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A NOTE ON CONNECTED INTEGRITY IN GRAPHS

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Abstract

Integrity is an effective measure of vulnerability of networks and is defined as $I(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\}$ where $m(G - S)$ denotes the order of the largest component of $G - S$. In this paper, we initiate the study of a new parameter of measuring the vulnerability named Connected Integrity. The Connected Integrity of a graph $G(V, E)$ denoted by $CI(G)$, is defined as the minimum value of the sum, $|S| + m(G - S)$, where minimum runs over connected subset S of V and $m(G - S)$ denotes the order of the largest component of $G - S$. We prove some characterization theorems here and establish some bounds for this parameter.

1. Introduction

Throughout this paper, we consider only finite, simple graphs. Basic notations and terminology not mentioned here can be referred from [13]. A graph $G(V, E)$ is connected if every two vertices of G are at a finite distance. The minimum and maximum degree of vertices in a graph are denoted by δ and Δ respectively. For any vertex v in a graph G, the open neighbourhood $N(v)$ is defined to be the set of all vertices adjacent to v in G and the closed

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neighbourhood $N[v]$ is defined to be $N(v) \cup \{v\}$. If v is a vertex of G, its eccentricity $e(v)$ is defined by $e(v) = \max\{d(u, v)/u \in V(G)\}$. The radius of G, $rad(G)$, is the minimum eccentricity of G. Center of G, $cen(G)$, is the set of all central vertices of G. The vertex connectivity of a graph is the minimum number of vertices whose removal disconnects the graph or results in a trivial graph which we denote by κ. $\langle S \rangle$ denotes the induced sub graph which is nothing but the graph $(V \cap S, E \cap (S \times S))$. The number of components of a graph G is denoted by $\omega(G)$. For a nontrivial connected graph G, the connected cut set connectivity $c\kappa(G)$ is defined to be the minimum cardinality of a vertex cut set S of G for which $\langle S \rangle$ is connected. If G is trivial or disconnected, then $c\kappa(G)$ is defined to be 0. Then $\kappa(G) \leq c\kappa(G)$ for every graph G. Bistar $B_{n,n}$ is the graph obtained by joining the centre (apex) vertices of two copies of $K_{1,n}$. A chord is an edge which joins two non adjacent vertices in a hamilton cycle of a graph. The n-cube graph, denoted by Q_n is the graph whose vertex set is the set of ordered n-tuples of $0s$ and $1s$ and where two vertices are adjacent if their ordered n-tuples differ in exactly one coordinate. The join of two graphs G and H, denoted by $G \vee H$ is formed from disjoint copies of G and H by connecting every vertex of G to every vertex of H.

Integrity is a fascinating topic in research relevant to Graph Theory. In this digital era, the versatility of graphs make them essential in network designing and analysis. At the same time, the study on the vulnerability of a communication network gains much of research attention, as the destruction of few nodes results in no members being able to communicate with others, thereby affecting the whole system in this technical world.

To overcome such risk of designing weak vulnerable networks, the concept of Integrity was introduced by Barefoot [1]. For any graph G, the Integrity is defined as $I(G) = \min_{S \subset V(G)} \{| S | + m(G - S)\}$ where $m(G - S)$ denotes the order of the largest component of $G - S$.

Wayne Goddard and Henda C. Swart laid the ground work on Integrity as a specific graphical parameter [2]. Also some other measures of vulnerability such as Domination Integrity [8], Hub Integrity [9], Edge Integrity [10], Vertex Neighbor Integrity [11], Neighbor Toughness [12] are available in literature.

For further reference on measures of vulnerability of graphs we suggest reading on [3, 4, 5].

