Advances and Applications in Mathematical Sciences
Volume 21, Issue 3, January 2022, Pages 1217-1228

RADIO LABELING OF SOME SPLITTING GRAPHS

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Abstract

Let $G = (V, E)$ be a simple graph with p vertices and q edges. For a connected graph G of diameter d, a radio labeling is a one to one mapping f from $V(G)$ to $N \cup \{0\}$ satisfying the condition $d(u, v) + | f(u) - f(v) | \geq 1 + diam(G)$ for every $u, v \in V(G)$. The span of a labeling f is the maximum integer that f maps to a vertex of G. The radio number of G, $rn(G)$ is the lowest span taken over all radio labelings of the graph G. In this paper, we analyze some splitting graphs for radio labeling.

I. Introduction

The graph labeling problem is one of the recent developing area in graph theory. Alex Rosa first introduced this problem in 1967 [8]. Radio labeling is motivated by the channel assignment problem introduced by W. K. Hale in 1980 [3]. In 2001, Gary Chartrand defined the concept of radio labeling of G [1]. Liu and Zhu first determined the radio number in 2005 [4]. Ponraj et al. [6] introduced the notion of radio mean labeling of graphs and investigated radio mean number of some graphs [7].

2020 Mathematics Subject Classification: 05C78.
Keywords: Radio, Radio number, Radio labeling.
Received July 8, 2021; Accepted October 11, 2021

Radio Labeling is used for X-ray, crystallography, coding theory, network security, network addressing, channel assignment process, social network analysis such as connectivity, scalability, routing, computing, cell biology etc.,

The following results are used in the subsequent section.

II. Definitions

Definition 2.1. Let $G = (V, E)$ be a simple graph with p vertices and q edges. For a connected graph G of diameter d, a radio labeling is a one to one mapping f from $V(G)$ to $N \cup \{0\}$ satisfying the condition $d(u, v) + | f(u) - f(v) | \geq 1 + diam(G)$ for every $u, v \in V(G)$. The span of a labeling f is the maximum integer that f maps to a vertex of G. The radio number of G, $rn(G)$ is the lowest span taken over all radio labelings of the graph G. In this paper, we analyze some splitting graphs for radio labeling.

Definition 2.2. For a graph G, the splitting graph $S'(G)$ of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.

Definition 2.3. Walk is an alternating sequence of vertices and edges starting and ending with vertices. A walk in which all the vertices are distinct is called a path. A closed path is called a cycle.

Definition 2.4. A Y-tree Y_{n+1} is a graph obtained from the path P_n by appending an edge to a vertex of the path P_n adjacent to an end point.

III. Main Results

Theorem 3.1. $(\dot{S}'(K_{1,n}) = 4n + 1$ *for* $n > 1$.

Proof. Let G be the splitting graph of the star graph.

Let $V(G) = \{v_1, v_2, v_3, \ldots, v_n, v'_1, v'_2, v'_3, \ldots, v'_n, v, v'\}$

Then $E(G) = \{vv'_i / 1 \leq i \leq n\} \cup \{vv_i / 1 \leq i \leq n\} \cup \{v_i v' / 1 \leq i \leq n\}$

Let $diam(S'(K_{1,n})) = 3$

Define $f : V(G) \to N \cup \{0\}$ by

$f(v'_1) = 0, f(v'_i) = 2i - 2, 2 \leq i \leq n.$

$f(v) = 4n + 1, f(v') = 1, f(v_i) = 2n + 2i - 2, \leq i \leq n.$

Claim: *f* is a valid radio labeling.

Since $(S'(K_{1,n})) = 3,$ to prove that *f* is a radio labeling, we need to prove that

$$d(x, y) + | f(x) - f(y) | \geq 4 \ldots (1)$$

for every pair of vertices (x, y) where $x \neq y.$

But, it is enough to prove (1) for pair of vertices with minimum f values and minimum $d(x, y)$ values.

Hence, the proof involves the following cases.

