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ON FUZZY FRACTION DENSE SPACES

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Abstract

In this paper, the notion of fuzzy fraction dense space is introduced. Several characterizations of fuzzy fraction dense spaces are established. It is established that fuzzy closed sets are not fuzzy nowhere dense sets and fuzzy regular closed sets are fuzzy F_σ-sets in fuzzy fraction dense and fuzzy P-spaces. It is obtained that fuzzy fraction dense spaces are not fuzzy hyper connected and fuzzy nodec spaces and fuzzy fraction dense and fuzzy DG_δ-spaces are fuzzy Baire, fuzzy σ-Baire and fuzzy Volterra spaces. A condition which ensures the existence of fuzzy σ-Baireness in fuzzy fraction dense spaces is obtained.

1. Introduction

The concept of fuzzy set as a new approach for modelling uncertainties was introduced by L. A. Zadeh [20] in 1965. The concept of fuzzy topological spaces was introduced by C. L. Chang [3] in 1968. Chang's works paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. The concept of fraction dense spaces was introduced by A. W. Hager and J. Martinez [6] and the same was studied as "cozero approximated spaces" by Gary Gruenhage [4].

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Keywords: fuzzy G_δ-set, fuzzy F_σ-set, fuzzy simply open set, fuzzy regular closed set, fuzzy Baire space, fuzzy σ-Baire space, fuzzy P-space, fuzzy DG_δ-space, fuzzy ∂-space, fuzzy hyper connected space, fuzzy Volterra space.

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In this paper, the notion of fuzzy fraction dense space is introduced. Several characterizations of fuzzy fraction dense spaces are established. It is established that fuzzy closed sets are not fuzzy nowhere dense sets and fuzzy open sets are not fuzzy dense sets in fuzzy fraction dense spaces. It is obtained that fuzzy regular closed sets are fuzzy F_σ-sets and fuzzy regular open sets are fuzzy G_δ-sets in fuzzy fraction dense and fuzzy P-spaces. Also it is established that fuzzy F_σ-sets are fuzzy somewhere dense sets and fuzzy G_δ-sets are fuzzy cs dense sets in fuzzy fraction dense and fuzzy P-spaces. It is obtained that fuzzy fraction dense spaces are not fuzzy hyper connected and fuzzy nodec spaces. In fuzzy fraction dense and fuzzy DG_δ-spaces, fuzzy nowhere dense sets become fuzzy F_σ-sets and fuzzy first category sets.

Fuzzy fraction dense and fuzzy DG_δ-spaces are found to be fuzzy Baire, fuzzy σ-Baire and fuzzy Volterra spaces. A condition which ensures the existence of fuzzy σ-Baireness in fuzzy fraction dense spaces is obtained by means of fuzzy boundary of fuzzy simply open sets. Also a condition which ensures the existence of fuzzy σ-Baireness in fuzzy fraction dense and fuzzy ∂-spaces is obtained by means of the fuzzy boundary of fuzzy G_δ-sets. It is established that fuzzy fraction dense and fuzzy G_δ-spaces are fuzzy nodef spaces and are not fuzzy P-spaces. It is found that fuzzy fraction dense spaces are not fuzzy nodec spaces. A condition under which fuzzy globally disconnected spaces become fuzzy fraction dense spaces is also obtained in this paper.

2. Preliminaries

Some basic notions and results used in the sequel, are given in order to make the exposition self-contained. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). Let X be a non-empty set and I the unit interval [0, 1]. A fuzzy set λ in X is a mapping from X into I. The fuzzy set 0_X is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set 1_X is defined as $1_X(x) = 1$, for all $x \in X$.

Definition 2.1 [3]. Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). The interior, the closure and the complement of λ are defined respectively as follows:

(i) $\text{int}(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$

(ii) $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

(iii) $\lambda'(x) = 1 - \lambda(x)$, for all $x \in X$.

