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COMMON FIXED POINT THEOREMS FOR α-ADMISSIBLE MAPPINGS IN NEUTROSOPHIC METRIC SPACES

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Abstract

In this paper, by using the concept of α-admissible mappings, we prove common fixed point in Neutrosophic metric space. We also introduce the notion of $\alpha - (\phi, \psi)$ weak contraction mappings in Neutrosophic metric spaces. The presented theorems extend, generalize and improve the corresponding results which given in the literature.

1. Introduction

In 1965, the concept of fuzzy set was introduced by Zadeh [15] in domain X and $[0, 1]$. In 1986, Atanasov [2] introduced the notion of an intuitionistic fuzzy metric space. Afterward, Park [8] gave the notion of an intuitionistic fuzzy metric space and generalized the notion of a fuzzy metric space due to George and Veeramani [4]. In 2008, Sadati et al. [12] modified the idea of an intuitionistic fuzzy metric space and presented the new notion of an intuitionistic fuzzy metric space.

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In 1998, Smarandache [9] characterized the new idea called neutrosophic set. In general the notion of fuzzy set and IFS deal with degree of membership and non-membership respectively. Neutrosophic set is a generalized state of Fuzzy and Intuitionistic Fuzzy Set by incorporating degree of indeterminacy. In addition, several researchers contributed significantly to develop the neutrosophic theory. Recently, in 2019, Kirisci et al. [7] defined neutrosophic metric space as a generalization of IFMS and brings about fixed point theorems in complete neutrosophic metric space. In 2020, Sowndrarajan and Jeyaraman et al. [13] proved some fixed point results in neutrosophic metric spaces.

In this paper, we introduced the concept of $\alpha - (\phi, \psi)$ weak contraction mappings in Neutrosophic metric space and prove some common fixed point results. In particular, the presented theorems extend, generalize and improve the results.

2. Preliminaries

Definition 2.1[13]**.** A 6-tuple $(\sum, \Xi, \Theta, \Upsilon, *, \lozenge)$ is said to be a Neutrosophic Metric Space (shortly NMS), if Σ is an arbitrary set, $*$ is a neutrosophic CTN, $\lozenge$ is a neutrosophic CTC and Ξ, Θ and Y are neutrosophic on $\sum^3 \times \mathbb{R}^+$ satisfying the following conditions: For all $\zeta, \eta, \delta \in \Sigma, t \in \mathbb{R}^+$.

(i) $0 \leq \Xi(\zeta, \eta, t) \leq 1; 0 \leq \Theta(\zeta, \eta, t) \leq 1; 0 \leq \Upsilon(\zeta, \eta, t) \leq 1;$

(ii) $\Xi(\zeta, \eta, t) + \Theta(\zeta, \eta, t) + \Upsilon(\zeta, \eta, t) \leq 3;$

(iii) $\Xi(\zeta, \eta, t) = 1$ if and only if $\zeta = \eta;$

(iv) $\Xi(\zeta, \eta, t) = \Xi(\eta, \zeta, t)$ for $t > 0;$

(v) $\Xi(\zeta, \eta, t) * \Xi(\eta, \delta, \mu) \leq \Xi(\zeta, \delta, t + \mu),$ for all $t, \mu > 0;$

(vi) $\Xi(\zeta, \eta, \delta, \cdot) : [0, \infty) \to [0, 1]$ is neutrosophic continuous;

(vii) $\lim_{t \to \infty} \Xi(\zeta, \eta, t) = 1$ for all $t > 0;$

(viii) $\Theta(\zeta, \eta, t) = 0$ if and only if $\zeta = \eta;$

(ix) $\Theta(\zeta, \eta, t) = \Theta(\eta, \zeta, t)$ for $t > 0;$

(x) $\Theta(\zeta, \eta, t) \lozenge \Theta(\eta, \delta, \mu) \geq \Theta(\zeta, \delta, t + \mu)$, for all $t, \mu > 0$;

(xi) $\Theta(\zeta, \eta, \cdot) : [0, \infty) \to [0, 1]$ is neutrosophic continuous;

(xii) $\lim_{t \to \infty} \Theta(\zeta, \eta, t) = 0$ for all $t > 0$;

(xiii) $\Upsilon(\zeta, \eta, t) = 0$ if and only if $\zeta = \eta$;

(xiv) $\Upsilon(\zeta, \eta, t) = \Upsilon(\eta, \zeta, t)$ for $t > 0$;

(xv) $\Upsilon(\zeta, \eta, t) \lozenge \Upsilon(\eta, \delta, \mu) \geq \Upsilon(\zeta, \delta, t + \mu)$, for all $t, \mu > 0$;

(xvi) $\Upsilon(\zeta, \eta, \cdot) : [0, \infty) \to [0, 1]$ is neutrosophic continuous;

(xvii) $\lim_{t \to \infty} \Upsilon(\zeta, \eta, t) = 0$ for all $t > 0$;

(xviii) If $t \leq 0$ then $\Xi(\zeta, \eta, t) = 0$, $\Theta(\zeta, \eta, t) = 1$, $\Upsilon(\zeta, \eta, \lambda) = 1$.

