyond the usual convergence. Schoenberg [22] related this with summability. Fridy [7] has defined the image of statistically convergent sequences and Connor [4] worked on statistical p-Cesàro convergence of sequences. Then later on generalization was given by many researchers in different forms of sequences and spaces. The term λ-statistical convergence was given by Mursaleen [17] using a non-decreasing sequence $\lambda = (\lambda_n)$ of positive real numbers tending to

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∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. Further Çolak [2, 3] extended the work on λ-statistical convergence for order α for single as well as for double sequences given by Mursaleen [17, 18, 20]. An important concept which is the generalization of metric space was presented by Menger [16] (named statistical metric space). This type of metric is important to use in the spaces where the exact distance between two points is not possible to find. Schweizer [23] also worked on Statistical Metric spaces. The impression of 2-normed space and n-normed space was presented by Gähler [8]. Diminie and White [5] has studied the linear 2-normed space with non-expansive mappings. Serstnev [24, 25] has introduced the concept of randomness and defined random normed spaces. Alsina et al. [1] has studied the probabilistic norm's continuity properties and Karakus [13] defined the statistical convergence on probabilistic normed spaces. Savaş [14] has given λ-statistical convergence in random 2-normed space. Later, Meenakshi et al. [15] extended the idea of λ-statistical convergence for order α on random 2-normed space and studied some properties. In this paper our objective is to specify the generalized statistical convergence of order α on random n-normed space. Here we are using only real sequences. First we take some basic definitions for our study as follows:

Definition 1.1 [6]. A sequence $x = (x_k)$ is called statistically convergent to a number l if for every $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |x_k - l| \geq \epsilon\}| = 0,$$

where vertical bars represents the order of the involved set. Here, we denote it as $S - \lim_{k \to \infty} x_k = l$.

Definition 1.2 [17]. A sequence $x = (x_k)$ is called λ-statistically convergent to a number l if for every $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - l| \geq \epsilon\}| = 0,$$

where $I_n = [n + 1 - \lambda_n, n]$ with $n = 1, 2, \ldots$

Definition 1.3 [19]. Let $\lambda = (\lambda_n)$ be a sequence of real numbers tending to ∞ such that $0 < \lambda_0 < \lambda_1 < \lambda_2 \ldots\ldots$ and $\lambda_k \to \infty$ as $k \to \infty$. Then a sequence $x = (x_k)$ of real numbers is called Λ-convergent to a number l if $\Lambda x_k \to l$ as $k \to \infty$, where $\Lambda x_k = \frac{1}{\lambda_k} \sum_{j=0}^{k} (\lambda_j - \lambda_{j-1}) x_j$.

Next definitions describe random n-normed space.

Definition 1.4 [9]**.** Let X be a real linear space of dimension d such that $d \geq n$, $n \in \mathbb{N}$. A function $\| \cdot, \cdot, \ldots, \cdot \| : X^n \to \mathbb{R}$ is called a random n-norm on X, if

(i) $\| x_1, x_2, \ldots\ldots, x_n \| = 0$ iff all $x_i \in X$, $i = 1, 2, \ldots n$ are linearly dependent,

(ii) $\| x_1, x_2, \ldots\ldots, x_n \|$ is invariant under the permutation for all $x_i \in X$, $i = 1, 2, \ldots n$,

(iii) $\| \alpha x_1, x_2, \ldots\ldots, x_n \| = | \alpha | \| x_1, x_2, \ldots\ldots, x_n \|$, for any $\alpha \in \mathbb{R}$, all $x_i \in X$, $i = 1, 2, \ldots, n$,

(iv) $\| x_1 + y_1, x_2, \ldots\ldots, x_n \| \leq \| x_1, x_2, \ldots\ldots, x_n \| + \| y_1, x_2, \ldots\ldots, x_n \|$ for all $x_1, y_1, x_2, \ldots\ldots, x_n \in X$

Also $(X, \| \cdot, \cdot, \ldots, \cdot, \cdot \|)$ is called a n-normed space.

