Advances and Applications in Mathematical Sciences
Volume 21, Issue 2, December 2021, Pages 755-762

DUAL OF GENERALIZED ALMOST DISTRIBUTIVE FUZZY LATTICES

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Abstract

The Dual of Generalized Almost Distributive Fuzzy Lattices (Dual-GADFL's) as a generalization of Generalized Almost Distributive Fuzzy Lattices (GADFL's) is investigated in this article. An appropriate and essential condition for a Dual-GADFL's is transformed into dual distributive fuzzy lattices. In this fuzzy lattice analogue of the result of classical dual-GADFL's are established.

1. Introduction

Rao, Ravi Kumar and Rafi [1] developed the concept of a Generalized Almost Distributive Lattice (GADL) as a generalisation of an Almost Distributive Lattice (ADL) [2], which was a common abstraction of almost all existing ring theoretic generalisations of a Boolean algebra on the one hand and distributive lattice on the other. The class of GADLs inherits almost all the properties of distributive lattice except possibly the commutativity of, the right distributivity of either of the operations over the other. GADLs properly contain the class of ADLs and retain many of the fundamental aspects of ADLs. On the other hand, L. A. Zadeh [3] introduced the concept of fuzzy set

2020 Mathematics Subject Classification: 06D72, 06B10, 06D75.
Keywords: Dual Generalized Almost Distributive Fuzzy Lattices (Dual-GADFL's), Generalized Almost Distributive Fuzzy Lattices (GADFL's), dual distributive fuzzy lattices, dual fuzzy poset.
Received May 17, 2021; Accepted June 7, 2021

in 1965 to mathematically describe vagueness in its most abstract form and to solve such problems by assigning a value to each possible individual in the universe of discourse that represents their grade of membership in the fuzzy set. Zadeh [4] defined fuzzy ordering as a generalisation of the concept of or dering, Ajmal and Thomas [5] described fuzzy sublattices and defined a fuzzy lattice as a fuzzy algebra in 1994. Chon [6] established a new notion of fuzzy lattice and researched the level sets of fuzzy lattices in 2009, based on Zadeh's fuzzy order concept.

In this paper, the Dual of Generalized Almost Distributive Fuzzy Lattices (Dual-GADFL's) as a generalization of Generalized Almost Distributive Fuzzy Lattices (GADFL's) is investigated in this article. An appropriate and essential condition for a Dual-GADFL's is transformed into dual distributive fuzzy lattices. In this fuzzy lattice analogue of the result of classical dual-GADFL's are established.

2. Preliminaries

First, we recall certain definitions of Generalized Almost Distributive Fuzzy Lattices that are required in this paper.

Definition 2.1. Let (R, V, Λ) be an algebra type $(2, 2)$ and (R, A) be a fuzzy poset. Then we call (R, A) is a Generalized Almost Distributive Fuzzy Lattice if it satisfies the following axioms.

1. $A((a \wedge b) \wedge c, a \wedge (b \wedge c)) = A(a \wedge (b \wedge c), (a \wedge b) \wedge c) = 1$

2. $A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1$

3. $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = 1$

4. $A(a \wedge (a \vee b), a) = A(a, a \wedge (a \vee b)) = 1$

5. $A((a \vee b) \wedge a, a) = A(a, (a \vee b) \wedge a) = 1$

6. $A((a \wedge b) \vee b, b) = A(b, (a \wedge b) \wedge b) = 1$ for all $a, b, c \in R.$

Next, the dual fuzzy partial order relation, dual fuzzy lattice, and dual distributive fuzzy lattice are all defined.

Definition 2.2. Let R be a set. A function $A : R * R \to [0, 1]$ is called a dual fuzzy relation in R. The dual fuzzy relation A in R is reflexive if and only if $A(a, a) = 1$ for all $a \in R$, A is transitive if and only if A is transitive if and only if $A(a, c) \leq \inf_{b \in R} \max(A(a, b), A(b, c))$, and A is symmetric if and only if $A(a, b) > 0$ implies $A(b, a) > 0$.

Definition 2.3. A dual fuzzy relation is called a dual fuzzy partial order relation if is reflexive, symmetric and transitive. A dual fuzzy partial order relation is a dual fuzzy total order relation if and only if $A(b, a) > 0$ or $A(a, b) > 0$.