Connectedness is a major issue in spreading of contagious diseases. Even in situations like COVID-19, which involves the attack of a network on the basis of its connectivity, the need of analyzing the connected integrity of the network arises. In defense, the concept of connectedness plays a vital role in demolishing terrorism. Breaking the communication chain by assassinating the leader and those who are continually in close proximity to him will be successful in breaking the network. During the war between Russia and Ukraine, the Russian Government initially tried to interrupt the internet connectivity and telecast of TV news in Ukraine, so that they can segregate the public and military officials. In this paper, we initiate the study on new parameter of measuring vulnerability of networks named Connected Integrity in graphs.

The Connected Integrity of a graph G denoted by $CI(G)$ is defined by $CI(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\}$ where $\langle S \rangle$ is connected and $m(G - S)$ denotes the order of the largest component of $G - S$.

We also aim to find out the properties of Connected Integrity and the interrelations among Connected Integrity and other graphical parameters. In addition we prove some realization theorems and characterization theorems for Connected Integrity.

2. Main Results

The following facts and propositions from [6] can be easily verified.

Fact 1. $2 \leq CI(G) \leq n$, for any nontrivial connected graph G.

Fact 2. For any complete graph K_n, $(n \geq 2)$, $CI(K_n) = n$.

Fact 3. The Connected Integrity of complete bipartite graph, $CI(K_{m,n}) = \min\{m, n\} + 2$ where $m, n \geq 2$.

Fact 4. For any cycle C_n, $n \geq 3$, $CI(C_n) = n$.

Fact 5. For $n \geq 2$, $CI(K_{1,n}) = 2$.

Fact 6. Connected Integrity of Petersen Graph is 8.

Fact 7. Connected Integrity of Q_3 is 7.

Proposition 2.1. *Let T be a tree. Then* $CI(T) = n$ *iff* $T \cong P_2$.

Proposition 2.2. *A connected graph G has* $I(G) = 2$ *iff* $CI(G) = 2$.

Proof. *It is obvious that* $I(G) = 2 \Leftrightarrow |S| = 1$ and $m(G - S) = 1 \Leftrightarrow G$ has a cut vertex and $G - S$ is a null graph $\Leftrightarrow G$ is a Star $\Leftrightarrow CI(G) = 2$. ■

Theorem 2.3. *The Connected Integrity of path* P_n *is* $CI(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Proof. *Let* $P_n : v_1 v_2 \ldots v_n$. Take $S = \{v_{\lceil \frac{n}{2} \rceil}\}$. Then $|S| + m(P_n - S) = 1 + \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n+1}{2} \right\rceil$. This forces $CI(P_n) \leq \left\lceil \frac{n+1}{2} \right\rceil$.

For any other set S', $|S'| + m(P_n - S') > \left\lceil \frac{n+1}{2} \right\rceil$ provided n is not even and $S' \neq \{v_{\frac{n}{2}}, v_{\frac{n}{2}+1}\}$. In the later case also, $CI(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$. Hence we conclude that $CI(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$. ■

Theorem 2.4. $CI(W_{1,n}) = \lceil 2\sqrt{n} \rceil$, *for any wheel graph* $W_{1,n}$.

Proof. Suppose $n = xy$. If we choose $|S| = x + 1$ which is formed by choosing 'x' almost equidistant vertices in the rim and the single central vertex of the wheel, then we get $m(W_{1,n} - S)$ to be $y - 1$. We have to minimize $CI = x + 1 + y - 1$.

Let $g(x) = x + y = x + \frac{n}{x} \Rightarrow g'(x) = 1 - \frac{n}{x^2}$.

$$g'(x) = 0 \Rightarrow 1 - \frac{n}{x^2} = 0 \Rightarrow x^2 = n \Rightarrow x = \sqrt{n}.$$

Therefore, minimum value of $CI = \sqrt{n} + \frac{n}{\sqrt{n}} - 1 + 1 = 2\sqrt{n}$. Hence $CI(W_{1,n}) = \lceil 2\sqrt{n} \rceil$.