Example [10]**.** Consider n-normed space $X = \mathbb{R}^n$ with Euclidean n-norm

$$\| X_1, X_2, \ldots\ldots, X_n \| = | \det (X_1, X_2, \ldots\ldots, X_n | \text{ where}$$

$$X_i = (x_{i1}, x_{i2}, \ldots\ldots, x_{in}) \in \mathbb{R}^n, \ i = 1, 2, \ldots, n.$$

The following definition of probabilistic n-normed space is obtained from Definition 1.4.

Definition 1.5. Consider X be a real linear space with dimension $d \geq n$, $n \in \mathbb{N}$, $*$ be a triangular norm and $\wp$ be a map on Cartesian product of X by itself n times into D^+ which is the set of all distribution functions. Then $\wp$ and $(X, \wp, *)$ is said to be probabilistic norm and probabilistic n-normed space respectively, if

(i) $\wp(w_1, w_2, \ldots, w_n; t) = H_0(t)$ {where $H_0(t) = 0$ if $t \leq 0$ and $H_0(t) = 1$ if $t > 0$} if $w_1, w_2, \ldots, w_n \in X$ are linearly dependent,

(ii) $\wp(w_1, w_2, \ldots, w_n; t)$ is invariant under any permutation of $w_1, w_2, \ldots, w_n \in X$,

(iii) $\wp(\alpha w_1, w_2, \ldots, w_n; t) \geq \wp(w_1, w_2, \ldots, w_n; t/|\alpha|)$ for every $t > 0$, nonzero $\alpha \in \mathbb{R}$ and $w_1, w_2, \ldots, w_n \in X$,

(iv) $\wp(w_1 + w_1', w_2, \ldots, w_n; t) \geq \wp(w_1, w_2, \ldots, w_n; t) * \wp(w_1', w_1, w_2, \ldots, w_n; t)$ for every $t > 0$ and $w_1, w_1', w_2, \ldots, w_n \in X$.

The condition (iv) can be formulated using t-norm as follows

(v) $\wp(w_1 + w_1', w_2, \ldots, w_n; s + t) \geq \wp(w_1, w_2, \ldots, w_n; s) * \wp(w_1', w_1, w_2, \ldots, w_n; t)$ for $w_1, w_1', w_2, \ldots, w_n \in X$ and $s, t \in [0, \infty)$.

If (i), (ii), (iii) and (v) are satisfied then $(X, \wp, *)$ is called generalized probabilistic n-normed space of Menger type or random n-normed space (RnN-space).

The following definitions associate the convergence of sequences for usual as well as statistical.

Definition 1.6 [11]. A sequence $x = (x_k)$ in a RnN-space $(X, \wp, *)$ is called convergent to $l \in X$ with respect to norm $\wp$ if for every $\varepsilon > 0, t \in (0, 1)$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$, there exists $K_0 \in \mathbb{Z}^+$ such that $\wp(w_1, w_2, \ldots, w_{n-1}, x_k - l; \varepsilon) > 1 - t$ whenever $k \geq K_0$. It is denoted by $\wp - \lim x_k = l$.

Definition 1.7 [11]. A sequence $x = (x_k)$ in a RnN-space $(X, \wp, *)$ is called Cauchy with respect to norm $\wp$ if for every $\varepsilon > 0, t \in (0, 1)$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$, there exists $K_0 \in \mathbb{Z}^+$ such that $\wp(w_1, w_2, \ldots, w_{n-1}, x_k - x_m; \varepsilon) > 1 - t$ whenever $k, m \geq K_0$.

Definition 1.8 [11]. A sequence $x = (x_k)$ in a RnN-space $(X, \wp, *)$ is called statistically convergent or S^{RnN}-convergent to $l \in X$ with respect to norm $\wp$ if for every $\varepsilon > 0, t \in (0,1)$ and non-zero elements $w_1, w_2, \ldots, w_{n-1} \in X$, we have $\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : \wp(w_1, w_2, \ldots, w_{n-1}, x_k - l; \varepsilon) \leq 1 - t\}| = 0$.

Here, we write $S^{RnN} - \lim x_k = l$ and l is termed as statistical limit.

II. Main Results

In the present part, we describe the term Λ-statistically convergence for the sequences in RnN-space $(X, \wp, *)$. Also, we outlined some basic results of this convergence in RnN-space.