Then, (Ξ, Θ, Υ) is called a Neutrosophic sets on Σ. The functions Ξ, Θ and Υ denote degree of closedness, neturalness and non-closedness between ζ and η with respect to t respectively.

Remark 2.2 [13]. In NMS, $\Xi(\zeta, \eta, \cdot)$ is non-decreasing $\Theta(\zeta, \eta, \cdot)$ is non-increasing and $\Upsilon(\zeta, \eta, \cdot)$ is non-increasing for all $\zeta, \eta \in \Xi$.

Definition 2.3 [13]. A sequence $\{\zeta_n\}$ in a NMS is said to be a Cauchy sequence if and only if for each $r \in (0, 1)$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\Xi(\zeta_n, \zeta_m, t) > 1 - r$, $\Theta(\zeta_n, \zeta_m, t) < r$ and $\Upsilon(\zeta_n, \zeta_m, t) < r$ for all $n, m \geq n_0$.

Definition 2.4 [13]. A sequence $\{\zeta_n\}$ in a NMS is called convergent to $\zeta \in \Sigma$ if for each $t > 0$, we have $\lim_{n \to \infty} \Xi(\zeta_n, \zeta, t) = 1$, $\lim_{n \to \infty} \Theta(\zeta_n, \zeta, t) = 0$ and $\lim_{n \to \infty} \Upsilon(\zeta_n, \zeta, t) = 0$.

Definition 2.5 [13]. A NMS is complete if and only if every Cauchy sequence is convergent.

Definition 2.6 [13]. A NMS is compact if every sequence contains a convergent subsequence.

Definition 2.7. Let $(\sum, \Xi, \Theta, \Upsilon, *, \Diamond)$ be a NMS. A mapping $f : \Sigma \to \Sigma$ is Neutrosophic contractive if there exists $k \in (0, 1)$ such that $\frac{1}{\Xi(f\zeta, f\eta, t)} - 1 \leq k\left(\frac{1}{\Xi(\zeta, \eta, t)} - 1\right)$, $\frac{1}{\Theta(f\zeta, f\eta, t)} - 1 \geq \frac{1}{k}\left(\frac{1}{\Theta(\zeta, \eta, t)} - 1\right)$ and $\frac{1}{\Upsilon(f\zeta, f\eta, t)} - 1 \geq \frac{1}{k}\left(\frac{1}{\Upsilon(\zeta, \eta, t)} - 1\right)$ for all $\zeta, \eta \in \Sigma$ and $t > 0$.

Definition 2.8. Let Σ be a nonempty set. Two mapping $f, T : \Sigma \to \Sigma$ are said to be weakly compatible if $fT\zeta = Tf\zeta$ for all $\zeta \in \Sigma$ which $f\zeta = T\zeta$.

3. Main Results

Definition 3.1. Let $(\sum, \Xi, \Theta, \Upsilon, *, \Diamond)$ be a NMS and $f, T : \Sigma \to \Sigma$ be two mappings. We say that T is α-admissible if there exists three function $\alpha : \Sigma \times \Sigma \times (0, \infty) \to [0, \infty)$ such that, for all $t > 0$ and $\zeta, \eta \in \Sigma$, we have $\alpha(f\zeta, f\eta, t) \geq 1 \Rightarrow \alpha(T\zeta, T\eta, t) \geq 1$.

Definition 3.2. Let $(\sum, \Xi, \Theta, \Upsilon, *, \Diamond)$ be a NMS and let $f, T : \Sigma \to \Sigma$ be two mappings. The mappings T is called Neutrosophic $\alpha - (\psi)$ weak contraction with respect to f, if there exist two functions $\alpha : \Sigma \times \Sigma \times (0, \infty) \to [0, \infty)$ and $\psi : (0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ such that $\alpha(f\zeta, f\eta, t)\left(\frac{1}{\Xi(T\zeta, T\eta, t)} - 1\right) \leq \left(\frac{1}{\Xi(f\zeta, f\eta, t)} - 1\right) - \psi\left(\frac{1}{\Xi(f\zeta, f\eta, t)} - 1\right)$, $\alpha(f\zeta, f\eta, t)\ (\Theta(T\zeta, T\eta, t)) \leq \Theta(f\zeta, f\eta, t) - \psi(\Theta(f\zeta, \eta, t))$ and $\alpha(f\zeta, f\eta, t)(\Upsilon(T\zeta, T\eta, t)) \leq \Upsilon(f\zeta, f\eta, t) - \psi(\Upsilon(f\zeta, f\eta, t))$, for all $\zeta, \eta \in \Sigma$ and $t > 0$.

