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FUZZY CONGRUENCE RELATIONS AND FUZZY FILTERS IN GENERALIZED ALMOST DISTRIBUTIVE FUZZY LATTICES

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Abstract

The relationship between Fuzzy Congruence Relations (FCR's) and Fuzzy Filters (FF's) in Generalized Almost Distributive Fuzzy Lattices is the subject of this research (GADFL). In GADFL, some theorems for FCR's and FF's underpinning the structure of distributive fuzzy lattices are derived. GADFL also discusses a characterization theorem for FCR's and FF's.

1. Introduction

Rao, Ravi Kumar, and Rafi [1] proposed the concept of a Generalized Almost Distributive Lattice (GADL) as a generalisation of an Almost Distributive Lattice (ADL). [2] Except for maybe the right distributivity of any of the binary operations or and the commutativity of $\wedge, \vee$ over the other the GADLs inherit essentially all of the properties of the distributive lattice. GADLs are properly classified as ADLs and retain several of the ADL's key characteristics. L. A. Zadeh [3] examined the fuzzy set concept to mathematically represent ambiguity in its most abstract form and also in [4] a generalisation of the fuzzy ordering, i.e., a transitive fuzzy relation is a

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fuzzy ordering. Chon [5] proposed a novel researched the fuzzy lattices [6]. A fuzzy congruence relation [7] on generic algebraic equivalency relation that is consistent with all of the algebra's fundamental operations [8]. Recent research on fuzzy sub algebraic structures includes fuzzy ideals and fuzzy filters [9-10], various generalisations of FI's in distributive lattices [11-13], FI's and FF's of partially ordered sets [14, 15], and FI's of universal algebras [16-17]. In GADFL, some theorems for FCR's and FF's underpinning the structure of distributive fuzzy lattices are derived. GADFL also discusses a characterization theorem.

2. Preliminaries

Definition 2.1 [18]**.** Let $L(R, \vee, \wedge)$ be an algebra type and $L(R, A)$ be a fuzzy poset. Then we call $L(R, A)$ is a GADFL if it satisfies the following axioms.

1. $A((a \wedge b) \wedge c, a \wedge (b \wedge c)) = A(a \wedge (b \wedge c), (a \wedge b) \wedge c) = 1$
2. $A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1$
3. $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = 1$
4. $A(a \wedge (a \vee b), a) = A(a, a \wedge (a \vee b)) = 1$
5. $A((a \vee b) \wedge a, a) = A(a, (a \vee b) \wedge a) = 1$
6. $A((a \wedge b) \vee b, b) = A(b, (a \wedge b) \vee b) = 1$ for all $a, b, c \in R$.

Definition 2.2. An equivalence relation θ on (R, A) is a called fuzzy congruence relations (FCR's) on (R, A), if the following are satisfied.

1. $\theta(x, x) = 1$ for all $x \in R$
2. $\theta(x, x) = \theta(y, x)$ for all $x, y \in R$
3. $\theta(x, z) \geq \theta(x, y) \wedge \theta(y, z)$ for all $x, y, z \in R$
4. $\theta(x \vee z, y \vee w) \wedge \theta(x \wedge z, y \wedge w) \geq \theta(x, y) \wedge \theta(z, w)$ for all $x, y, z, w \in R$.

We denote the set of all FCR's on for all $L(R, A)$.

Definition 2.3. A fuzzy subset is a called fuzzy filter (FF's) of (R, A), if

1. $A(1) = 1$ and

2. $A(x \wedge y) = A(x) \vee A(y)$, for all $x, y \in R$.

3. Fuzzy Congruence Relations (FCR's) and Fuzzy Filters (FF's) in GADFL

Definition 3.1. Let $L(R, A)$ be a GADFL. An equivalence θ relation on $L(R, A)$ is called a FCR's θ on $L(R, A)$ if, for $x, y, z, w \in R$ holds.

$$\theta((x, y) \wedge (z, w)) \geq \theta((x \vee z), (y \vee w) \wedge (x \wedge z), (y \wedge w)).$$

Definition 3.2. A fuzzy subset of a GADFL $L(R, A)$ is called a FF's of $L(R, A)$ if

1. $\mu_\theta(1) = 1$

2. $\mu_\theta(x) = \sup\{\theta(a \vee x, x) = \theta(x, a \vee x) = 1; a \in R\}$, for all $x \in R$.

Example 3.1. Let $R = \{0, a, b, c\}$ and let $\vee$ and $\wedge$ be binary operations on a GADFL $L(R, A)$.

Then $(R, \vee, \wedge, 0)$ is a GADFL (a discrete GADFL).