Definition 2.1. A sequence $x = (x_k)$ in a random n-normed space $(X, \wp, *)$ is called Λ-statistically convergent or S_{Λ}^{RnN}-convergent to some $l \in X$ with respect to norm $\wp$ if for every $\varepsilon > 0, t \in (0, 1)$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} | \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; \varepsilon) \leq 1 - t\} | = 0$$

or

$$\lim_{n\to\infty} \frac{1}{\lambda_n} | \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; \varepsilon) > 1 - t\} | = 1.$$

Here, we write $S_{\Lambda}^{RnN} - \lim x_k = l$ and l is termed as Λ-statistical limit.

Definition 2.2. A sequence $x = (x_k)$ in a RnN-space $(X, \wp, *)$ is called Λ-statistically Cauchy with respect to norm $\wp$ if for every $\varepsilon > 0, t \in (0, 1)$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$, there exists $K_0 \in \mathbb{Z}^+$, we have $\lim_{n\to\infty} \frac{1}{\lambda_n} (k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - \Lambda x_m; \varepsilon) \leq 1 - t\} | = 0,$ for all $k, m \geq K_0$.

Definition 2.3. Let sequence $\lambda = (\lambda_n)$ be non decreasing positive real numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. A sequence $x = (x_k)$ in a RnN-space $(X, \wp, *)$ with respect to norm $\wp$ is called Λ-statistically convergent of order $\alpha (0 < \alpha \leq 1)$ to $l \in X$ if for every $\varepsilon > 0, t \in (0, 1)$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; \varepsilon) \leq 1 - t\} | = 0$$

or

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; \varepsilon) > 1 - t\} | = 1$$

i.e. we write $S_\Lambda^{RnN^\alpha} - \lim\limits_{k\to\infty} x_k = l$. $S_\Lambda^{RnN^\alpha}$ represents the class of all Λ-statistical convergent sequence of order α in a RnN- space $(X, \wp, *)$.

Definition 2.4. Let sequence $\lambda = (\lambda_n)$ be non-decreasing positive real numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. A sequence $x = (x_n)$ in a RnN- space $(X, \wp, *)$ is called Λ-statistically Cauchy of order $\alpha(0 < \alpha \leq 1)$, if for every $\varepsilon > 0, t \in (0, 1)$ and for nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - \Lambda x_m; \varepsilon) \leq 1 - t\} | = 0$$

or

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - \Lambda x_m; \varepsilon) \leq 1 - t\} | = 1.$$

On the basis of above definitions we give the next Lemma 2.5.

Lemma 2.5. *Let $(X, \wp, *)$ be a RnN- space and sequence $\lambda = (\lambda_n)$ of non-decreasing positive real number tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. Then for every $\varepsilon > 0, t \in (0, 1), \alpha(0 < \alpha \leq 1)$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$, the following statements are equivalent for the sequence $x = (x_k)$ in X:*

(i) $S_\Lambda^{RnN^\alpha} - \lim\limits_{k\to\infty} x_k = l$.

(ii) $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; \varepsilon) \leq 1 - t\} | = 0$

(iii) $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; \varepsilon) > 1 - t\}| = 1$

(iv) $S_\Lambda^{RnN^\alpha} - \lim_{k\to\infty} \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; \varepsilon) = 1.$

Now we come to the following results using above definitions and lemma.

Theorem 2.6. *Let $x = (x_k)$ be a sequence in RnN-space $(X, \wp, *)$ and $0 < \alpha \le 1$. If $S_\Lambda^{RnN^\alpha} - \lim_{k\to\infty} x_k = l$, then l must be unique.*

Proof. Suppose $S_\Lambda^{RnN^\alpha} - \lim_{k\to\infty} x_k = m$, where $l \neq m$. For $\varepsilon > 0, t > 0$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$, choose $\rho > 0$ such that $(1-\rho) * (1-\rho) > 1 - \varepsilon$. Define

$$K_1(\rho, t) = \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t/2) \le 1 - \rho\}$$

$$K_2(\rho, t) = \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - m; t/2) \le 1 - \rho\}.$$

Since $S_\Lambda^{RnN^\alpha} - \lim_{k\to\infty} x_k = l$ and $S_\Lambda^{RnN^\alpha} - \lim_{k\to\infty} x_k = m$. Hence, for every $t > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K_1(\rho, t)| = 0 \text{ and } \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K_2(\rho, t)| = 0.$$

