Advances and Applications in Mathematical Sciences
Volume 21, Issue 12, October 2022, Pages 6695-6702

FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN FUZZY METRIC SPACES

P. SUTHA MERLIN, G. UTHAYA SANKAR and P. NAMASIVAYAM

[1]Research Scholar, [3]Associate Professor
The M. D. T. Hindu College
Tirunelveli, Tamil Nadu, India – 627010
Affiliated to Manonmaniam Sundaranar
University, Abishekapatti, Tirunelveli, India-627 – 012
E-mail: sudhamerlin.p@gmail.com
vasuhe2010@gmail.com

[2]Assistant Professor
Department of Mathematics
Manonmaniam Sundaranar University College
Naduvakurichi, Sankarankoil, India – 627 862
E-mail: uthayaganapathy@yahoo.com

Abstract

In this study, we show some fixed point theorems for multivalued mappings in complete fuzzy metric spaces.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [19] in 1965. Kramosil and Michalek [11] introduced the concept of fuzzy metric space and modified by George and Veeramani [4]. Kubiaczyk and Sushil Sharma [12] introduced the notion of multivalued mappings for fuzzy metric space in the sense of Kramosil and Michalek [11] and extended the result of Grabiec [5]. Romaguera [17] introduced Hausdorff fuzzy metric on a set of nonempty closed and bounded subsets of a given fuzzy metric space. Kiany [10] et al. proved fixed point theorems for multivalued fuzzy contraction maps in fuzzy

2020 Mathematics Subject Classification: 54H25, 47H10.

Keywords: Fuzzy metric space, Fixed point.

Received February 3, 2022; Accepted March 2, 2022

metric spaces and obtained generalization of Banach contraction theorem in fuzzy metric spaces. In this study, we show some fixed point theorems for multivalued mappings in complete fuzzy metric spaces.

2. Preliminaries

Definition 2.1. A 3-tuple $(X, M, *)$ is said to be fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for each $u, v, w \in X$ and $r, s > 0$

(1) $M(u, v, s) > 0$

(2) $M(u, v, s) = 1$ for all $s > 0 \leftrightarrow u = v$

(3) $M(u, v, s) = M(v, u, s)$

(4) $M(u, v, s) * M(v, w, r) \leq M(u, w, s + r)$

(5) $M(u, v, .) : (0, \infty) \to [0, 1]$ is continuous,

(6) $\lim_{t \to \infty} M(u, v, s) = 1$.

Example 2.2. Let (X, d) be a metric space. Define $a * b = ab$ (or $a * b = \min\{a, b\}$) and for all $u, v \in X$ and $s > 0$, $M(u, v, s) = \frac{s}{s + d(u, v)}$.

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric space M induced by the metric d as the standard fuzzy metric.

Definition 2.3. Let $(X, M, *)$ be a fuzzy metric space.

A sequence $\{u_n\}$ in X is said to be Cauchy sequence if $\lim_{n \to \infty} M(u_{n+p}, u_n, s) = 1$ for all $s > 0$ and $p > 0$.

A sequence $\{u_n\}$ in X converges to u in X if $\lim_{n \to \infty} M(u_n, u, s) = 1$ for all $s > 0$.

A fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent to a point in it.

Lemma 2.4 [18]. *Let* $\{u_n\}$ *be a sequence in a fuzzy metric space* $(X, M, *)$ *with the condition* (6). *If there exists a number* $q \in (0, 1)$ *such that* $M(u_{n+2}, u_{n+1}, qs) \geq M(u_{n+1}, u_n, s)$ *for all* $s > 0$ *and* $n = 1, 2, 3, \ldots,$ *then* $\{u_n\}$ *is a Cauchy sequence in X.*

Lemma 2.5 [18]. *Let* $(X, M, *)$ *be a fuzzy metric space. If there exists a number* $q \in (0, 1)$ *such that* $M(u, v, qs) \geq M(u, v, s)$ *for all* $u, v \in X$ *and* $s > 0,$ *then* $u = v.$