Now, μ_θ of $L(R, A)$ by $\mu_\theta(1)$, $\mu_\theta(a) = \mu_\theta(b) = 0.8$ and 0.6.

Thus μ_θ is a fuzzy filter of $L(R, A)$.

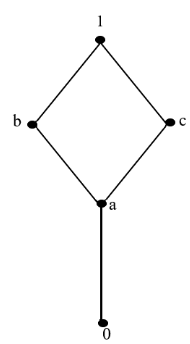

Figure 1. Hasse diagram, discrete GADFL $R = \{0, a, b, c\}$.

Fuzzy relation θ on $L(R, A)$.

$\theta(0, 0) = \theta(a, a) = \theta(b, b) = \theta(c, c) = 1,$

$\theta(0, c) = \theta(c, 0) = 0.6$

$\theta(a, b) = \theta(b, a) = \theta(a, c) = \theta(c, a) = \theta(b, c) = \theta(c, b)$

$= \theta(0, a) = \theta(a, 0) = \theta(0, b) = \theta(b, 0) = 0.8$

Thus θ is a FCR's on GADFL $L(R, A)$.

Theorem 3.1. *In GADFL, '$L(R, A)$' let be a FCR's. Then* μ_θ *is a FF's of* $L(R, A)$.

Proof of the theorem 3.1

The is the mapping defined by and is defined by $\theta : A \to B$ for all $\mu_\theta(a)$ Therefore, we get $\mu_\theta(1) = \theta(1, 0) = 1.$ For any $x, y \in \theta$ this implies,

$\mu_\theta(x \wedge y) = \theta((x \wedge y), 0)$

$= \sup\{\theta(x, a) \vee \theta(y, b) \mid a, b \in A\}$

$\mu_\theta(x \wedge y) \leq \{\theta(x, a) \vee \theta(y, b) \mid a, b \in A\}$

Assume $b = 1,$ we get,

$\mu_\theta(x \wedge y) \leq \{\theta(x, a) \vee \theta(y, 1)\}, \mu_\theta(x \wedge y) \leq 1 \wedge \mu_\theta(y), \mu_\theta(x \wedge y) \leq \mu_\theta(y)$

Similarly, we find

$\mu_\theta(x \vee y) = \theta((x \wedge y), 0)$

$= \sup\{\theta(x, a) \vee \theta(y, b) \mid a, b \in A\}.$ Assume $a = 1,$ we get,

$\mu_\theta(x \vee y) \leq \theta\{(x \wedge 1) \vee \theta(y, b)\}$

$\mu_\theta(x \vee y) \leq \mu_\theta(x, 1) \vee 1, \mu_\theta(x \vee y) \leq \mu_\theta(x, 1), \mu_\theta(x \vee y) \leq \mu_\theta(x)$

Therefore, we get $\mu_\theta(x \vee y) \leq \mu_\theta(x) \vee \mu_\theta(y).$ This implies $\mu_\theta(a)$ is a FF's of GADFL '$L(R, A)$.' Hence Proved.

Theorem 3.2. *Let* θ *be a FCR's on GADFL* $L(R, A)$. *A fuzzy subset* $\vartheta_\theta(x) = \sup\{\theta(a \vee x, x) = \theta(x, a \vee x) = 1 : a \in R\}$, *for all* $x \in R$ *is a FF's on GADFL* $L(R, A)$.

Proof of the theorem 3.2

$\vartheta_\theta(1) = \sup\{\theta(a \vee 1, 1) = \theta(1, a \vee 1) = 1 : a \in R\} = \theta(1, 1) = 1$

For any $x, y \in R$, consider

$\vartheta_\theta(x \wedge y) = \sup\{\theta(a \vee (a \vee y), (x \wedge y) = \theta((x \wedge y), a \vee (x \wedge y)) = 1; a \in R)\}$,

$= \sup\{\theta[(a \vee x) \wedge (a \vee y), (x \wedge y)] = \theta[(x \wedge y), (a \vee x) \wedge (a \vee y)] = 1 : a \in R\}$

$\leq \sup\{\theta[(a \vee x, x) \vee (a \vee y, y)] = \theta[(x, (a \vee x) \vee (y, a \vee y)] = 1 : a \in R\}$

$= \vartheta_\theta(x) \vee \vartheta_\theta(y)$. Also consider,

$\leq \sup\{\theta[(a \vee x, x) \vee (a \vee y, y)] = \theta[(y, y) \vee (a \vee x, x)] = 1 : a \in R\}$

$= \vartheta_\theta(x)$

In the similar fashion $\vartheta(x \vee y) \leq \vartheta_\theta(y)$.