Let $K(\rho, t) = K_1(\rho, t) \cup K_2(\rho, t)$ then $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K(\rho, t)| = 0$, which implies that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K^c(\rho, t)| = 1.$

Let $k \in K^c(\rho, t) = K_1^c(\rho, t) \cap K_2^c(\rho, t)$. We write,

$$\wp(w_1, w_2, \ldots, w_{n-1}, l - m; t) \ge \wp\left(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; \frac{t}{2}\right)$$

$$* \wp\left(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - m; \frac{t}{2}\right) > (1-\rho) * (1-\rho) > 1 - \varepsilon.$$

Here $\varepsilon > 0$ is arbitrary small positive number which implies that $\wp(w_1, w_2, \ldots, w_{n-1}, l - m; t) = 1$, for all $t > 0$ and non-zero elements $w_1, w_2, \ldots, w_{n-1} \in X$.

Hence, $l = m$.

Next result gives us the elementary properties of Λ-statistical convergence on RnN- spaces.

Theorem 2.7. *Let* $(X, \wp, *)$ *be a* RnN*-space and* $0 < \alpha \le 1$. *Let* $x = (x_k)$ *and* $y = (y_k)$ *be two sequences in X. Then*

(i) *If* $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} x_k = l$ *and* $c \in \mathbb{R}$, *then* $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} cx_k = cl$.

(ii) *If* $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} x_k = l$ *and* $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} y_k = m$ *then* $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} (x_k + y_k) = l + m$.

Proof (i)**.** Let $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} x_k = l$ and $c \in \mathbb{R}$. Then for $\varepsilon > 0$, $t \in (0, 1)$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$, define $K(\varepsilon, t) = \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t) \le 1 - \varepsilon\}$, then $\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} | K(\varepsilon, t) | = 0$.

Take $k \in K^c(\varepsilon, t)$, then we have

$$\wp(w_1, w_2, \ldots, w_{n-1}, \Lambda cx_k - cl; t) = \wp(w_1, w_2, \ldots, w_{n-1}, c(\Lambda x_k - l); t)$$

$$= \wp\left(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; \frac{t}{|c|}\right) \ge \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t)$$

$$* \wp\left(w_1, w_2, \ldots, w_{n-1}, 0; \frac{t}{|c|} - t\right) \ge \wp\left(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; \frac{t}{2}\right) * 1$$

$$> 1 - \varepsilon.$$

Therefore, $\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} | K^c(\varepsilon, t) | = 1$ i.e. $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} cx_k = cl$.

(ii) Let $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} x_k = l$ and $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} y_k = m$. Then for $\varepsilon > 0, t > 0$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$, choose $\rho > 0$ such that $(1-\rho) * (1-\rho) > 1 - \varepsilon$. Define

$$K_x(\rho, t) = \{k \in I_n : \wp(w_1, w_2, \ldots w_{n-1}, \Lambda x_k - l; t/2) \leq 1 - \rho\}$$

$$K_y(\rho, t) = \{k \in I_n : \wp(w_1, w_2, \ldots w_{n-1}, \Lambda x_k - m; t/2) \leq 1 - \rho\}.$$

Hence, for every $t > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} | K_x(\rho, t) | = 0 \text{ and } \lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} | K_y(\rho, t) | = 0.$$

Let $K(\rho, t) = K_1(\rho, t) \cup K_2(\rho, t)$ then $\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} | K(\rho, t) | = 0$.

For $k \in K^c(\rho, t)$

$$\wp(w_1, w_2, \ldots, w_{n-1}, \Lambda(x_k + y_k) - (l + m); t)$$

$$\geq \wp\left(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; \frac{t}{2}\right) * \wp\left(w_1, w_2, \ldots, w_{n-1}, \Lambda y_k - m; \frac{t}{2}\right)$$

$$> (1-\rho) * (1-\rho) > 1 - \varepsilon.$$

Therefore, $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} (x_k + y_k) = l + m$.