Theorem 2.5. *If G is a cycle with a chord, then $CI(G) = n - g(G) + 2$ where $g(G)$ denotes the girth of G.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $v_i v_j$ be the chord of G. Let $S = \{v_i, v_j\}$. Then $m(G - S) = n - g(G)$. $CI(G) \le |S| + m(G - S) = 2 + n - g(G)$. Therefore, $CI(G) \le n - g(G) + 2$. It is clear that no other subset S' *of* $V(G)$ will be the CI set of G. Hence $CI(G) = n - g(G) + 2$. ∎

For Example, let G be the graph C_8 with a chord whose girth is 4 as given in Figure 1

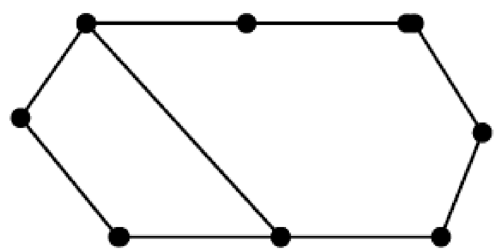

Figure 1

Then $CI(G) = 6$.

Theorem 2.6. *For any graph G, $CI(G) \ge \delta(G) + 1 \ge \kappa(G) + 1$. The bound is sharp for $K_{1,n}$.*

Proof. Let S be the *CI-set* of G. Then $m(G - S) \ge \delta(G - S) + 1 \ge \delta(G) - |S| + 1$. $CI(G) = |S| + m(G - S) \ge \delta(G) + 1 \ge \kappa(G) + 1$. ∎

An I-set need not be connected. But a CI-set is necessarily an I-set which shows that $I(G) \le CI(G)$ for any graph G. Also for a graph with a full vertex, $I(G) = CI(G)$.

Theorem 2.7. *For any given* $k \geq 4$, *there exists a graph G of order k and maximum degree less than* $k-1$ *such that* $CI(G) = I(G)$.

Proof. Let $k \geq 4$ be given. Consider the Bistar $B_{m, k-m}$ for any integer $m, 1 \leq m \leq k-1$. We claim that Bistar is the required graph G. We know $I(B_{m, k-m}) = 3$. Now the central vertices form a Connected Integrity set of Bistar forcing $CI(B_{m, k-m}) \leq 3$. Also any other set S', if it contains a single cut vertex has $m(G - S') \geq 2$, implying that $CI(B_{m, k-m}) \geq 3$. Hence we conclude that $B_{m, k-m}$ is a graph of order k with $CI(G) = I(G)$. ∎

Theorem 2.8. *Given any integer* $k \geq 3$, *there do exist a* 2*-connected graph G with* $I(G) = k+1$ *and* $CI(G) = I(G) + 2$.

Proof. Consider two copies of K_k. Let $\{v_1, v_2, \ldots, v_k\}$ be vertices of first copy of K_k and $\{w_1, w_2, \ldots, w_k\}$ be those of second copy. Construct a new graph G from two copies of K_k with vertices $\{v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k\}$ and deleting the edges v_1v_k and w_1w_k and adding the edges v_1w_1 and v_kw_k. Now G is 2-connected of order $2k$, particularly with $\kappa = 2$. Now $S = \{v_1, v_i, v_k\}$ for some i, forms a Connected Integrity set forcing $CI(G) \leq k+3$. Also $S_1 = \{v_1, w_k\}$ is an Integrity Set of G and thereby $I(G) \leq k-1+2 = k+1$. It is easy to verify that no other Connected Integrity Set S' exists with $|S'| < |S|$ and hence $CI(G) = k+3$. Similarly we can check that $I(G) = k+1$. We find that $CI(G) - I(G) = 2$. Hence the graph G is constructed such that it meets all requirements. ∎

For example, when $k = 5$, the constructed graph G is given in Figure 2

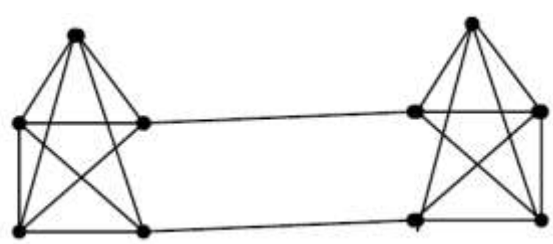

Figure 2

Theorem 2.9. *There do exist a graph G of given order $k \geq 6$ with $CI(G) = I(G) + 1$.*