Definition 2.4. Let (R, A) be a dual fuzzy poset and let $B \subset R$. An element is said to be an upper bound for a subset B iff $A(x, b) > 0$ for all $b \in B$. An upper bound x_0 for B is the least upper bound of B if and only if $A(x, x_0) > 0$ for every upper bound x for B. An element $y \in R$ is said to be a lower bound for a subset B if and only if $A(b, y) > 0$ for all $b \in B$. A lower bound y_0 for B is the greatest lower bound of B if and only if $A(y_0, y) > 0$ for every lower bound y for B. We denote the least upper bound of the set $\{a, b\}$ by $a \wedge b$ and denote the greatest lower bound of the set $\{a, b\}$ by $a \vee b$.

Definition 2.5. Let (R, A) be a dual fuzzy poset. (R, A) is a dual fuzzy lattice if $a \wedge b$ and $a \wedge b$ only if and exist for all $a, b \in R$.

Definition 2.6. Let (R, A) be a dual fuzzy lattice. (R, A) is a dual distributive if and only if $a \vee (b \wedge c) = (a \vee b) \vee (a \wedge c) = a \wedge (b \vee c)$.

3. Dual of Generalized Almost Distributive Fuzzy Lattices

The definitions of Dual Generalized Almost Distributive Fuzzy Lattices are given in this section. First let us give the definition of a Dual Almost Distributive Fuzzy Lattices

Definition 3.1. Let $(R, \wedge, \vee, 0)$ be an algebra of type $(2, 2, 0)$ and $L_1(R, A)$ be a dual fuzzy poset. Then we call $L_1(R, A)$ is a Dual Almost Distributive Fuzzy Lattice (Dual-ADFL) if the following axioms are satisfied.

1. $A(a \wedge 0, a) = A(a, a \wedge 0) = 1$

2. $A(0 \wedge a, 0) = A(0, 0 \wedge a) = 1$

3. $A((a \vee c) \wedge (b \vee c), (a \wedge b) \vee c)) = A((a \wedge b) \vee c, (a \vee c) \wedge (b \vee c)) = 1$

4. $A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = 1$

5. $A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = 1$

6. $A(b, (a \wedge b) \vee b) = A((a \wedge b) \vee b, b) = 1$ for all $a, b, c \in R$.

Definition 3.2. The dual fuzzy poset $L_1(R, A)$ is directed above if and only if the dual poset $(R, \geq)$ is directed above. Now we give the definition of a Dual Generalized Almost Distributive Fuzzy lattices as follows.

Definition 3.3. Let $(R, \wedge, \vee)$ be an algebra of type $(2, 2)$ and $L_1(R, A)$ be a dual fuzzy poset. Then we call $L_1(R, A)$ is a Dual Generalized Almost Distributive Fuzzy Lattice if it satisfies the following axioms.

1. $A(a \vee (b \vee c), (a \vee b) \vee c) = A((a \vee b) \vee c, a \vee (b \vee c)) = 1$

2. $A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = 1$

3. $A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = 1$

4. $A(a, a \vee (a \wedge b)) = A(a \vee (a \wedge b), a) = 1$

5. $A(a, (a \wedge b) \vee a) = A((a \wedge b) \vee a, a) = 1$

6. $A(b, (a \vee b) \vee b) = A((a \vee b) \wedge b, b) = 1$

Definition 3.4. Let $L_1(R, A)$ be a Dual-GADFL. Then for any $a, b \in R, a \geq b$ if and only if $A(b, a) > 0$.

Theorem 3.1. *Let $L_1(R, A)$ be a Dual-GADFL. Then for any $a, b \in R$, the following are equivalent.*

1. $A(a, a \vee (b \wedge a)) > 0$ *and* $A(a \vee (b \wedge a), a) > 0$

2. $A(a, (a \vee b) \wedge a) > 0$ *and* $A((a \vee b) \wedge a, a) > 0$

3. $A(b, b \vee (a \wedge b)) > 0$ *and* $A(b \vee (a \wedge b), b) > 0$

4. $A(b, (b \vee a) \wedge b) > 0$ *and* $A((b \vee a) \wedge b, b) > 0$

5. $A(b \vee a, a \vee b) > 0$ *and* $A(a \vee b, b \vee a) > 0$

6. $A(b \wedge a, a \wedge b) > 0$ *and* $A(a \wedge b, b \wedge a) > 0$

Proof of the theorem 3.1.

Let $L_1(R, A)$ be a Dual-GADFL and $a, b \in R$

$(1) \Leftrightarrow (2) A(a, a \vee (b \wedge a)) > 0$

$\Leftrightarrow A((a, (a \vee b) \wedge (a \vee a)) > 0 \Leftrightarrow A(a, (a \vee b) \wedge a) > 0$

And $A(a \vee (b \wedge a), a) > 0 \Leftrightarrow A((a \vee b) \wedge (a \vee a), a) > 0$

$\Leftrightarrow A((a \vee b) \wedge a, a) > 0.$

Hence (1) and (2) are equivalent.