Definition 2.6. Let $(X, M, *)$ be a fuzzy metric space and $CB(X)$ is the collection of all nonempty closed and bounded subsets of X. Define a function M_∇ on $CB(X) \times CB(X) \times (0, \infty)$ by $M_\nabla(A, B, s) = \min\{\inf_{a \in A} M^\nabla(a, B, s), \inf_{b \in B} M^\nabla(A, b, s)\}$ for all $A, B \in CB(X)$ and $s > 0,$ where $M^\nabla(a, B, s) = \sup\{M(a, b, s) : b \in B\}.$ Clearly M_∇ is metric on $CB(X).$

Remark 2.7. Obviously $M_\nabla(A, B, s) \leq M^\nabla(x, B, s)$ whenever $x \in A.$

3. Main Results

Theorem 3.1. *Take a complete fuzzy metric space* $(X, M, *)$ *with continuous t-norm* $*$ *defined by* $a * b = \min\{a, b\}$ *and* $T : X \to CB(X)$ *is a multivalued mapping satisfying the following condition for all* $x, y \in X$ *and* $t > 0.$

$$M_\nabla(Tx, Ty, qt) \geq \min\{M(x, y, t), M^\nabla(x, Tx, t) * M^\nabla(y, Ty, t), M^\nabla(y, Tx, t)\} \tag{3.1.1}$$

where $q \in (0, 1).$ Then T has a unique fixed point.

Proof of theorem 3.1. Let $x_0 \in X$

Choose $x_1 \in X$ such that $x_1 \in Tx_0$

Continuing this process we get $x_n \in Tx_{n-1}$

$M(x_n, x_{n+1}, qt) \geq M_{\nabla}(Tx_{n-1}, Tx_n, qt)$

$\geq \min\{M(x_{n-1}, x_n, t), M^{\nabla}(x_{n-1}, Tx_{n-1}, t) * M^{\nabla}(x_n, Tx_n, t),$

$M^{\nabla}(x_n, Tx_{n-1}, t)\}$

$\geq \min\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t), M(x_n, x_n, t)\}$

$= \min\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t), 1\}$

$= M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t)$

Therefore, $M(x_n, x_{n+1}, qt) \geq M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t)$

Hence $M(x_n, x_{n+1}, qt) \geq M(x_{n-1}, x_n, t)$, for each $t > 0$

Therefore by Lemma 2.4, $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete, sequence $\{x_n\}$ converges to a point $x \in X$

Now,

$M^{\nabla}(x_{n+1}, Tx, qt) \geq M_{\nabla}(Tx_n, Tx, qt)$

$\geq \min\{M(x_n, x, t), M^{\nabla}(x, Tx_n, t) * M^{\nabla}(x, Tx, t), M^{\nabla}(x, Tx_n, t)\}$

$\geq \min\{M(x_n, x, t), M(x_n, x_{n+1}, t) * M^{\nabla}(x, Tx, t), M(x, x_{n+1}, t)$

Taking limit as $n \to \infty$ we get

$M^{\nabla}(x, Tx, qt) \geq \min\{M(x, x, t), M(x, x, t) * M^{\nabla}(x, Tx, t), M(x, x, t)$

$= \min\{1, 1 * M^{\nabla}(x, Tx, t), 1\}$

$= \min\{1, M^{\nabla}(x, Tx, t)\}$

$= M^{\nabla}(x, Tx, t)$

Hence $M^{\nabla}(x, Tx, qt) \geq M^{\nabla}(x, Tx, t)$, for each $t > 0$

Therefore by Lemma 2.5, $x \in Tx$

Thus x is a fixed point of T.

Uniqueness: Let y be another fixed point of T. Then

$M(x, y, qt) \geq M_{\nabla}(Tx, Ty, qt)$

$\geq \min\{M(x, y, t), M^{\nabla}(x, Tx, t) * M^{\nabla}(y, Ty, t), M^{\nabla}(y, Tx, t)\}$

$\geq \min\{M(x, y, t), M(x, x, t) * M(y, y, t), M(y, x, t)\}$

$= \min\{M(x, y, t), 1 * 1, M(x, y, t)\}$

$= \min\{M(x, y, t), 1\}$

$= M(x, y, t).$

Hence $M(x, y, qt) \geq M(x, y, t),$ for each $t > 0$

Therefore by Lemma 2.5, $y = x$

Thus T has a unique fixed point.