For a family $\{\lambda_i / i \in I\}$ of fuzzy sets in (X, T), the union $\Psi = \vee_i(\lambda_i)$ and

The intersection $\delta = \wedge_i(\lambda_i)$, are defined respectively as

(iv) $\Psi(x) = \sup_i\{\lambda_i(x)/x \in X\}$.

(v) $\delta(x) = \inf_i\{\lambda_i(x)/x \in X\}$.

Lemma 2.1 [1]. *For a fuzzy set* λ *of a fuzzy topological space X,*

(i) $1 - \text{int}(\lambda) = cl(1 - \lambda)$, *and* (ii) $1 - cl(\lambda) = \text{int}(1 - \lambda)$.

Definition 2.2. A fuzzy set λ in a fuzzy topological space (X, T) is called a

(1) fuzzy regular - open if $\lambda = \text{int}\,cl(\lambda)$ and fuzzy regular-closed if $\lambda = cl\,\text{int}(\lambda)$ [1].

(2) fuzzy G_δ-set if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ where $\lambda_i \in T$ for $i \in I$ [2].

(3) fuzzy F_σ-set if $\lambda = \vee_{i=1}^{\infty}(\lambda_k)$, where $1 - \lambda_i \in T$ for $i \in I$ [2].

(4) fuzzy semi-open if $\lambda \leq cl\,\text{int}(\lambda)$ and fuzzy semi-closed if $\text{int}\,cl(\lambda) \leq \lambda$ [1].

Definition 2.3. A fuzzy set λ in a fuzzy topological space (X, T), is called a

(i) fuzzy dense set if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$. That is, $cl(\lambda) = 1$, in (X, T) [9].

(ii) fuzzy nowhere dense set if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $\text{int}\,cl(\lambda) = 0$, in (X, T) [9].

(iii) fuzzy first category set if $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category [9].

(iv) fuzzy residual set if $1 - \lambda$ is a fuzzy first category set in (X, T) [11].

(v) fuzzy somewhere dense set if there exists a non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $\operatorname{int} cl(\lambda) \neq 0$, in (X, T) [10] and $1 - \lambda$ is called a fuzzy complement of fuzzy somewhere dense set in (X, T) and is denoted as fuzzy cs dense set in (X, T) [19].

(vi) fuzzy σ-nowhere dense set if λ is a fuzzy F_σ-set with $\operatorname{int}(\lambda) = 0$, in (X, T) [13].

(vii) fuzzy simply open set if $Bd(\lambda)$ is a fuzzy nowhere dense set in (X, T).

That is, λ is a fuzzy simply open set in (X, T) if $[cl(\lambda) \wedge cl(1 - \lambda)]$, is a fuzzy nowhere dense set in (X, T) [15].

Definition 2.4. A fuzzy topological space (X, T) is called a

(i) fuzzy P-space if each fuzzy G_δ-set in (X, T) is fuzzy open in (X, T) [8]

(ii) fuzzy Baire space if $\operatorname{int}(\vee_{i=1}^{\infty} (\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T) [11].

(iii) fuzzy σ-Baire space if $\operatorname{int}(\vee_{i=1}^{\infty} (\lambda_i)) = 0$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T) [13].

(iv) fuzzy Volterra space if $cl(\wedge_{i=1}^{N} (\lambda_i)) = 1$, where (λ_i)'s are fuzzy dense and fuzzy G_δ-sets in (X, T) [14].

(v) fuzzy hyper connected space if every non-null fuzzy open subset of (X, T) is fuzzy dense in (X, T) [7].

(vi) fuzzy ∂-space if each fuzzy G_δ-set is a fuzzy simply open set in

(X, T) [18].

(vii) fuzzy DG_δ-space if each fuzzy dense (but not fuzzy open) set in (X, T) is a fuzzy G_δ-set in (X, T) [17].

(viii) fuzzy globally disconnected space if each fuzzy semi-open set in (X, T) is fuzzy open in (X, T) [16].

(ix) fuzzy nodef space if each fuzzy nowhere dense set is a fuzzy F_σ-set in (X, T) [17].