Case a: Here, we consider the pairs (v'_i, v_i) for $1 \leq i \leq n$ and $\min(d(v'_i, v_i)) = 2$

$d(v'_i, v_i) + | f(v'_i) - f(v_i) | = 2 + | 2i - 2 - 2n - 2i + 2 | > 4$

Case b: Here, we consider the pairs (v'_1, v) and $d(v'_1, v) = 1$

$d(v'_1, v) + | f(v'_1) - f(v) | = 1 + | 0 - 4n - 1 | > 4$

Case c: Here, we consider the pairs (v'_1, v') and $d(v'_1, v) = 3$

$d(v'_1, v) + | f(v') - f(v') | = 3 + | 0 - 1 | = 4$

Case d: Here, we consider the pairs (v_n, v) and $d(v_n, v) = 1$

$d(v'_n, v) + | f(v) - f(v_n) | = 1 + | 4n - 2 - 4n - 1 | = 4$

Case e: Here, we consider the pairs (v_i, v') for $1 \leq i \leq n$ and $\min(d(v_i, v')) = 1$

$d(v_i, v') + | f(v_i) - f(v) | = 1 + | 2n + 2i - 2 - 1 | > 4.$

Case f: Here, we consider the pairs (v, v') and $d(v, v') = 2.$

$d(v', v) + |(v') - f(v)| = 2 + |1 - 4n - 1| > 4.$

Hence, by all the above cases, the radio condition is satisfied by f.

Further, f attains its maximum corresponding to v and so $rn(S'(K_{1,n}))$ $= 4n + 1$ for $n > 1$.

Theorem 3.2.

$$rn(S'(P_n)) = \begin{cases} 6 & for\ n = 2 \\ 9 & for\ n = 3 \\ 4(nk - k^2 - 1) - 3k + n & for\ n > 3\ and\ n\ is\ odd \\ k(4n - 4k + 3) - n - 6 & for\ n > 2\ and\ n\ is\ even \end{cases}$$

Proof. Let G be the splitting graph of the path. Let $V(G) = \{v_1, v_2, v_3, \ldots, v_n, v_1', v_2', v_3', \ldots, v_n'\}$. Then $E(G) = \{v_i v_{i+1} / 1 \le i \le n-1\}$ $\cup \{v_i v_{i+1}' / 1 \le i \le n-1\} \cup \{v_i' v_{i+1} / 1 \le i \le i \le n-1\}$.

Case 1: $n = 2$

Let $diam(S'(P_n)) = 3$

Define $f : V(G) \to N \cup \{0\}$ by

$f(v_1) = 3, f(v_2) = 6, f(v_1') = 1, f(v_2') = 0$

Claim: f is a valid radio labeling.

Since $diam(S'(P_n)) = 3$, to prove that f is a radio labeling, we need to prove that

$$d(x, y) + |(x) - f(y)| \ge 4 \ldots (1)$$

for every pair of vertices (x, y) where $x \ne y$.

When $|f(x) - f(y)|$ is minimum $d(x, y) = 3$.

Similarly, when $d(x, y)$ is minimum $|f(x) - f(y)| = 3$.

In both the cases, $d(x, y) + |f(x) - f(y)| = 4 = 1 + diam(G)$.

Hence, f attains its maximum corresponding to v_2 and so $rn(S'(P_n) = 6$ for $n = 2$.

Case 2: $n = 3$

Let $diam\,(S'(P_n) = 3$

Define $f : V(G) \to N \cup \{0\}$ by

$f(v_1) = 7,\ f(v_2) = 0,\ f(v_3) = 9$

$f(v_2') = 3,\ f(v_2') = 2,\ f(v_3') = 5$

Claim: *f* is a valid radio labeling.