By interchanging the rules of a and b we get the equivalence of (3) and (4)

$(2) \Leftrightarrow (5)$

Suppose (2), $A(b \vee a, a \vee b) = A(b \vee \{(a \vee b) \wedge a, a \vee b)$

$= A(\{b \vee (a \vee b)\} \wedge (b \vee a), a \vee b)) = A((a \vee b) \wedge (b \vee a), a \vee b)$

$= A((a \vee b) \wedge \{a \vee (b \vee a)\}, a \vee b) = A(a \vee (b \wedge (b \vee a)) a \vee b)$

$= A(a \vee b, a \vee b) = 1$

□ $\mathrm{A}(b \vee a, a \vee b) > 0$

Similarly, $\mathrm{A}(a \vee b, b \vee a) > 0$

Suppose (5), $\mathrm{A}(a, (a \vee b) \wedge a) = A(a, (b \vee a) \wedge a) = 1$

Hence $\mathrm{A}(a, (a \vee b) \wedge a) > 0$

Similarly, $\mathrm{A}((a \vee b) \wedge a, a) > 0$

Therefore (2) and (5) are equivalent. By interchanging the rules of a and b, we get $(4) \Leftrightarrow (5)$.

Thus (1) to (5) are equivalent.

We complete the proof by proving the equivalence of (6) and (1).

$(6) \Leftrightarrow (1)$ is direct.

Suppose (1),

$A(b \wedge a, a \wedge b) = A(b \wedge \{(a \wedge b) \vee a\}, a \wedge b)$

$= A((a \wedge b) \vee (b \wedge a), a \wedge b) = A((a \wedge b) \vee \{a \wedge (b \wedge a)\}, a \wedge b)$

$= A((a \wedge \{b \vee (b \wedge a)\}, a \wedge b) = A(a \wedge b, a \wedge b) = 1.$

Hence $A(b \wedge a, a \wedge b) > 0.$ Similarly, $A(a \wedge b, b \wedge a) > 0.$ Hence the proof

Definition 3.6. Let $L_1(R, A)$ be a Dual-GADFL. For any $a, b \in R,$ we say that a is greater than or equal to b and we write $a \geq b$ if $A(a, a \vee b) > 0$ and $(a \vee b, a) > 0$ (or) $A(b, a \wedge b) > 0$ and $(a \wedge b, b) > 0.$

Finally, we conclude this section with the following theorem which gives the equivalent conditions for a Dual-GADFL to become a Dual-ADFL.

Theorem 3.2. *Let* $L_1(R, A)$ *be a Dual-GADFL. Then the following are equivalent.*

1. $L_1(R, A)$ *is a Dual Almost Distributive Fuzzy Lattice.*

2. $A((a \vee c) \wedge (b \vee c), (a \wedge b) \vee c) > 0$ *and* $A((a \wedge b) \vee c, (a \vee c) \wedge (b \vee c)) > 0$

3. $A(b, (a \wedge b) \vee b) > 0$ *and* $A((a \vee b) \vee b, b) > 0$

4. $A((b \wedge a) \vee c, (a \wedge b) \vee c) > 0$ *and* $A((a \wedge b) \vee c, (b \wedge a) \vee c) > 0$ *for all* $a, b, c \in R.$

Proof of the theorem 3.2.

Let $L_1(R, A)$ be a Dual-GADFL and $a, b, c \in R,$ then $(1) \Leftrightarrow (2).$

Suppose $L_1(R, A)$ is a Dual Almost Distributive Fuzzy Lattice. Then we have,

$A((a \vee c) \wedge (b \vee c), (a \wedge b) \vee c) = A((a \wedge b) \vee c, (a \vee c) \wedge (b \vee c)) = 1$

$\Rightarrow A((a \vee c) \wedge (b \vee c), (a \wedge b) \vee c) > 0$ and $A((a \wedge b) \vee c, (a \wedge c), (b \vee c)) > 0$

Hence (2) holds true.