(x) fuzzy nodec space if each fuzzy nowhere dense set is a fuzzy closed set in (X, T) [12].

Theorem 2.1 [15]. *If* λ *is a fuzzy simply open set in a fuzzy topological space* (X, T), *then* $\lambda \wedge (1 - \lambda)$ *is a fuzzy nowhere dense set in* (X, T).

Theorem 2.2 [1]. *In a fuzzy topological space*,

(a) *The closure of a fuzzy open set is a fuzzy regular closed set*.

(b) *The interior of a fuzzy closed set is a fuzzy regular open set*.

Theorem 2.3 [19]. *If* λ *is a fuzzy somewhere dense set in a fuzzy topological space* (X, T), *then there exists a fuzzy regular closed set* η *in* (X, T) *such that* $\eta \leq cl(\lambda)$.

Theorem 2.4 [17]. *If* λ *is a fuzzy nowhere dense* (*but not fuzzy closed*) *set in a fuzzy* DG_δ*-space* (X, T), *then* λ *is a fuzzy* F_σ*-set in* (X, T).

Theorem 2.5 [14]. *If each fuzzy nowhere dense set is a fuzzy* F_σ*-set in a fuzzy topological space* (X, T), *then* (X, T) *is a fuzzy Volterra space*.

Theorem 2.6 [14]. *If each fuzzy nowhere dense set is a fuzzy* F_σ*-set in a fuzzy topological space* (X, T), *then* (X, T) *is a fuzzy Baire space*.

Theorem 2.7 [13]. *If* λ *is a fuzzy* σ*-nowhere dense set in a fuzzy topological space* (X, T), *then* $1 - \lambda$ *is a fuzzy residual set in* (X, T).

Theorem 2.8 [17]. *If* λ *is a fuzzy dense and fuzzy open set in a fuzzy nodef space* (X, T), *then* λ *is a fuzzy residual set in* (X, T).

Theorem 2.9 [17]. *If a fuzzy topological space (X, T) is a fuzzy nodef and fuzzy P-space, then (X, T) is a fuzzy nodec space.*

Theorem 2.10 [16]. *If λ is a fuzzy first category set in a fuzzy globally disconnected space (X, T), then λ is a fuzzy F_σ-set in (X, T).*

3. Fuzzy Fraction Dense Spaces

Motivated by the works of Hager and Martinez [6] and Gary Gruenhage [4], the concept of fuzzy fraction dense space is introduced and studied in this paper.

Definition 3.1. A fuzzy topological space (X, T) is called a fuzzy fraction dense space if for each fuzzy open set λ in (X, T), $cl(\lambda) = cl(\mu)$, where μ is a fuzzy F_σ-set in (X, T).

That is, (X, T) is a fuzzy fraction dense space if for each fuzzy open set λ in (X, T), $cl(\lambda) = cl(\vee_{i=1}^{\infty} (\mu_i))$, where $1 - \mu_i \in T$.

Example 3.1. Let $X = \{a, b, c\}$. Let $I = [0, 1]$. The fuzzy sets α, β and γ are defined on X as follows:

$\alpha : X \to I$ is defined by $\alpha(a) = 0.5, \alpha(b) = 0.6, \alpha(c) = 0.4,$

$\beta : X \to I$ is defined by $\beta(a) = 0.4, \beta(b) = 0.5, \beta(c) = 0.6,$

$\gamma : X \to I$ is defined by $\gamma(a) = 0.6, \gamma(b) = 0.4, \gamma(c) = 0.5.$

Then, $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \beta \vee [\alpha \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \alpha \wedge [\beta \vee \gamma], \beta \wedge [\alpha \vee \gamma], \gamma \wedge [\alpha \vee \beta], (\alpha \wedge \beta \wedge \gamma), (\alpha \vee \beta \vee \gamma), 1\}$ is a fuzzy topology on X.