Definition 3.3. Let $(\sum, \Xi, \Theta, \Upsilon, *, \Diamond)$ be a NMS and let $f, T : \Sigma \to \Sigma$ be two mappings. The mappings T is called Neutrosophic $\alpha - (\phi, \psi)$ weak contraction with respect to f, if there exist three functions $\alpha : \Sigma \times \Sigma \times (0, \infty) \to [0, \infty)$, $\phi : (0, \infty) \to [0, \infty)$ and $\psi : (0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ such that

$\alpha(f\zeta, f\eta, t)\phi\left(\frac{1}{\Xi(T\zeta, T\eta, t)} - 1\right) \leq \phi\left(\frac{1}{\Xi(f\zeta, f\eta, t)} - 1\right) - \psi\left(\frac{1}{\Xi(f\zeta, f\eta, t)} - 1\right),$

$\alpha(f\zeta, f\eta, t)\ \phi(\Theta(T\zeta, T\eta, t)) \leq \phi(\Theta(f\zeta, f\eta, t)) - \psi(\Theta(f\zeta, \eta, t))$ and

$\alpha(f\zeta, f\eta, t)\phi(\Upsilon(T\zeta, T\eta, t)) \leq \phi(\Upsilon(f\zeta, f\eta, t)) - \psi(\Upsilon(f\zeta, f\eta, t))$ for all $\zeta, \eta \in \Sigma$ and $t > 0$. (3.3.1)

Example 3.4. Let $\Sigma = \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \cup \{0, 4\}$ and $*$ be a minimum t-norm and $\lozenge$ be a maximum t-conorm. Let Ξ, Θ, Υ be defined by

$$\Xi(\zeta, \eta, t) = \begin{cases} \frac{t}{t + |\zeta - \eta|}, & \text{if } t > 0 \\ 0, & \text{if t} = 0 \end{cases}, \quad \Theta(\zeta, \eta, t) = \begin{cases} \frac{|\zeta - \eta|}{t + |\zeta - \eta|}, & \text{if } t > 0 \\ 1, & \text{if t} = 0 \end{cases} \text{ and}$$

$$\Upsilon(\zeta, \eta, t) = \begin{cases} \frac{|\zeta - \eta|}{t}, & \text{if } t > 0 \\ 1, & \text{if t} = 0. \end{cases}$$

Define the mapping $T : \Sigma \to \Sigma$ by $T(\zeta) = \begin{cases} \frac{\zeta}{4}, & \text{if } \zeta \neq 0 \\ 1, & \text{if } \zeta = 4 \end{cases}$ and define the function $\alpha : \Sigma \times \Sigma \times (0, \infty) \to [0, \infty)$ by $\alpha(f\zeta, f\eta, t) = \begin{cases} 1, & \text{if } \zeta, \eta \in \Sigma \setminus \{4\} \\ 0, & \text{otherwise} \end{cases}$

Also, define $\phi, \psi : (0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{t}{2}$, $\psi(t) = \frac{t}{8}$ and let $f(\zeta) = \frac{\zeta}{2}$

In fact, if at least one between ζ and η is equal to 4, then $\alpha(f\zeta, f\eta, t) = 0$ and so holds trivially. Otherwise, if both ζ and η are in $\Sigma \setminus \{4\}$ then $\alpha(f\zeta, f\eta, t) = 1$ and so (3.1.1). Then, we have

$$\phi\left(\frac{1}{\Xi(T\zeta, T\eta, t)} - 1\right) - \psi\left(\frac{1}{\Xi(f\zeta, f\eta, t)} - 1\right) = \phi\left(\frac{1}{\Xi\left(\frac{\zeta}{2}, \frac{\eta}{2}, t\right)} - 1\right) - \psi\left(\frac{1}{\Xi\left(\frac{\zeta}{2}, \frac{\eta}{2}, t\right)} - 1\right)$$

$$= \phi\left(\frac{\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{t}-1\right)-\psi\left(\frac{\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{t}-1\right)=\frac{\frac{\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{t}}{2}-\frac{\frac{\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{t}}{8}=\frac{\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{2t}-\frac{\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{8t}$$

$$=\frac{4\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|-\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{8t}=\frac{3\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{8t}\geq\frac{2\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{8t}\geq\frac{\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{4t}=\frac{\left|\frac{\zeta}{4}-\frac{\eta}{4}\right|}{2t}=\phi\left(\frac{\left|\frac{\zeta}{4}-\frac{\eta}{4}\right|}{t}\right)$$

$$=\phi\left(\frac{1}{\Xi\left(\frac{\zeta}{4},\frac{\eta}{4},t\right)}-1\right)=1,\ \phi\left(\frac{1}{\Xi\left(\frac{\zeta}{4},\frac{\eta}{4},t\right)}-1\right)=\alpha(f\zeta,f\eta,t)\phi\left(\frac{1}{\Xi\left(\frac{\zeta}{4},\frac{\eta}{4},t\right)}-1\right),$$