Since $diam(S'(P_n) = 3,$ to prove that *f* is a radio labeling, we need to prove that

$$d(x,\ y) + |\ f(x) - f(y)\ | \geq 4 \ldots (1)$$

for every pair of vertices $(x,\ y)$ where $x \neq y.$

When $|\ f(x) - f(y)\ |$ is minimum $d(x,\ y) = 3.$

Similarly, when $d(x,\ y)$ is minimum $|\ f(x) - f(y)\ | = 3.$

In both the cases, $d(x,\ y) + |\ f(x) - f(y)\ | = 4 = 1 + diam(G).$

Hence, *f* attains its maximum corresponding to v_3 and so $rn(S'(P_n)) = 9$ for $n = 3.$

Case 3: $n > 3$ and *n* is odd

If *n* is odd, then $n = 2k + 1$

Let $diam(S'(P_n)) = n - 1$

Define $f : V(G) \to N \cup \{0\}$ by

For $0 \leq i \leq k - 1$

$f(v_{k-i}) = 4(ni - i + n - 1 - i^2) - 5(i + 1)$

$f(v_{k-i}) = 4(ni - i - i^2) + 2(n - 1) - 3$

For $2 \leq i \leq k + 1$

$f(v_{k+i}) = 4(ni - i - i^2) + 13i - 5n - 9$

$f(v_{k+i}) = 4(ni - i - i^2) + 9i - 3n - 5$

$f(v_{k+1}) = 0,\ f(v_{k+1}) = n - 2$

Claim: f is a valid radio labeling.

Since $diam(S'(P_n)) = n - 1$, to prove that f is a radio labeling, we need to prove that

$$d(x, y) + |f(x) - (y)| \geq n \ldots (1)$$

for every pair of vertices (x, y) where $x \neq y$.

But, it is enough to prove (1) for pair of vertices with minimum f values and minimum $d(x, y)$ values.

Hence, the proof involves the following subcases.

Subcase 3a: Here, we consider the pairs (v_k, v_{k+1}) and $d(v_k, v_{k+1}) = 1$

$d(v_k, v_{k+1}) + |f(v_k) - f(v_{k+1})| = 1 + |4n - 9| > n$

Subcase 3b: Here, we consider the pairs (v_{k+1}, v'_{k+1}) and $d(v_k, v'_{k+1}) = 2$

$d(v_{k+1}, v'_{k+1}) + |f(v_{k+1}) - f(v'_{k+1})| = 2 + |n - 1 - 1| = n.$

Subcase 3c: Here, we consider the pairs (v'_k, v'_{k+1}) and $d(v'_k, v'_{k+1}) = 3$

$d(v'_k, v'_{k+1}) + |f(v'_k) - f(v'_{k+1})| = 3 + |2n - 2 - 3 - n + 1 + 1| = n.$

Hence, by all the above cases, the radio condition is satisfied by f.

Further, f attains its maximum corresponding to v_n and so $rn(S'(P_n))$ $= 4(nk - k^2 - 1) - 3k + n$ for $n > 3$ and n is odd.

Case 4: $n > 2$ and n is even

If n is even, then $n = 2k$

Let $diam\,(S'(P_n)) = n - 1$

Define $f : V(G) \to N \cup \{0\}$ by

$f(v'_1) = k(4n - 4k + 3) - n - 6$

For $0 \leq i \leq k - 1$

$f(v_{k-i}) = 4(ni - i - i^2 + n - 1) - 5(i + 1)$

For $0 \leq i \leq k - 2$

$f(v'_{k-i}) = 4(ni - i - i^2) - i + 2n - 5$

For $2 \leq i \leq k$

$f(v'_{k-i}) = 4(ni - i - i^2) + 13i - 5(n - 1) - 14$

$f(v_{k-i}) = 4(ni - i - i^2) - 9i + 3(n - 1) - 8$

$f(v_{k+i}) = 0,\ f(v'_{k+1}) = n - 2$

Claim: f is a valid radio labeling.

Since $(S'(P_n)) = n - 1$, to prove that f is a radio labeling, we need to prove that

$$d(x,\ y) + |\ f(x) - f(y)\ | \geq n \ldots (1)$$

for every pair of vertices $(x,\ y)$ where $x \neq y$.

But, it is enough to prove (1) for pair of vertices with minimum f values and minimum $d(x,\ y)$ values.

Hence, the proof involves the following subcases.