By computation one can find that the fuzzy F_σ-sets in (X, T) are $\alpha \vee \beta, \alpha \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \beta \vee [\alpha \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \alpha \wedge [\beta \vee \gamma], \beta \wedge [\alpha \vee \gamma], \gamma \wedge [\alpha \vee \beta], (\alpha \wedge \beta \wedge \gamma)$ and $(\alpha \vee \beta \vee \gamma)$. Also one can find that

$cl(\alpha) = 1 - [\gamma \wedge [\alpha \vee \beta]] = cl(1 - [\gamma \wedge [\alpha \vee \beta]]) = cl(\alpha \vee [\beta \wedge \gamma]),$

$cl(\beta) = 1 - [\alpha \wedge [\beta \vee \gamma]] = cl(1 - [\alpha \wedge [\beta \vee \gamma]]) = cl(\beta \vee [\alpha \wedge \gamma]),$

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$cl(\alpha \wedge \beta \wedge \gamma) = 1 - [\alpha \vee \beta \vee] = cl(1 - (\alpha \vee \beta \vee \gamma)) = cl(\alpha \wedge \beta \wedge \gamma),$

$cl(\alpha \vee \beta \vee \gamma) = 1 - [\alpha \wedge \beta \wedge \gamma] = cl(1 - (\alpha \wedge \beta \wedge \gamma)) = cl(\alpha \vee \beta \vee \gamma).$

Hence (X, T) is a fuzzy fraction dense space.

Example 3.2. Let $X = \{a, b, c\}$. Let $I = [0, 1]$. The fuzzy sets α, β and γ are defined on X as follows:

$\alpha : X \to I$ is defined by $\alpha(a) = 0.4,\ \alpha(b) = 0.6,\ \alpha(c) = 0.4,$

$\beta : X \to I$ is defined by $\beta(a) = 0.6,\ \beta(b) = 0.4,\ \beta(c) = 0.6,$

$\gamma : X \to I$ is defined by $\gamma(a) = 0.5,\ \gamma(b) = 0.5,\ \gamma(c) = 0.5.$

Then, $T_1 = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, 1\}$ is a fuzzy topology on X. By computation one can find that the fuzzy F_σ-sets in (X, T) are $1 - \beta, 1 - \gamma, 1 - (\alpha \wedge \beta), 1 - (\alpha \wedge \gamma)$ and $1 - (\beta \wedge \gamma)$. Now for the fuzzy open set β, $cl(\beta) = 1 - \alpha \neq cl(\theta)$, where $\theta = 1 - \beta, 1 - \gamma, 1 - (\alpha \wedge \beta), 1 - (\alpha \wedge \gamma),$

$1-(\beta \wedge \gamma)$. Hence (X, T_1) is not a fuzzy fraction dense space.

Proposition 3.1. *A fuzzy topological space (X, T) is a fuzzy fraction dense space if and only if for each fuzzy regular closed set μ in (X, T), $\mu = cl(\eta)$, where η is a fuzzy F_σ-set in (X, T).*

Proof. Let μ be a fuzzy regular closed set in (X, T). Then, $cl\operatorname{int}(\mu) = \mu$, in (X, T). Let $\lambda = \operatorname{int}(\mu)$. Then, λ is a fuzzy open set in (X, T). Since (X, T) is a fuzzy fraction dense space, $cl(\lambda) = cl(\eta)$, where η is a fuzzy F_σ-set in (X, T). Thus, $\mu = cl\operatorname{int}(\mu) = cl(\lambda) = cl(\eta)$ and $\mu = cl(\lambda)$, in (X, T).

Conversely, let δ be a fuzzy open set in (X, T). Then, $cl(\delta)$ is a fuzzy regular closed set in (X, T) (Theorem 2.2). By hypothesis, $cl(\delta) = cl(\eta)$, where λ is a fuzzy F_σ-set in (X, T) and thus (X, T) is a fuzzy fraction dense space.