$$\phi(\Theta(f\zeta,f\eta,t))-\psi(\Theta(f\zeta,f\eta,t))=\phi\left(\frac{\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{t+\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}\right)-\psi\left(\frac{\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{t+\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}\right)$$

$$=\left(\frac{\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{2\left(t+\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|\right)}\right)-2\left(\frac{\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{8\left(t+\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|\right)}\right)=\frac{3\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{8\left(t+\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|\right)}$$

$$\geq\frac{2\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{8\left(t+\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|\right)}\geq\frac{\left|\frac{\zeta}{4}-\frac{\eta}{4}\right|}{2\left(t+\left|\frac{\zeta}{4}-\frac{\eta}{4}\right|\right)}=\phi\left(\frac{\left|\frac{\zeta}{4}-\frac{\eta}{4}\right|}{2\left(t+\left|\frac{\zeta}{4}-\frac{\eta}{4}\right|\right)}\right)=\phi\left(\frac{\left|\frac{\zeta}{4}-\frac{\eta}{4}\right|}{t+\left|\frac{\zeta}{4}-\frac{\eta}{4}\right|}\right)$$

$$=\phi\left(\Theta\left(\frac{\zeta}{4},\frac{\eta}{4},t\right)\right)=1.\ \phi(\Theta(T\zeta,T\eta,t))=\alpha(f\zeta,f\eta,t)\phi((T\zeta,T\eta,t))$$

$$\phi(\Upsilon(f\zeta,f\eta,t))-\psi(\Upsilon(f\zeta,f\eta,t))=\phi\left(\Upsilon\left(\frac{\zeta}{2},\frac{\eta}{2},t\right)\right)-\psi\left(\Upsilon\left(\frac{\zeta}{2},\frac{\eta}{2},t\right)\right)$$

$$=\phi\left(\frac{\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{t}\right)-\psi\left(\frac{\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{t}\right)=\frac{3\left|\frac{\zeta}{2}-\frac{\eta}{2}\right|}{8t}-\frac{\left|\frac{\zeta}{4}-\frac{\eta}{4}\right|}{2t}$$

$$= \phi\left(\frac{\left|\frac{\zeta}{4} - \frac{\eta}{4}\right|}{t}\right) = \phi\left(\Upsilon\left(\frac{\zeta}{4}, \frac{\eta}{4}, t\right)\right) = 1, \phi\left(\Upsilon\left(\frac{\zeta}{4}, \frac{\eta}{4}, t\right)\right) = \alpha(f\zeta, f\eta, t)\phi(\Upsilon(f\zeta, f\eta, t)).$$

Therefore T is Neutrosophic $\alpha - (\phi, \psi)$ weak contraction with respect to f.

Theorem 3.5. *Let* $(\Sigma, \Xi, \Theta, \Upsilon, *, \lozenge)$ *be a NMS. Let T and f be self-mappings on* Σ *such that the range of f contains the range of* $T(T\Sigma \subseteq f\Sigma)$ *and* $f(\Sigma)$ *or* $T(\Sigma)$ *is a complete subset of* Σ *and* $\alpha : \Sigma \times \Sigma \times (0, \infty) \to [0, \infty)$. *Suppose that T is Neutrosophic* $\alpha - (\phi, \psi)$ *weak contraction with respect to f and the following conditions hold*:

(i) *T is* α-*admissible*

(ii) *There exists* $\zeta_0 \in \Sigma$ *such that* $\alpha(f\zeta_0, T\zeta_0, t) \geq 1$, *for all* $t > 0$

(iii) *T is continuous*

Then, *T and f have a coincidence point in* Σ. *If T and f* are *weakly compatible*, *then T and f have a unique common fixed point in* Σ.

Proof. Let $\zeta_0 \in \Sigma$ such that $\alpha(f\zeta_0, T\zeta_0, t) \geq 1$, for all $t > 0$ and choose a point ζ_1 *in* Σ such that $T\zeta_0 = f\zeta_1$. Define the sequence $\{\zeta_n\}$ and $\{\eta_n\}$ in Σ such that $\eta_n = T\zeta_n = f\zeta_{n+1}$, $n \in \mathbb{N} \cup \{0\}$. In particular, if $\eta_n = \eta_{n+1}$, then η_{n+1} is a point of coincidence of T and f. Consequently, we assume that $\eta_n \neq \eta_{n+1}$ for all $n \in \mathbb{N}$. By condition (ii), we have $\alpha(T\zeta_0, T\zeta_1, t) = \alpha(f\zeta_1, f\zeta_2, t) \geq 1$, $\alpha(T\zeta_1, T\zeta_2, t) = \alpha(f\zeta_2, f\zeta_3, t) \geq 1$. By induction, we get $\alpha(f\zeta_n, f\zeta_{n+1}, t) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. Now, by (3.3.1) with $\zeta = \zeta_n$, $\eta = \eta_{n+1}$, we have