$(2) \Rightarrow (3)$ Suppose (2)

$A(b, (a \wedge b) \vee b) = A(b, (a \vee b) \wedge (b \vee b))$

$= A(b, (a \vee b) \wedge b) = 1$

Hence $A(b, (a \wedge b) \vee b, b) > 0$

Similarly, $A((a \wedge b) \vee b, b) > 0$

$(3) \Rightarrow (4)$

Since $((a \wedge b) \vee c) \wedge c = c$ and $((b \wedge a) \vee c) \wedge c = c$

We have $c \geq (a \wedge b) \vee c$ and $c \geq (b \wedge a) \vee c$

$\Rightarrow A(c, (a \wedge b) \vee c) > 0$ and $A(c, (b \wedge a) \vee c) > 0$

We have $A([(b \wedge a) \vee c] \vee [(a \wedge b) \vee c], [(a \wedge b) \vee c] \vee [(b \wedge a) \vee c]) > 0$ and $A([(a \wedge b) \vee c] \vee [(b \wedge a) \vee c], [(b \wedge a) \vee c] \vee [(a \wedge b) \vee c]) > 0$

$\Rightarrow A((b \wedge a) \vee [c \vee ((a \wedge b) \vee c)], (a \wedge b) \vee [c \vee ((b \wedge a) \vee c)]) > 0$ and

$A((a \wedge b) \vee [c \vee ((b \wedge a) \vee c)], (b \wedge a) \vee [c \vee ((a \wedge b) \vee c)]) > 0$

$\Rightarrow A((b \wedge a) \vee (a \wedge b) \vee c, (a \wedge b) \vee (b \wedge a) \vee c) > 0$ and

$A((a \wedge b) \vee (b \wedge a) \vee c, (b \wedge a) \vee (a \wedge b) \vee c) > 0$

$\Rightarrow A(([(b \wedge a) \vee a] \wedge [(b \wedge a) \vee b)] \vee c, ([(a \wedge b) \vee b] \wedge [(a \wedge b) \vee a]) \vee c] > 0$

$A((a \wedge b) \vee b] \wedge [(a \wedge b) \vee a]) \vee c, ([(b \wedge a) \vee a] \wedge [(b \wedge a) \vee b]) \vee c] > 0$

$\Rightarrow A((a \wedge b) \vee c, (b \wedge a) \vee c) > 0$ and $A((b \wedge a) \vee c, (a \wedge b) \vee c) > 0.$

Hence (4) holds true.

$(4) \Rightarrow (1)$: Suppose (4)

$A((a \vee c) \wedge (b \vee c), (a \wedge b) \vee c) = A([(a \vee c) \wedge b] \vee [(a \vee c) \wedge c], (a \wedge b) \vee c)$

$= A([(a \vee c) \wedge b] \vee c, (a \wedge b) \vee c) = A([b \wedge (a \vee c)] \vee c, (a \wedge b) \wedge c)$

$= A([(b \wedge a) \vee (b \vee c)] \vee c, (a \wedge b) \wedge c)$

$= A(([(b \wedge a) \vee b] \wedge [(b \vee a) \vee c]) \vee c, (a \wedge b) \wedge c)$

$= A([(b \wedge a) \vee c] \wedge [(b \wedge a) \vee b]) \vee c, (a \wedge b) \vee c)$

$= A([(b \wedge a) \vee (c \wedge b)] \vee c, (a \wedge b) \vee c) = A((b \wedge a) \vee [(c \wedge b)] \vee c], (a \wedge b) \vee c)$

$= A((b \wedge a) \vee c, (a \wedge b) \vee c) = A((a \wedge b) \vee c, (a \wedge b) \vee c) = 1$

Hence $A((a \vee c) \wedge (b \vee c), (a \wedge b) \vee c) = 1$

Similarly, $A((a \wedge b) \vee c, (a \vee c) \wedge (b \vee c)) = 1$

Therefore $A((a \vee c) \wedge (b \vee c), \wedge (a \wedge b) \vee c) = 1$

Also, $A(b, (a \wedge b) \vee b) = A((a \wedge b) \wedge b, b) = 1$

Hence $L_1(R, A)$ is a Dual Almost Distributive Fuzzy Lattice.

References

[1] G. Rao, R. Bandaru and N. Rafi, Generalized almost distributive lattices-I, Southeast Asian Bulletin of Mathematics 33(6) (2009).

[2] U. Swamy and G. Rao, Almost distributive lattices, Journal of the Australian Mathematical Society (Series A) 31(1) (1981), 77-91.

[3] L. Zadeh, Fuzzy sets, Information and Control 8(3) (1965), 338-353.

[4] Lotfi A. Zadeh, Similarity relations and fuzzy orderings, Information Sciences 3(2) (1971), 177-200.

[5] N. Ajmal and K. Thomas, Fuzzy lattices, Information Sciences 79(3-4) (1994), 271-291.

[6] I. Chon, Fuzzy partial order relations and fuzzy lattices, Korean J. Math. 17(4) (2009), 361-374.

[7] S. G. Karpagavalli and T. Sangeetha, Normal ideals in generalized almost distributive fuzzy lattices, AIP Conference Proceedings 2282, 020003 (2020).