Proposition 3.2. *If (X, T) is a fuzzy fraction dense space and δ is a fuzzy regular open set in (X, T), then $\delta = \operatorname{int}(\theta)$, where θ is a fuzzy G_δ-set in (X, T).*

Proof. Let δ be a fuzzy regular open set in (X, T). Then, $1-\delta$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.1, $1-\delta = cl(\lambda)$, where λ is a fuzzy F_σ-set in (X, T). Then, $\delta = 1 - cl(\lambda) = \operatorname{int}(1-\lambda)$ [2.1]. Let $\theta = 1-\lambda$ and then θ is a fuzzy G_δ-set in (X, T). Hence $\delta = \operatorname{int}(\theta)$, where θ is a fuzzy G_δ-set in (X, T).

Proposition 3.3. *If λ is a fuzzy open set in a fuzzy fraction dense space (X, T), then there exists a fuzzy F_σ-set μ in (X, T) such that $\mu \leq cl(\lambda)$.*

Proof. Let λ be a fuzzy open set in (X, T). Since (X, T) is a fuzzy fraction dense space, $cl(\lambda) = cl(\eta)$, where μ is a fuzzy F_σ-set in (X, T). Now $\mu \leq cl(\lambda)$, implies that $\mu \leq cl(\eta)$.

Proposition 3.4. *If λ is a fuzzy open set in a fuzzy fraction dense space (X, T), then there exists a fuzzy F_σ-set η in (X, T) such that $\lambda \leq cl(\eta)$.*

Proof. Let λ be a fuzzy open set in (X, T). Now $\lambda \leq cl(\lambda)$ in (X, T).

Since (X, T) is a fuzzy fraction dense space, $cl(\lambda) = cl(\mu)$, where λ is a fuzzy F_{σ}-set in (X, T). Thus there exists a fuzzy F_{σ}-set λ in (X, T) such that $\lambda \leq cl(\lambda)$.

Proposition 3.5. *If λ is a fuzzy open set in a fuzzy fraction dense space (X, T), then there exist fuzzy F_{σ}-sets μ and η in (X, T) such that $\mu \leq cl(\lambda) \leq cl(\eta)$.*

Proof. Let λ be a fuzzy open set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.3, there exists a fuzzy F_{σ}-set μ in (X, T) such that $\mu \leq cl(\lambda)$. Also, by Proposition 3.4, there exists a fuzzy F_{σ}-set λ in (X, T) such that $\lambda \leq cl(\eta)$. Then, $\mu \leq cl(\lambda) \leq cl(cl(\eta))$. Since $cl(cl(\eta)) = cl(\eta)$ in (X, T), for a fuzzy open set λ in (X, T), $\mu \leq cl(\lambda) \leq cl(\lambda\eta)$.

Proposition 3.6. *If δ is a fuzzy closed set in a fuzzy fraction dense space (X, T), then there exists a fuzzy G_{δ}-set θ in (X, T) such that $\operatorname{int}(\delta) \leq \theta$.*

Proof. Let δ be a fuzzy closed set in (X, T). Then, $1 - \delta$ is a fuzzy open set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.3, there exists a fuzzy F_{σ}-set μ in (X, T) such that $\mu \leq cl(1 - \delta)$. Then, $\mu \leq 1 - \operatorname{int}(\delta)$ [2.1]. This implies that $\operatorname{int}(\delta) \leq 1 - \mu$. Let $\theta = 1 - \mu$. Then θ is a fuzzy G_{δ}-set and $\operatorname{int}(\delta) \leq \theta$, in (X, T).

Proposition 3.7. *If δ is a fuzzy closed set in a fuzzy fraction dense space (X, T), then there exists a fuzzy G_{δ}-set ρ in (X, T) such that $\operatorname{int}(\rho) \leq \delta$.*

Proof. Let δ be a fuzzy closed set in (X, T). Then, $1 - \delta$ is a fuzzy open set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.4, there exists a fuzzy F_{σ}-set η in (X, T) such that $1 - \delta \leq cl(\lambda)$. Then, $1 - cl(\lambda) \leq \delta$ and [2.1] $\operatorname{int}(1 - \lambda) = 1 - cl(\lambda) \leq \delta$. Let $\rho = 1 - \lambda$ and ρ is a fuzzy G_{δ}-set in (X, T) and $\operatorname{int}(\rho) \leq \delta$ in (X, T).