$$\phi\left(\frac{1}{\Xi(\eta_n, \eta_{n+1}, t)} - 1\right) = \phi\left(\frac{1}{\Xi(T\zeta_n, T\zeta_{n+1}, t)} - 1\right) \leq \alpha(f\zeta_n, f\zeta_{n+1}, t)$$

$$\phi\left(\frac{1}{\Xi(T\zeta_n, T\zeta_{n+1}, t)} - 1\right) \leq \phi\left(\frac{1}{\Xi(T\zeta_n, T\zeta_{n+1}, t)} - 1\right) - \psi\left(\frac{1}{\Xi(T\zeta_n, T\zeta_{n+1}, t)} - 1\right)$$

$$= \phi\left(\frac{1}{\Xi(T\zeta_n, T\zeta_{n+1}, t)} - 1\right) - \psi\left(\frac{1}{\Xi(T\zeta_n, T\zeta_{n+1}, t)} - 1\right) < \phi\left(\frac{1}{\Xi(T\zeta_n, T\zeta_{n+1}, t)} - 1\right)$$

which considering that the ϕ function is non-decreasing implies that $\Xi(\eta_{n+1}, \eta_n, t) > \Xi(\eta_{n-1}, \eta_n, t)$ for all $n \in \mathbb{N}$ and hence $\Xi(\eta_{n-1}, \eta_n, t)$ is an increasing sequence of positive real numbers in $(0, 1]$.

Let $S(t) = \lim_{n\to\infty} \Xi(\eta_{n-1}, \eta_n, t)$ we show that $S(t) = 1$ for all $t > 0$. If not, there exists $t > 0$ such that $S(t) < 1$, then from the above inequality on taking $n \to \infty$, we obtain $\phi\left(\frac{1}{S(t)} - 1\right) \leq \phi\left(\frac{1}{S(t)} - 1\right) - \psi\left(\frac{1}{S(t)} - 1\right)$, a contradiction. Therefore, $\Xi(\eta_n, \eta_{n+1}, t) \to 1$ as $n \to \infty$. Now, for each positive p,

$$\Xi(\eta_n, \eta_{n+p}, t) \geq \Xi\left(\eta_n, \eta_{n+1}, \frac{t}{p}\right) * \Xi\left(\eta_{n+1}, \eta_{n+2}, \frac{t}{p}\right) * \ldots * \Xi\left(\eta_{n+p-1}, \eta_{n+p}, \frac{t}{p}\right)$$

It follows that $\lim\limits_{n\to\infty} \Xi(\eta_n, \eta_{n+p}, t) \geq 1 * 1 * \ldots 1 = 1$. We have

$$\phi(\Theta(\eta_n, \eta_{n+1}, t)) = \phi(\Theta(T\zeta_n, T\zeta_{n+1}, t)) \leq (\alpha(f\zeta_n, f\zeta_{n+1}, t))\phi(\Theta(T\zeta_n, T\zeta_{n+1}, t))$$

$$\leq \phi(\Theta(f\zeta_n, f\zeta_{n+1}, t)) - \psi(\Theta(f\zeta_n, f\zeta_{n+1}, t)) = \phi(\Theta(f\eta_{n-1}, \eta_n, t)) - \psi(\Theta(f\eta_{n-1}, \eta_n, t))$$

$$< \phi(\Theta(\eta_{n-1}, \eta_n, t)),$$

which considering that the ϕ function is non-decreasing implies that $\Theta(\eta_n, \eta_{n+1}, t) < \Theta(\eta_{n-1}, \eta_n, t)$ for all $n \in \mathbb{N}$ and hence $\Theta(\eta_{n-1}, \eta_n, t)$ is a decreasing sequence of positive real number in $[0, 1)$. Let $R(t) = \lim_{n\to\infty} \Theta(\eta_{n-1}, \eta_n, t)$ we show that $R(t) = 0$ for all $t > 0$. If not, there exists $t > 0$ such that $R(t) > 0$, then from the above inequality on taking $n \to \infty$, we obtain $\phi(R(t)) \leq \phi(R(t)) - \psi(R(t))$, a contradiction. Therefore, $\Theta(\eta_n, \eta_{n+1}, t) \to 0$ as $n \to \infty$.

Now, for each positive integer p,

$$\phi(\Upsilon(\eta_n, \eta_{n+1}, t)) = \phi(\Upsilon(T\zeta_n, T\zeta_{n+1}, t)) \leq \alpha(f\zeta_n, f\zeta_{n+1}, t)\,\phi(\Upsilon(T\zeta_n, T\zeta_{n+1}, t))$$

$$\leq \phi(\Upsilon(f\zeta_n, f\zeta_{n+1}, t)) - \psi(\Upsilon(f\zeta_n, f\zeta_{n+1}, t)) = \phi(\Upsilon(f\eta_{n-1}, \eta_n, t))$$

$$- \psi(\Upsilon(f\eta_{n-1}, \eta_n, t)) < \phi(\Upsilon(\eta_{n-1}, \eta_n, t))$$