Theorem 3.2. *Take a complete fuzzy metric space* $(X, M, *)$ *and* $T : X \to CB(X)$ *a multivalued mapping satisfying the following condition for all* $x, y \in X$ *and* $t > 0$

$$M_{\nabla}(Tx, Ty, qt) \geq \min\{M(x, y, t), M^{\nabla}(x, Tx, t), M^{\nabla}(y, Ty, t), M^{\nabla}(y, Tx, t)$$

$$\frac{M^{\nabla}(x, Tx, t) + M^{\nabla}(y, Tx, t)}{1 + M(x, y, t)}\} \tag{3.2.1}$$

where $q \in (0, 1)$. *Then T has a unique fixed point.*

Proof of theorem 3.2. Let $x_0 \in X$

Choose $x_1 \in X$ such that $x_1 \in Tx_0$

Continuing this process we get $x_n \in Tx_{n-1}$

$M(x_n, x_{n+1}, qt) \geq M_{\nabla}(Tx_{n-1}, Tx_n, qt)$

$\geq \min\{M(M(x_{n-1}, x_n, t), M^{\nabla}(x_{n-1}, Tx_n, t), M^{\nabla}(x_n, Tx_n, t),$

$$M^{\nabla}(x_n, Tx_{n-1}, t)\frac{M^{\nabla}(x_{n-1,}, Tx_{n-1}, t) + M^{\nabla}(x_n, Tx_{n-1}, t)}{1 + M(x_{n-1}, x_n, t)}\}$$

$$\geq \min\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t)\ M(x_n, x_{n-1}, t), M(x_n, x_n, t),$$

$$\frac{M(x_{n-1}, x_n, t) + M(x_n, x_n, t)}{1 + M(x_{n-1}, x_n, t)}\}$$

$$= \min\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_{n-1}, x_{n-1}, t), 1, 1$$

$$= M(x_{n-1}, x_n, t)$$

Hence $M(x_n, x_{n+1}, qt) \geq M(x_{n-1}, x_n, t)$, for each $t > 0$

Therefore by Lemma 2.4, $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete, sequence $\{x_n\}$ converges to a point $x \in X$

Now, $M^{\nabla}(x_{n+1}, Tx, qt) \geq M_{\nabla}(Tx_n, Tx, qt)$

$$\geq \min\{M(x_n, x, t), M^{\nabla}(x_n, Tx, t), M^{\nabla}(x, Tx, t), M^{\nabla}(x, Tx_n, t),$$

$$\frac{M^{\nabla}(x_n, Tx, t) + M^{\nabla}(x, Tx, t)}{1 + M(x_n, x, t)}\}$$

$$\geq \min\{M(x_n, x, t), M(x_n, x_{n+1}, t), M^{\nabla}(x, Tx, t), M(x, x_{n+1}, t)$$

$$\frac{M(x_n, x_{n+1}, t) + M(x, x_{n+1}, t)}{1 + M(x_n, x, t)}\}$$

Taking limit as $n \to \infty$ we get

$$M^{\nabla}(x, Tx, qt) \geq \min\{M(x, x, t), M(x, x, t), M^{\nabla}(x, Tx, t), M(x, x, t),$$

$$\frac{M(x, x, t) + M(x, x, t)}{1 + M(x, x, t)}\}$$

$$M^{\nabla}(x, Tx, qt) \geq \min\{1, M^{\nabla}(x, Tx, t)\}$$

$$= M^{\nabla}(x, Tx, t)$$

Hence $M^{\nabla}(x, Tx, qt) \geq M^{\nabla}(x, Tx, t)$, for each $t > 0$. Therefore by Lemma 2.5, $x \in Tx$. Thus x is a fixed point of T.

Uniqueness:

Let Y be another fixed point of T. Then

$$M(x, y, qt) \geq M_{\nabla}(Tx, Ty, qt)$$

$$\geq \min\{M(x, y, t), M^{\nabla}(x, Tx, t), M^{\nabla}(y, Ty, t), M^{\nabla}(y, Tx, t),$$

$$\frac{M^{\nabla}(x, Tx, t) + M^{\nabla}(y, Tx, t)}{1 + M(x, y, t)}\}$$

$$\geq \min\{M(x, y, t), M(x, x, t), M(y, y, t), M(y, x, t),$$

$$\frac{M(x, x, t) + M(y, x, t)}{1 + M(x, y, t)}\}$$

$$= \min\{M(x, y, t), 1\}$$

$$= M(x, y, t)$$

Hence $M(x, y, qt) \geq M(x, y, t)$, for each $t > 0$. Therefore by Lemma 2.5, $y = x$. Thus T has a unique fixed point.