Proposition 3.8. *If δ is a fuzzy closed set in a fuzzy fraction dense space (X, T), then there exist fuzzy G_{δ}-sets ρ and θ in (X, T) such that $\operatorname{int}(\rho) \leq \operatorname{int}(\delta) \leq \theta$, in (X, T).*

Proof. Let δ be a fuzzy closed set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.6, there exists a fuzzy G_δ-set θ in (X, T) such that $\text{int}(\delta) \leq \theta$. Also, by Proposition 3.7, there exists a fuzzy G_δ-set ρ in (X, T) such that $\text{int}(\rho) \leq \delta$. Then, $\text{int}(\text{int}(\rho)) \leq \text{int}(\delta) \leq \theta$, in (X, T). This implies that $\text{int}(\rho) \leq \text{int}(\delta) \leq \theta$, in (X, T).

Proposition 3.9. *If* δ *is a fuzzy regular open set in a fuzzy fraction dense space* (X, T), *then there exists a fuzzy* G_δ*-set* θ *with* $\text{int}\, cl(\theta) \neq 0$ *in* (X, T) *such that* $\delta \leq \theta$.

Proof. Let δ be a fuzzy regular open set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.2, there exists a fuzzy G_δ-set θ in (X, T). such that $\delta = \text{int}(\theta)$. Now $\text{int}(\theta) \leq \text{int}\, cl(\theta)$, implies that $\delta \leq \text{int}\, cl(\theta)$ and thus $\text{int}\, cl(\theta) \neq 0$. Thus, there exists a fuzzy G_δ-set θ with $\text{int}\, cl(\theta) \neq 0$ in (X, T) such that $\delta \leq \theta$.

Corollary 3.1. *If* δ *is a fuzzy regular open set in a fuzzy fraction dense space* (X, T), *then there exists a fuzzy somewhere dense set* θ *in* (X, T) *such that* $\delta \leq \theta$.

Proof. Let δ be a fuzzy regular open set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.9, there exists a fuzzy G_δ-set θ with $\text{int}\, cl(\theta) \neq 0$ in (X, T) such $\delta \leq \theta$. Now $\text{int}\, cl(\theta) \neq 0$ implies that θ is a fuzzy somewhere dense set in (X, T).

Proposition 3.10. *If* λ *is a fuzzy regular closed set in a fuzzy fraction dense space* (X, T), *then there exists a fuzzy* F_σ*-set* δ *in* (X, T) *such that* $\delta \leq \lambda$.

Proof. Let λ be a fuzzy *regular* closed set in (X, T). Then, $1 - \lambda$ is a fuzzy regular open set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.9, there exists a fuzzy G_δ-set θ in (X, T) such that $1 - \lambda \leq \theta$ and then $1 - \theta \leq \lambda$. Let $\delta = 1 - \theta$. Then δ is a fuzzy F_σ-set in (X, T). Hence there exists a fuzzy F_σ-set δ in (X, T) such that $\delta \leq \lambda$.

Corollary 3.2. *If* λ *is a fuzzy regular closed set in a fuzzy fraction dense space* (X, T), *then there exists a fuzzy cs dense set* δ *in* (X, T) *such that* $\delta \leq \lambda$.

Proof. Let λ be a fuzzy regular closed set in (X, T). Then, $1-\lambda$ is a fuzzy regular open set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Corollary 3.1, there exists a fuzzy somewhere dense set θ such that $1-\lambda \leq \theta$, in (X, T). Then, $1-\theta \leq \lambda$. Let $\delta = 1-\theta$. Then, δ is a fuzzy cs dense set in (X, T) and $\delta \leq \lambda$.