Then $\vartheta(x \vee y) \leq \vartheta_\theta(x) \wedge \vartheta_\theta(y)$. Thus ϑ_θ is a FF's of GADFL $L(R, A)$.

Theorem 3.3. *Let* θ *be a FCR's on GADFL* $L(R, A)$. *Then* $\mu_\theta = \vartheta_\theta$.

Proof of the theorem 3.3.

For any FCR's on GADFL $L(R, A)$. We claim to show that $\mu_\theta = \vartheta_\theta$. For any $x \in R$, we have,

$\vartheta_\theta(x) = \sup\{\theta(a \vee x, x) = \theta(x, a \vee x) = 1 : a \in R\}$

Then $\vartheta_\theta(x) \geq \{\theta(a \vee x, x) = \theta(x, a \vee x) = 1\}$ for all $a \in R$.

In particular for $a = 1$.

$\vartheta_\theta(x) \geq \{\theta(1 \wedge x, x) = \theta(x, 1 \wedge x) = 1\}$

$= \theta(x, 1) = 1 = \mu_\theta(x)$. Therefore $\vartheta_\theta(x) \geq \mu_\theta(x)$ (1)

On the other hand, for any $a \in R$, consider,

$\{\theta(a \vee x, x) = \theta(x, a \vee x) = 1\} = \theta[a \vee x, (a \vee x) \wedge x] = \theta[(a \vee x) \wedge x, a \vee x] = 1$

Thus for all $\{\theta(a \vee x, x) = \theta(x, a \vee x) = 1\} \leq \mu_\theta(x)$ for all $a \in R$.

So $\vartheta_\theta(x) = \sup\{\theta(a \vee x, x) = \theta(a \vee x, x) = 1 : a \in R\}$

$\leq \mu_\theta(x)$

Therefore $\vartheta_\theta(x) \leq \mu_\theta(x)$ (2)

From (1) and (2) we get $\vartheta_\theta(x) \leq \mu_\theta(x)$ Hence $\mu_\theta = \vartheta_\theta$. Hence the proved.

Theorem 3.4. *Let μ_θ be a FF of a GADFL $L(R, A)$. Then the relation* $\varphi^{\mu\theta} = \{(x, y) \in R \times R \mid \theta(a \wedge x, a \wedge y) = \theta(a \wedge y, a \wedge x = 1)$ *for some* $a \in \mu_\theta\}$ *is a congruence relation on* $L(R, A)$.

Proof of the theorem 3.4

Clearly $\varphi^{\mu\theta}$ is an equivalence relation on $L(R, A)$.

Now, let $(x, y), (p, q) \in \varphi^{\mu\theta}$ then, $\theta(a \wedge x, a \wedge y) = \theta(a \wedge y, a \wedge x)$ and $\theta(b \wedge p, b \wedge q) = \theta(b \wedge q, b \wedge p) = 1$

For some $a, b \in \mu_\theta$. Hence $a, b \in \mu_\theta \Rightarrow a \wedge b \in \mu_\theta$ and

$\theta(a \wedge b \wedge x \wedge p, a \wedge b \wedge y \wedge q) = \theta(a \wedge x \wedge b \wedge p, a \wedge b \wedge y \wedge q)$

$= \theta(a \wedge y \wedge b \wedge p, a \wedge b \wedge y \wedge q) = \theta(a \wedge y \wedge b \wedge q, a \wedge b \wedge y \wedge q)$

$= \theta(a \wedge b \wedge y \wedge q, a \wedge b \wedge y \wedge q) = 1$

Similarly, $= \theta(a \wedge b \wedge y \wedge q, a \wedge b \wedge x \wedge p) = 1$.

Therefore $(x \wedge p, y \wedge q) \in \varphi^{\mu\theta}$, Also,

$\theta((a \wedge b) \wedge (x \vee p), (a \wedge b) \wedge (y \vee q)) = \theta((a \wedge b) \wedge (x \vee p), (a \wedge b) \wedge (y \vee q)) = 1$

Similarly, $\theta((a \wedge b) \wedge (y \vee q), (a \wedge b) \wedge (x \vee p)) = 1$

Therefore $(x \vee p, y \vee q) \in \varphi^{\mu\theta}$ and hence $\varphi^{\mu\theta}$ is a FCR's on $L(R, A)$.

Finally, the following theorem in GADFL provides characterisation of FCR's and FF's.

Theorem 3.5. *Let be a FCR's on GADFL $L(R, A)$. Then the following are equivalent.*

1. $L(R, A)$ *is a GADFL.*

2. *For any FF* V_θ *of* φV_θ *is a FCR's on* $L(R, A)$.

3. φa_θ *is a FCR's on* $L(R, A)$ $\forall a \in R$.