Theorem 3.3. *Take a complete fuzzy metric space* $(X, M, *)$ *and* $T : X \to CB(X)$ *a multivalued mapping satisfying the following condition for all* $x, y \in X$ *and* $t > 0$

$$M_{\nabla}(Tx, Ty, qt) \geq \min\{M(x, y, t), M^{\nabla}(x, Tx, t), M^{\nabla}(y, Ty, t), M^{\nabla}(y, Tx, t),$$

$$\frac{M^{\nabla}(y, Tx, t)[1 + M^{\nabla}(x, Tx, t)]}{M(x, y, t)}\} \tag{3.3.1}$$

where $q \in (0, 1)$. *Then T has a unique fixed point.*

Proof of theorem 3.3. Proof is similar.

References

[1] A. Deb Ray and P. K Saha, Fixed point theorems on Generalized Fuzzy Metric spaces, Hacettepe Journal of Math and Statistics 39(1) (2010), 1-9.

[2] B. C. Dhage, Generalised metric spaces and mappings with fixed point, Bull. Calcutta Math. Soc. 84(4) (1992), 329-336.

[3] J. X. Fang, On fixed point theorem in fuzzy metric spaces, Fuzzy sets and systems 46 (1992), 107-113.

[4] A. George and P. Veeramani, On some results in fuzzy metric space, Fuzzy Sets and Systems 64 (1994), 395-399.

[5] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems 27 (1988), 385-389.

[6] V. Gregori and A. Sapena, On fixed point theorem in fuzzy metric spaces, Fuzzy Sets and Systems 125 (2002), 245-252.

[7] Guang Peng Sun and Kai Yang, Generalized fuzzy metric spaces with Properties, Research Journal of Applied Sciences, Engineering and Technology 2(7) (2010), 673-678.

[8] Jong Seo Park, Some fixed point theorems and examples in M-fuzzy metric space, J. Korean Soc. Math. Education Series B: Pure Appl. Math. 17(3) (2010), 205-209.

[9] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. and Math. Sci. 9 (1986), 771-779.

[10] F. Kiany and Amini-Harandi, Fixed point and endpoint theorems for set-valued fuzzy contraction maps in fuzzy metric spaces, Fixed point Theory 2011(94) (2011). Appl.doi:10.1186/1687-1812-2011-94

[11] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975), 326-334.

[12] I. Kubiaczyk and Sushil Sharma, Common coincidence point in fuzzy metric space, J. Fuzzy Maths, (accepted).

[13] D. Mihet, A Banach contraction theorem in fuzzy metric spaces, Fuzzy Sets and Systems 144 (2004), 431-439.

[14] Z. Mustafa, H. Obiedat and F. Awawdeh, Some fixed point theorem for Mapping on complete G-metric spaces, Fixed point Theory and applications, 2008, article ID 189870, doi:10.1155/2008/189870

[15] S. V. R. Naidu, K. P. R. Rao and R. N. Srinivasa, On the convergent sequences and fixed point theorems in D-metric spaces, Internat. J. Math. and Math. Sci. 12 (2005), 1969-1988.

[16] J. H. Park, J. S. Park and Y. C. Kwun, Fixed point in M-fuzzy metric spaces, Optimization and Decision Making 7(4) (2008), 305-315.

[17] J. Rodriguez-Lopez and S. Romaguera, The Hausdorff fuzzy metric on compact sets, Fuzzy Sets and Systems 147 (2004), 273-283.

[18] R. Saini, K. Vishal and S. B. Singh, Fuzzy Versions of some fixed point theorems on expansion type maps in fuzzy FM metric spaces, Thai Journal of Mathematics 5(2) (2007).

[19] L. A. Zadeh, Fuzzy Sets, Information and control 8 (1965), 338-353.