Theorem 2.8. *Let $x = (x_k)$ be a sequence in a RnN-space $(X, \wp, *)$ and $0 < \alpha \leq 1$. If $S_{\Lambda}^{RnN} - \lim_{k\to\infty} x_k = l$ then $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} x_k = l$.*

Proof. Since $S_{\Lambda}^{RnN} - \lim_{k\to\infty} x_k = l$ then for $\varepsilon > 0, t \in (0, 1)$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$, there exists $k_0 \in \mathbb{Z}^+$ shut that

$\wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t) > 1 - \varepsilon$ for all $k \geq k_0$.

Hence the set $K(\varepsilon, t) = \{n \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t) \leq 1 - \varepsilon\}$

$\subset \{1, 2, 3, \ldots, k_0 - 1\}$, for which $\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |K(\varepsilon, t)| = 0$. This shows that $S_{\Lambda}^{RnN} - \lim_{k\to\infty} x_k = l$.

Theorem 2.9. *Let* $(X, \wp, *)$ *be a* RnN*-space and* $0 < \alpha \le \beta \le 1$. *Then* $S_{\Lambda}^{RnN^{\alpha}}(X) \subseteq S_{\Lambda}^{RnN^{\beta}}(X)$ *and this inclusion is strict for some* α *and* β *such that* $\alpha < \beta$.

Proof. If $0 < \alpha \le \beta \le 1$ and $S_{\Lambda}^{RnN^{\alpha}}(X) \subseteq S_{\Lambda}^{RnN^{\beta}}(X)$. Then for every $\varepsilon > 0$, $t > 0$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$, we have

$$\frac{1}{\lambda_n^{\beta}} |\{n \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t) \le 1 - \varepsilon\}|$$

$$\le \frac{1}{\lambda_n^{\alpha}} |\{n \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t) \le 1 - \varepsilon\}|$$

which immediately implies that $S_{\Lambda}^{RnN^{\alpha}}(X) \subseteq S_{\Lambda}^{RnN^{\beta}}(X)$.

Theorem 2.10. *Let* $x = (x_k)$ *be a sequence in* RnN*-space* $(X, \wp, *)$ *and* $0 < \alpha \le 1$. *Then* $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} x_k = l$ *if and only if there exists a set* $K = \{k_m : k_1 < k_2 < \ldots\} \subset \mathbb{N}$ *with* $\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |K| = 1$ *such that* $S_{\Lambda}^{RnN^{\alpha}} - \lim_{n\to\infty} x_{k_n} = l$.

Proof. First suppose that $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} x_k = l$ then for any $t > 0$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$, $\rho \in N$ if we define

$$A(\rho, t) = \left\{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t) \le 1 - \frac{1}{\rho}\right\}$$

$$B(\rho, t) = \left\{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t) > 1 - \frac{1}{\rho}\right\}.$$

Then $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | A(\rho, t) | = 0.$ For $\rho = 1, 2, 3, \ldots$

$$B(1, t) \supset B(2, t) \supset \ldots B(j, t) \supset B(j+1, t) \supset \ldots \tag{2.1}$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | B(\rho, t) | = 1. \tag{2.2}$$

Now, it sufficient to prove that $S_\Lambda^{RnN^\alpha} - \lim_{n\to\infty} x_{k_n} = l$ over $B(\rho, t)$ for getting the result. Assume for $k \in A(\rho, t)$ sequence $x = (x_k)$ is not convergent to l with respect to norm $\wp$. Then, for $\eta > 0$ we have

$\{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t) \le 1 - \eta\}$ has infinitely many terms x_k.

Let $B(\eta, t) = \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t) > 1 - \eta\}$ and $\eta > \frac{1}{\rho}$, for $\rho = 1, 2, 3, \ldots$. This implies that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | B(\eta, t) | = 0.$

Also from (2.1) we have $B(\rho, t) \subset B(\eta, t)$, that gives $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | B(\rho, t) | = 0$ which is contradiction to (2.2).