which considering that the ϕ function is non-decreasing implies that $(\Upsilon(\eta_n, \eta_{n+1}, t)) < \Upsilon(\eta_{n-1}, \eta_n, t)$ for all $n \in \mathbb{N}$ and hence $\Upsilon(\eta_{n-1}, \eta_n, t)$ is a decreasing sequence of positive real number in $[0, 1)$. Let $Q(t) = \lim_{n\to\infty} \Upsilon(\eta_{n-1}, \eta_n, t)$ we show that $Q(t) = 0$, for all $t > 0$. If not, there exists $t > 0$ such that $Q(t) > 0$, then from the above inequality on taking $n \to \infty$, we obtain $\phi(Q(t)) \leq \phi(Q(t)) - \psi(Q(t))$, a contradiction. Therefore $\Upsilon(\eta_n, \eta_{n+1}, t) \to 0$ as $n \to \infty$. Now, for each positive integer Υ by Definition (2.1) (ii) must be $\Xi(\eta_n, \eta_{n+p}, t) + \Theta(\eta_n, \eta_{n+p}, t) + \Upsilon(\eta_n, \eta_{n+p}, t) \leq 3$ and then $\lim_{n\to\infty}(\Xi(\eta_n, \eta_{n+p}, t) + \Theta(\eta_n, \eta_{n+p}, t) + \Upsilon(\eta_n, \eta_{n+p}, t) \leq 3)$. It follows that $\lim_{n\to\infty} \Theta(\eta_n, \eta_{n+p}, t) = 0$ and $\lim_{n\to\infty} \Upsilon(\eta_n, \eta_{n+p}, t) = 0$. Hence, η_n is a Cauchy sequence. If $f(\zeta)$ is complete, then there exists $q \in f(\zeta)$ such that $\eta_n \to q$ as $n \to \infty$. The same holds if $T(\Sigma)$ is complete with $q \in T(\Sigma)$. Let $p \in \Sigma$ be such that $fp = q$. Now, we show that p is a coincidence point of f and T. In fact, we have

$$\phi\left(\frac{1}{\Xi(Tp, f\zeta_{n+1}, t)} - 1\right) = \phi\left(\frac{1}{\Xi(Tp, T\zeta_n, t)} - 1\right) \leq \alpha(fp, f\zeta_n, t)$$

$$\phi\left(\frac{1}{\Xi(Tp, T\zeta_n, t)} - 1\right) \leq \phi\left(\frac{1}{\Xi(Tp, T\zeta_n, t)} - 1\right) - \psi\left(\frac{1}{\Xi(fp, f\zeta_n, t)} - 1\right),$$

for every $t > 0$, which on taking $n \to \infty$ gives that

$$\lim_{n\to\infty} \Xi(Tp, f\zeta_{n+1}, t) = \lim_{n\to\infty} \Xi(Tp, T\zeta_n, t) = \Xi(Tp, fp, t) = 1.$$

$$\phi\left(\frac{1}{\Theta(Tp, f\zeta_{n+1}, t)} - 1\right) = \phi\left(\frac{1}{\Xi(Tp, T\zeta_n, t)} - 1\right) \geq \alpha(fp, f\zeta_n, t)$$

$$\phi\left(\frac{1}{\Theta(Tp, T\zeta_n, t)} - 1\right) \geq \phi\left(\frac{1}{\Xi(Tp, T\zeta_n, t)} - 1\right) - \psi\left(\frac{1}{\Xi(Tp, f\zeta_n, t)} - 1\right),$$

for every $t > 0$, which on taking $n \to \infty$ gives that

$$\lim_{n\to\infty}\Theta(Tp, T\eta_{n+1}, t) = \lim_{n\to\infty}\Theta(Tp, T\eta_n, t) = \Theta(Tp, fp, t) = 0.$$

$$\phi\left(\frac{1}{\Upsilon(Tp, f\zeta_{n+1}, t)} - 1\right) = \phi\left(\frac{1}{\Upsilon(Tp, T\zeta_n, t)} - 1\right) \geq \alpha(fp, f\zeta_n, t)\phi\left(\frac{1}{\Upsilon(Tp, T\zeta_n, t)} - 1\right)$$

$$\geq \phi\left(\frac{1}{\Upsilon(Tp, T\zeta_n, t)} - 1\right) - \psi\left(\frac{1}{\Upsilon(Tp, f\zeta_n, t)} - 1\right),$$

for every $t > 0$, which on taking $n \to \infty$ gives that $\lim_{n\to\infty}\Upsilon(Tp, f\eta_{n+1}, t)$ $= \lim_{n\to\infty}\Upsilon(Tp, T\zeta_n, t) = \Upsilon(Tp, fp, t) = 0.$ i.e., $fp = Tp = q$ and so q is a point of coincidence of T and f. Now, we show that $fq = q$. Now, if q is a point of coincidence of T and f as T and f are weakly compatible, then we prove that q is common fixed point of T and f. Since $fp = Tp = q$ and f and T then $fq = Tq$. Using (3.3.1) and suppose that $fq \neq q$ then consider