Subcase 4a: Here, we consider the pairs $(v_k,\ v_{k+1})$ and $d(v_k,\ v_{k+1}) = 1$

$$d(v_k,\ v_{k+1}) + |\ f(v_k) - f(v_{k+1})\ | = 1 + |\ 4n - 9\ | > n$$

Subcase 4b: Here, we consider the pairs $(v_{k+1},\ v'_{k+1})$ and $d(v'_k,\ v'_{k+1}) = 3$

$$d(v_{k+1},\ v'_{k+1}) + |\ f(v'_k) - f(v'_{k+1})\ | = 2 + |\ - n + 2\ | = n$$

Subcase 4c: Here, we consider the pairs (v'_k, v'_{k+1}) and $d(v'_k, v'_{k+1}) = 3$

$$d(v'_k, v'_{k+1}) + | f(v'_k) - f(v'_{k+1}) = 3 + | 2n - 5 - n + 2 | = n.$$

Hence, by all the above cases, the radio condition is satisfied by f.

Further, f attains its maximum corresponding to v'_1 and so $rn(S'(P_n)) = k(4n - 4k + 3) - n - 6$ for $n > 2$ and n is even.

Theorem 3.3.

$$rn(S'(Y_{n+1})) = \begin{cases} 13 & \text{for } n = 3 \\ k(2n + 1) + 2n - 8 & \text{for } n > 3 \text{ and } n \text{ is odd for } n > 2 \\ 4k(n - k) + 5k - n - 9 & \text{for } n > 2 \text{ and } n \text{ is even} \end{cases}$$

Proof. Let G be the splitting graph of Y-tree

If n is odd, then $n = 2k + 1$

Let $V(G) = \{v_1, v_2, v_3, \ldots, v_n, v'_1, v'_2, v'_3, \ldots, v'_n, v, v'\}$

Then $E(G) = \{v_i v_{i+1} / 1 \le i \le n-1\} \cup \{v_i v'_{i+1} / 1 \le n-1\} \cup \{v'_i v_{i+1} / 1 \le i \le n-1\}$

$\cup \{v'_2 v\} \cup \{v_2 v\} \cup \{v_2 v'\}$

Case 1: $n = 3$

Let $diam\,(S'(Y_{n+1})) = 3$

Define $f : V(G) \to N \cup \{0\}$ by

$f(v_1) = 7,\ f(v_2) = 0,\ f(v_3) = 9$

$f(v'_1) = 3,\ f(v'_2) = 2,\ f(v'_3) = 5$

$f(v) = 11,\ f(v') = 13.$

Claim: f is a valid radio labeling.

Since $diam\,(S'(Y_{n+1})) = 3,$ to prove that f is a radio labeling, we need to prove that

$$d(x, y) + | f(x) - f(y) | \ge 4 \ldots (1)$$

for every pair of vertices (x, y) where $x \neq y$.

When $| f(x) - f(y) |$ is minimum $d(x, y) = 3$.

Similarly, when $d(x, y)$ is minimum $| f(x) - f(y) | = 3$.

In both the cases, $d(x, y) + | f(x) - f(y) | = 4 = 1 + diam(G)$.

Hence, f attains its maximum corresponding to v' and so $rn(S'(Y_{n+1})) = 13$ for $n = 3$

Case 2: $n > 3$ and n is odd

If n is odd, then $n = 2k + 1$

Let $diam\,(S'(Y_{n+1})) = n - 1$

Define $f : V(G) \to N \cup \{0\}$ by

For $0 \leq i \leq k - 1$

$f(v_{k-i}) = 4(n_i - i + n - 1 - i^2) - 5(i + 1)$

$f(v_{k-i}) = 4(ni - i - i^2) - i + 2(n - 1) - 3$

For $2 \leq i \leq k + 1$

$f(v'_{k-i}) = 4(ni - i - i^2) + 13i - 5n - 9$

$f(v_{k-i}) = 4(ni - i - i^2) + 9i - 3n - 5$

$f(v_{k-1}) = 0,\ f(v'_{k+1}) = n - 2$

$f(v) = k(2n + 1) + n - 6,\ f(v') = k(2n + 1) + 2n - 8$

Claim: *f* is a valid radio labeling.

Since $diam\,(S'(Y_{n+1})) = n - 1$, to prove that *f* is a radio labeling, we need to prove that

$$d(x, y) + | f(x) - f(y) | \geq n \ldots (1)$$

for every pair of vertices (x, y) where $x \neq y$.