Hence $S_\Lambda^{RnN^\alpha} - \lim_{n\to\infty} x_{k_n} = l.$

Conversely, Suppose $K = \{k_m : k_1 < k_2 < \ldots\}$ is a subset of $\mathbb{N}$ with $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | k | = 1$ and $S_\Lambda^{RnN^\alpha} - \lim_{n\to\infty} x_{k_n} = l.$ Then for every $\varepsilon > 0, t > 0$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$, there exists $k_0 \in \mathbb{Z}^+$ *such that*

$$\{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t) > 1 - \varepsilon\}$$

for all $k \ge k_0$. Since the set $\{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t) \le 1 - \varepsilon\}$ is contained in $\mathbb{N} - \{k_0 + 1, k_0 + 2, k_0 + 3, \ldots\}$.

Therefore, $\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t) \le 1 - \varepsilon\}| = 0$

Hence, $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} x_k = l.$

Theorem 2.11. *A sequence* $x = (x_k)$ *in a* RnN*-space* $(X, \wp, *)$ *is statistically convergent of order* α *iff it is statistically Cauchy of order* α *where* $0 < \alpha \le 1$.

Proof. Let $x = (x_k)$ be statistically convergent sequence of order α. i.e. $S_{\Lambda}^{RnN^{\alpha}} - \lim_{k\to\infty} x_k = l.$ Then for every $\varepsilon > 0, t > 0$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$, choose $\rho > 0$ such that $(1-\rho) * (1-\rho) > 1 - \varepsilon.$

Define $K(\rho, t) = \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t/2) \le 1 - \rho\}.$

Then $K^c(\rho, t) = \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t/2) > 1 - \rho\}$ which gives by virtue of $S_{\alpha}^{RnN} - \lim_{k\to\infty} x_n = l.$ Thus

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |K(\rho, t)| = 0 \text{ and } \lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |K^c(\rho, t)| = 1.$$

Let $m \in K^c(\rho, t)$ then $\wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_m - l; t/2) > 1 - \rho.$

If we take $B(\varepsilon, t) = \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - \Lambda x_m; \ ) \le 1 - \varepsilon\}.$

For result, it is enough to prove that $B(\varepsilon, t) \subset K(\rho, t).$

Let $k \in B(\varepsilon, t)$ which gives $\wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - \Lambda x_m; t) \le 1 - \varepsilon.$ Suppose $k \notin K(\rho, t)$ then $\wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t/2) > 1 - \rho.$ Now we have

$$1 - \varepsilon \ge \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - \Lambda x_m; t) \ge \wp\left(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; \frac{t}{2}\right)$$

$$* \wp\left(w_1, w_2, \ldots, w_{n-1}, \Lambda x_m - l; \frac{t}{2}\right)$$

$$\ge (1-\rho) * (1-\rho) > 1 - \varepsilon.$$

This contradiction shows that $B(\varepsilon, t) \subset K(\rho, t)$.

Conversely, We prove this by contradiction. Let sequence $x = (x_k)$ be statistically Cauchy of order α but not statistically convergent of order α with respect to norm $\wp$. Then, for every $t > 0, \varepsilon > 0$ and nonzero elements $w_1, w_2, \ldots, w_{n-1} \in X$, there exists $m \in \mathbb{Z}^+$ such that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | K(\varepsilon, t) | = 0$,

where $K(\varepsilon, t) = \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - \Lambda x_m; t) \leq 1 - \varepsilon\}$. This implies that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | K^c(\varepsilon, t) | = 1$.

Choose $\mu > 0$ such that $(1 - \mu) * (1 - \mu) > 1 - \varepsilon$.

Let $B(r, t) = \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; t/2) > 1 - \eta\}$

Let $m \in B(r, t)$, then $\wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_m - l; t/2) > 1 - \eta$.

Since,

$$\wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - \Lambda x_m; t) \geq \wp\left(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - l; \frac{t}{2}\right)$$

$$* \wp\left(w_1, w_2, \ldots, w_{n-1}, \Lambda x_m - l; \frac{t}{2}\right)$$

$$\leq (1 - \mu) * (1 - \mu) > 1 - \varepsilon.$$

Therefore, $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | \{k \in I_n : \wp(w_1, w_2, \ldots, w_{n-1}, \Lambda x_k - \Lambda x_m; t) \leq 1 - \varepsilon\} | = 0$

i.e. $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} | K^c(\varepsilon, t) | = 0$ which leads to contradiction. Hence $x = (x_k)$ is statistically convergent of order α.

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