$$\frac{1}{\Xi(fq, q, t)} - 1 = \frac{1}{\Xi(Tq, Tp, t)} - 1 \leq \alpha(fq, fp, t)\left(\frac{1}{\Xi(Tq, Tp, t)} - 1\right)$$

$$\leq \left(\frac{1}{\Xi(fq, fp, t)} - 1\right) - \psi\left(\frac{1}{\Xi(fq, fp, t)} - 1\right) = \left(\frac{1}{\Xi(fq, q, t)} - 1\right) - \psi\left(\frac{1}{\Xi(fq, q, t)} - 1\right),$$

$$\frac{1}{\Theta(fq, q, t)} - 1 = \frac{1}{\Theta(Tq, Tp, t)} - 1 \geq \alpha(fq, fp, t)\left(\frac{1}{\Theta(Tq, Tp, t)} - 1\right)$$

$$\geq \left(\frac{1}{\Theta(fq, fp, t)} - 1\right) - \psi\left(\frac{1}{\Theta(fq, fp, t)} - 1\right) = \left(\frac{1}{\Theta(fq, q, t)} - 1\right) - \psi\left(\frac{1}{\Theta(fq, q, t)} - 1\right),$$

$$\frac{1}{\Upsilon(fq, q, t)} - 1 = \frac{1}{\Upsilon(Tq, Tp, t)} - 1 \geq \alpha(fq, fp, t)\left(\frac{1}{\Upsilon(Tq, Tp, t)} - 1\right)$$

$$\geq \left(\frac{1}{\Upsilon(fq, fp, t)} - 1\right) - \psi\left(\frac{1}{\Upsilon(fq, fp, t)} - 1\right) = \left(\frac{1}{\Upsilon(fq, q, t)} - 1\right) - \psi\left(\frac{1}{\Upsilon(fq, q, t)} - 1\right),$$

a contradiction which leads to the result, that is $fq = q$ and $fp = Tp = q$. Therefore T and f have a common fixed point in Σ. We will prove the uniqueness of a common fixed point of f and T. Let δ be another common fixed point of f and T. Let δ be another common fixed point of f and $T(\delta \neq q)$. Then, there exists $t > 0$, such that

$$\phi\left(\frac{1}{\Xi(q,\delta,t)}-1\right)=\phi\left(\frac{1}{\Xi(Tq,T\delta,t)}-1\right)\leq\alpha(fq,f\delta,t)\phi\left(\frac{1}{\Xi(Tq,T\delta,t)}-1\right)$$

$$\leq\alpha\left(\frac{1}{\Xi(fq,f\delta,t)}-1\right)-\psi\left(\frac{1}{\Xi(fq,f\delta,t)}-1\right)<\phi\left(\frac{1}{\Xi(q,\delta,t)}-1\right),$$

$$\phi\left(\frac{1}{\Theta(q,\delta,t)}-1\right)=\phi\left(\frac{1}{\Theta(Tq,T\delta,t)}-1\right)\geq\alpha(fq,f\delta,t)\phi\left(\frac{1}{\Theta(Tq,T\delta,t)}-1\right)$$

$$\geq\alpha\left(\frac{1}{\Theta(fq,f\delta,t)}-1\right)-\psi\left(\frac{1}{\Theta(fq,f\delta,t)}-1\right)>\phi\left(\frac{1}{\Theta(q,\delta,t)}-1\right),$$

$$\phi\left(\frac{1}{\Upsilon(q,\delta,t)}-1\right)=\phi\left(\frac{1}{\Upsilon(Tq,T\delta,t)}-1\right)\geq\alpha(fq,f\delta,t)\phi\left(\frac{1}{\Upsilon(Tq,T\delta,t)}-1\right)$$

$$\geq\alpha\left(\frac{1}{\Upsilon(fq,f\delta,t)}-1\right)-\psi\left(\frac{1}{\Upsilon(fq,f\delta,t)}-1\right)>\phi\left(\frac{1}{\Upsilon(q,\delta,t)}-1\right),$$

a contradiction which leads to the result. i.e., $q=\delta$. Therefore q is a unique common fixed point of f and T.

Theorem 3.6. *Let* $(\Sigma, \Xi, \Theta, \Upsilon, *, \Diamond)$ *be a NMS and* $\{f_i\}, \{T_k\}$ *where* $i\in(1, 2, \ldots)$ *and* $k\in(1, 2, \ldots, n)$, *be two finite families of self mappings on* Σ *with* $f=f_1f_2\ldots f_n$ *and* $T=T_1T_2\ldots T_m$. *Let T be an Neutrosophic* $\alpha-(\phi, \psi)$ *weak contraction with respect to f. If the range of f contains the range of* $T(T\Sigma\subseteq f\Sigma)$ *and* $f(\Sigma)$ *or* $T(\Sigma)$ *is a complete subset of* Σ *and* $\alpha:\Sigma\times\Sigma\times(0,\infty)\to[0,\infty)$ *then* T_k *and* T_i *have a unique common fixed point in* Σ.