Proof. Suppose $k \geq 6$ be the given integer. Let G be the graph derived from $B_{m, k-1-m} (1 \leq m \leq k-2)$ by subdividing, once, the edge joining the central vertices of $B_{m, k-1-m}$. Clearly G is of order k.

Let v be the central vertex of G. $N(v)$ is the minimum Integrity set of G which thereby implies $I(G) = 3$. Now $N[v]$ is a Connected Integrity set of G which forces $CI(G) \leq 4$. Since the value of k is at least 6, $S = \{v\}$ also results in $CI(G) \geq 4$. Hence we can easily conclude that $CI(G) = 4$. Now G proves the existence of a graph with order k and $CI(G) = I(G) + 1$. ∎

Note that the above theorem proves the existence of family of graphs for which $CI(G) - I(G) = 1$ but with fixed integrity values. ($I(G) = 3$ and $CI(G) = 4$). The following theorem proves even more by constructing a graph with given integrity value and $CI(G) - I(G) = 1$.

Theorem 2.10. *For any given $k \geq 4$, there do exist a graph G with $I(G) = m$ and $CI(G) = m + 1$.*

Proof. Construct a graph G with $V(G) = \{u_1, u_2, \ldots, u_{m-2}, w_1, w_2, \ldots, w_{m-2}, u'_1, u'_2, \ldots, u'_{m-2}, w'_1, w'_2, \ldots, w'_{m-2}\}$ of order $4m - 8$ and $E(G) = \{w_i w_j, u_i u_j, w'_i w'_j, u'_i u'_j 1 \leq i, j \leq m-2, i \neq j\} \cup \{w_1 w'_1, w'_1 u_1, u_1 u'_1, u'_1 w_1\}$. Now $\{w_1, u_1\}$ forms the minimum integrity set of G. Hence $I(G) = m$.

Also, $\{w_1, u_1, w'_1\}$ is the minimum Connected Integrity set of G. Therefore, $CI(G) = m + 1$ which proves the theorem. ∎

For Example, when $m = 7$, the constructed graph G is given in Figure 3

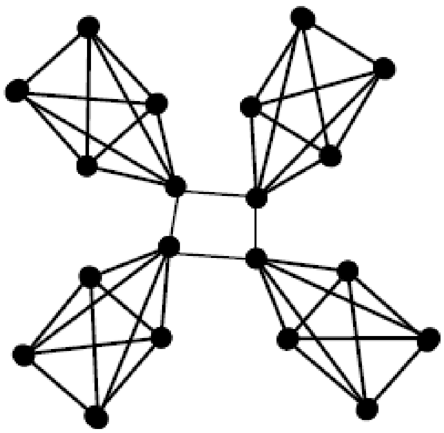

Figure 3

Center fusion of graphs with isomorphic centers

Let G_1 and G_2 be any two connected graphs with $cen(G_1) \cong cen(G_2)$. Then center fusion of G_1 and G_2, denoted by $G_1 \Theta G_2$ is defined to be a graph with vertex set $V = V_1 \cup V_2 \setminus (cen(G_1))$. If f is an isomorphism from $cen(G_1)$ to $cen(G_2)$ and $v_i \in cen(G_1)$, take $w_i = f(v_i)$, then edge set of $G_1 \Theta G_2$ is $E = E_1 \cup E_2 \cup \{ww_i / wv_i \in E_1 \text{ and } w \in V_1 \setminus V(cen(G_1))\}$.