But, it is enough to prove (1) for pair of vertices with minimum f values and minimum $d(x, y)$ values.

Hence, the proof involves the following subcases.

Subcase 2a: Here, we consider the pairs (v_k, v_{k+1}) and $d(v_k, v_{k+1}) = 1$

$$d(v_k, v_{k+1}) + | f(v_k) - f(v_{k+1}) | = 1 + | 4n - 9 | > n$$

Subcase 2b: Here, we consider the pairs (v_k, v'_{k+1}) and $d(v_{k+1}, v'_{k+1}) = 2$

$$d(v_k, v'_{k+1}) + | f(v_{k+1}) - f(v'_{k+1}) | = 2 + | n - 1 - 1 | = n$$

Subcase 2c: Here, we consider the pairs (v'_k, v'_{k+1}) and $d(v_k, v'_{k+1}) = 3$

$$d(v'_k, v'_{k+1}) + | f(v'_k) - f(v'_{k+1}) | = 3 + | 2n - 2 - 2 - 3n + 1 + 1 | = n$$

Hence, by all the above cases, the radio condition is satisfied by f.

Further, f attains its maximum corresponding to v' and so $rn(S'(Y_{n+1})) = k(2n + 1) + 2n - 8$ for $n > 3$ and n is odd

Case 3: $n > 2$ and n is even

If n is even, then $n = 2k$

Let $diam(S'(Y_{n+1})) = n - 1$

Define $f : V(G) \to N \cup \{0\}$ by $f(v'_1) = k(4n - 4k + 3) - n - 6$

For $0 \le i \le k - 1$

$$f(v_{k-i}) = 4(ni - i - i^2 + n - 1) - 5(i + 1)$$

For $0 \le i \le k - 2$

$$f(v_{k-i}) = 4(ni - i - i^2 + n - 1) - 5(i + 1)$$

For $0 \le i \le k - 2$

$$f(v'_{k-i}) = 4(ni - i - i^2) - i + 2n - 5$$

For $2 \le i \le k$

$f(v'_{k+i}) = 4(ni - i - i^2) + 13i - 5(n-1) - 14$

$f(v_{k+i}) = 4(ni - i - i^2) + 9i - 3(n-1) - 8$

$f(v) = 4k(n-k) + 5k - 2n - 7,\ f(v') = 4k(n-k) + 5k - n - 9$

Claim: f is a valid radio labeling.

Since $diam(S'(Y_{n+1})) = n - 1,$ to prove that f is a radio labeling, we need to prove that

$$d(x,\ y) + |\ f(x) - f(y)\ | \geq n \ldots (1)$$

for every pair of vertices $(x,\ y)$ where $x \neq y.$

But, it is enough to prove (1) for pair of vertices with minimum f values and minimum $d(x,\ y)$ values.

Hence, the proof involves the following subcases.

Subcase 3a: Here, we consider the pairs $(v_{k+1},\ v'_{k+1})$ and $d(v_k,\ v_{k+1}) = 1$

$$d(v_k,\ v_{k+1}) + |\ f(v_k) - f(v_{k+1})\ | = 1 + |\ 4n - 9\ | > n$$

Subcase 3b: Here, we consider the pairs $(v_{k+1},\ v'_{k+1})$ and $d(v_{k+1},\ v'_{k+1}) = 2$

$$d(v_{k+1},\ v'_{k+1}) + |\ f(v_{k+1}) - f(v'_{k+1})\ | = 2 + |\ -n + 2\ | = n.$$

Subcase 3c: Here, we consider the pairs $(v'_k,\ v'_{k+1})$ and $(v'_k,\ v'_{k+1}) = 3$

$$d(v'_k,\ v'_{k+1}) + |\ f(v'_k) - f(v'_{k+1}) = 3 + |\ 2n - 5 - n + 2\ | = n.$$

Hence, by all the above cases, the radio condition is satisfied by f.

Further, f attains its maximum corresponding to v' and so $rn(S'(Y_{n+1}))$ $= 4k(n-k) + 5k - n - 9$ for $n > 2$ and n is even.

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