Proof of the theorem 3.5.

Assume (1), Let V_θ be a FF of $L(R, A)$. Clearly, φV_θ is an equivalence relation

Let $(a, a) \in \varphi V_\theta$. Then $\theta(a \wedge x, a \wedge x) = 1$ and $\theta(a \wedge y, a \wedge y) = 1$ for some, $y \in V_\theta$.

Let $(a, b) \in \varphi V_\theta$. Then $\theta(a \wedge x, b \wedge x) = 1$ and $\theta(a \wedge y, b \wedge y) = 1$ for some $x, y \in V_\theta$.

Therefore $\theta(a, b) = \theta(b, a)$ $\forall$ $a, b \in \varphi V_\theta$. Let $(a, b, c) \in \varphi V_\theta$.

Then $\theta(a \wedge x, c \wedge x) \leq \theta(a \wedge x, b \wedge x) \wedge \theta(b \wedge x, c \wedge x) = 1$ and $\theta(a \wedge x, c \wedge y)$ $\leq \theta(a \wedge y, b \wedge y) \wedge \theta(b \wedge y, c \wedge y) = 1$ for some $x, y, \in V_\theta$.

Therefore $\theta(a, c) \leq (a, b) \wedge (b, c)$ $\forall$ $a, b, \in \varphi V_\theta$. Let $(a, b), (c, d) \in \varphi V_\theta$. Then $\theta(a \wedge x, b \wedge x) = \theta(b \wedge x, a \wedge x) = 1$ and $\theta(c \wedge y, d \wedge y) = \theta(d \wedge y, c \wedge y) = 1$ for some $x, y, \in V_\theta$.

Since V_θ is a fuzzy ideal of $L(R, A)$, $x, y \in V_\theta$

Now,

$$\theta((a \wedge c) \wedge x \wedge y, (b \wedge d) \wedge x \wedge y) = \theta((b \wedge d) \wedge x \wedge y, (b \wedge d) \wedge x \wedge y) = 1$$

Similarly, $\theta((b \wedge d) \wedge x \wedge y, (a \wedge c) \wedge x \wedge y) = 1$ and hence $(a \wedge c, b \wedge d)$ $\in \varphi V_\theta$.

Also,

$$\theta((a \vee c) \wedge x \wedge y, (b \wedge d) \wedge x \wedge y) = \theta((a \wedge x \wedge y) \vee (c \wedge x \wedge y), (b \vee d) \wedge x \wedge y)$$

$= \theta((b \wedge x \wedge y) \vee (x \wedge c \wedge y), (b \vee d) \wedge x \wedge y)$

$= \theta([b \wedge (x \wedge y)] \vee [(x \wedge d) \wedge y], (b \vee d) \wedge x \wedge y)$

$= \theta((b \vee d) \wedge x \wedge y, (b \vee d) \wedge x \wedge y) = 1$

Similarly, $\theta((b \vee d) \wedge x \wedge y, (a \vee c) \wedge x \wedge y) = 1$

Therefore, $(a \vee c, b \vee d) \in \varphi V_\theta$.

Thus φV_θ is a FCR's on $L(R, A)$.

(2) $\Rightarrow$ (3) It is obvious (3) $\Rightarrow$ (1). Assume (3) Let $a, b \in R$.

Since $\theta(a \vee b, (a \vee b) \vee b) = \theta((a \vee b) \vee b, a \vee b) = 1$. Then $(a, a \vee b) \in \varphi a_\theta$.

Also $\theta(b \vee b, b \vee d) = 1$. Hence $(b, b) \in \varphi a_\theta$. Since φa_θ is a FCR's on $L(R, A)$, $(a \wedge b, (a \vee b) \wedge b) \in \varphi a_\theta$.

Hence $\theta((a \wedge b) \vee b, [(a \vee b) \wedge b] \vee b) = \theta([(a \vee b) \wedge b] \vee b, (a \wedge b) \vee b) = 1$

$\Rightarrow \theta((a \wedge b) \vee b, b \vee b) = \theta(b \vee b, (a \wedge b) \vee b) = 1$

$\Rightarrow \theta((a \wedge b) \vee b, b) > 0$ and $\theta(b, (a \wedge b) \vee b) > 0$. Therefore $L(R, A)$ is a GADFL.

Conclusion

Many definitions of FF's and FCR's of a GADFL's are examined in this study. We also discovered filters that may or may not be FF's. In addition, we proved the theorems of FCR's and FF's using a distributive fuzzy lattice. Finally, a characterization theorem for FCR's and created FF's for GADFL. Fuzzy congruence relations, as well as some of the ideals and filters of GADFL, will be the focus of our future research.

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