For example, $K_{1,6} \Theta W_6$ is given in Figure 4

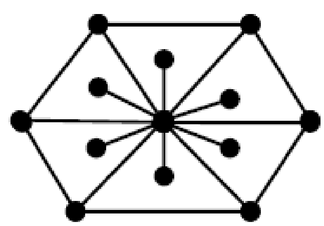

Figure 4

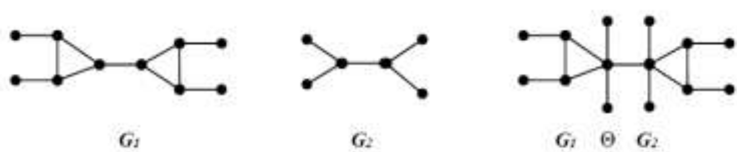

Figure 5

Friendship graph F_m is constructed by joining m copies of the cycle C_3 with a common vertex. That is a Friendship graph is nothing but $K_1 \vee mK_2$. The graph $K_2 \vee mK_1$ with $n \geq 2$ is denoted as Cr_n.

Theorem 2.11. *Let G be any connected graph. Then* $CI(G) = 3$ *if and only if G is isomorphic to any one of the following*: (i) $S_1(K_{1,n})$ (ii) F_m (iii) $F_m \Theta S_1(K_{1,n})$ (iv) $F_m \Theta K_{1,n}$ (v) $F_m \Theta S_1(K_{1,n}) \Theta K_{1,r}$ (vi) $S_1(K_{1,n}) \Theta K_{1,r}$ (vii) Cr_n (viii) *Bistar* (ix) Cr_n *with some pendant vertices at both or one of its central vertices*.

Proof. Let G be any graph with $CI(G) = 3$. Then G contains a connected cut set say S of cardinality 1 or 2.

Case 1. $|S| = 1$. Then we have $m(G - S) = 2$. This forces each component of $G - S$ to be K_1 or K_2, with the number of components isomorphic to K_2 being at least 1.

Subcase (i). All components of $G - S$ are isomorphic to K_2.

I. If both vertices of K_2 in each component are adjacent to the vertex in S, then F_m results.

II. If exactly one vertex of K_2 all components are adjacent to the vertex in S, then $S_1(K_{1,n})$ results.

III. If a combination of above two situations occurs, then $G \cong F_m \Theta S_1(K_{1,n})$.

Subcase (ii) Some components of $G - S$ are isomorphic to K_2.

Condition I along with some K_1 components produces $F_m \Theta K_{1,n}$.

Condition II along with some K_1 components results in $S_1(K_{1,n}) \Theta K_{1,r}$.

Condition III along with some K_1 components results in $F_m \Theta S_1(K_{1,n}) \Theta K_{1,r}$.

Case 2. $|S| = 2.$

Then we have $G - S \cong nK_1$. The various combinations of adjacency pattern of these 'n' components with S other than the possibility of being a star $K_{1,n}$ forms a graph isomorphic to Cr_n or Bistar or Cr_n with some pendant vertices at one or both of its central vertices. This concludes the theorem. ■

Connected Integrity for Trees

Theorem 2.12. *For any tree T of order* n, $CI(T) \leq n - rad(T) + 1$.

Proof. Let T be any tree and let $S = cen(T)$. Then S is nonempty with $|S| \leq 2$. For any S, $\omega(T - S) \geq 2$ with each component of $T - S$ having at least $rad(T) - 1$ vertices. Hence the largest component has at most $n - |S| - rad(T) + 1$ vertices. Now $CI(T) = \min\{|S| + m(T - S)\} \leq |S| + n - |S| - rad(T) + 1 \leq n - rad(T) + 1$.

Also, we know that for any graph G, $rad(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$. Therefore, $CI(T) \leq n - rad(T) + 1 = n - \left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lceil \frac{n}{2} \right\rceil + 1$. Thus we have the following corollary.

Corollary 2.13. *For any tree,* $CI(T) \leq \left\lceil \frac{n}{2} \right\rceil + 1$. *The inequality is sharp and it is attained by even paths.*

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