Proof. Using Theorem (3.5), we conclude that q is unique common fixed point of T and f. Now, we will show that q remains the fixed point of all the component mappings. Consider $T(T_iq)=((T_1, T_2, \ldots, T_m)T_i)_q$ $=((T_1, T_2, \ldots, T_{m-1})(T_mT_i)_q)=(T_1, T_2, \ldots, T_{m-1})(T_iT_mq)\ldots T_1T_i(T_2, T_3, T_4, \ldots, T_mq)=T_iT_1(T_2, T_3, T_4, \ldots, T_mq)=T_i(T_q)=T_iq$. Similarly, we can show that $T(f_kq)=f_k(T_q)=f_kq$, $f(f_kq)=f_k(f_q)=f_kq$ and $f(T_iq)=T_i(f_q)=T_iq$ which implies that, for all i and k, T_iq and f_kq are other fixed point of the pair $\{T, f\}$. Now appealing to the uniqueness of a common fixed point of mappings T and f we get $q=T_iq=f_kq$ which shows that q is a common fixed point of f_i and T_k for all i and k.

Definition 3.7. Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \Diamond)$ be a NMS. Let $T, f : \Sigma \to \Sigma$ be two mappings. The mappings T is called Neutrosophic $\alpha - (\phi, \psi)$ weak contraction of integral type with respect to f, if there exist three functions $\alpha : \Sigma \times \Sigma \times (0, \infty) \to [0, \infty)$, $\phi : (0, \infty) \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ such that

$$\alpha(f\zeta, f\eta, t)\phi\left(\int_0^{\frac{1}{\Xi(T\zeta, T\eta, f)}-1} \phi(s)\,ds\right) \le \phi\left(\int_0^{\frac{1}{\Xi(T\zeta, T\eta, f)}-1} \phi(s)\,ds\right)$$

$$- \psi\left(\int_0^{\frac{1}{\Xi(T\zeta, T\eta, f)}-1} \phi(s)\,ds\right),$$

$$\alpha(f\zeta, f\eta, t)\phi\left(\int_0^{\frac{1}{\Theta(T\zeta, T\eta, f)}-1} \phi(s)\,ds\right) \ge \phi\left(\int_0^{\frac{1}{\Theta(T\zeta, T\eta, f)}-1} \phi(s)\,ds\right)$$

$$- \psi\left(\int_0^{\frac{1}{\Theta(T\zeta, T\eta, f)}-1} \phi(s)\,ds\right), \text{ and}$$

$$\alpha(f\zeta, f\eta, t)\phi\left(\int_0^{\frac{1}{\Upsilon(T\zeta, T\eta, f)}-1} \phi(s)\,ds\right) \ge \phi\left(\int_0^{\frac{1}{\Upsilon(T\zeta, T\eta, f)}-1} \phi(s)\,ds\right)$$

$$- \psi\left(\int_0^{\frac{1}{\Upsilon(T\zeta, T\eta, f)}-1} \phi(s)\,ds\right), \tag{3.7.1}$$

for all $\zeta, \eta \in \Sigma$ and $t > 0$, where $\phi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function which is summable on each compact subset of $[0, \infty)$ and such that for all $\in > 0$, $\int_0^{\in} \phi(s) ds$.

Theorem 3.8. *Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \Diamond)$ be a NMS. Let T and f be self-mappings on Σ such that the range of f contains the range of $T(T\Sigma \subseteq f\Sigma)$ and $f(\Sigma)$ or $T(\Sigma)$ is a complete subset of Σ and $\alpha : X \times X \times (0, \infty) \to [0, \infty)$.*

Suppose that T is Neutrosophic $\alpha - (\phi, \psi)$ *weak contraction of integral type with respect to f and the following conditions hold*;

(i) T *is* $f - \alpha -$ *admissible*;

(ii) *There exists* $\zeta_0 \in \Sigma$ *such that* $\alpha(f(\zeta_0), T(\zeta_0), t) \geq 1$ *for all* $t > 0$;

(iii) T *is continuous*.

Then T and f have a coincidence point in Σ. *If T and f are weakly compatible*, *then T and f have a unique common fixed point in* Σ.

Proof. Define $\Gamma : [0, \infty) \to [0, \infty)$ by $\Gamma = \phi\int_0^{\Sigma} \phi(s)ds.$ So, condition (3.7.1) reduces to condition (3.3.1) and condition (3.7.1) reduces to condition (3.3.1) as $\phi \circ \Gamma$ is an altering distance function and $\psi \circ \Gamma : [0, \infty) \to [0, \infty)$ with $\psi(\Gamma(r)) > 0$ for $r > 0$ and $\psi(\Gamma(0)) = 0$. Therefore, the conclusion follows immediately by Theorem (3.5).

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