Proposition 3.11. *If* (X, T) *is a fuzzy fraction dense space*, *then there exist a fuzzy* F_σ*-set* λ *and a fuzzy* G_δ*-set* θ *in* (X, T) *such that* $\lambda \leq cl(\theta)$.

Proof. Let δ be a fuzzy regular open set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.9, there exists a fuzzy G_δ-set θ in (X, T) such that $\delta \leq \theta$ and then $cl(\delta) \leq cl(\theta)$, in (X, T). Since a fuzzy regular open set is a fuzzy open set in a fuzzy topological space [Th.2.2], by Proposition 3.3, there exists a fuzzy F_σ-set λ in (X, T) such that $\lambda \leq cl(\delta)$. Then, $\lambda \leq cl(\delta) \leq cl(\theta)$ and thus $\lambda \leq cl(\theta)$, in (X, T).

The following propositions show that the fuzzy closed sets are not fuzzy nowhere dense sets and the fuzzy open sets are not fuzzy dense sets in fuzzy fraction dense spaces.

Proposition 3.12. *If* μ *is a fuzzy closed set in a fuzzy fraction dense space* (X, T), *then* μ *is not a fuzzy nowhere dense set in* (X, T).

Proof. Let μ be a fuzzy closed set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.7, there exists a fuzzy G_δ-set ρ in (X, T) such that $\text{int}(\rho) \leq \mu$. Then, $\text{int}(\mu) \neq 0$ and $\text{int}(\mu) \leq \text{int}\, cl(\mu)$ implies that $\text{int}\, cl(\mu) \neq 0$, in (X, T). Hence μ is not a fuzzy nowhere dense set in (X, T).

Proposition 3.13. *If* λ *is a fuzzy open set in a fuzzy fraction dense space* (X, T), *then* λ *is not a fuzzy dense set in* (X, T).

Proof. Let λ be a fuzzy open set in (X, T). Suppose that $cl(\lambda) = 1$, in (X, T). Then, $\operatorname{int} cl(1-\lambda) = 1 - cl \operatorname{int}(\lambda) = 1 - cl(\lambda) = 1 - 1 = 0$. This implies that the fuzzy closed set $1-\lambda$ is a fuzzy nowhere dense set in the fuzzy fraction dense space (X, T), a contradiction [by Proposition 3.12]. Hence λ is not a fuzzy dense set in (X, T).

Proposition 3.14. *If* μ *is a fuzzy closed set in a fuzzy fraction dense space* (X, T), *then there exists a fuzzy regular closed set* λ *in* (X, T) *such that* $\lambda \leq \mu$.

Proof. Let μ be a fuzzy closed set in (X, T) and then, $cl(\mu) = \mu$. Since (X, T) is a fuzzy fraction dense space, by Proposition 3.12, μ is not a fuzzy nowhere dense set in (X, T) and then $\operatorname{int} cl(\mu) \neq 0$, in (X, T). Now $\operatorname{int}(\mu) = \operatorname{int} cl(\mu) \neq 0$ and then there exists a fuzzy open set λ in (X, T) such that $\lambda \leq \mu$. Then, $cl(\lambda) \leq cl(\mu) = \mu$ and $cl(\lambda)$ is a fuzzy regular closed set in (X, T) [Th.2.2]. Let $\lambda = cl(\lambda)$. Hence there exists a fuzzy regular closed set λ in (X, T) such that $\lambda \leq \mu$.

Proposition 3.15. *If* λ *is a fuzzy open set in a fuzzy fraction dense space* (X, T), *then there exists a fuzzy regular open set* θ *in* (X, T) *such that* $\lambda \leq \theta$.

Proof. Let λ be a fuzzy open set in (X, T). Then, $1-\lambda$ is a fuzzy closed set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.14, there exists a fuzzy regular closed set λ in (X, T) such that $\lambda \leq 1-\lambda$. Then, $\lambda \leq 1-\lambda$. Let $\theta = 1-\lambda$ and θ is a fuzzy regular open set in (X, T) and $\lambda \leq \theta$.

Proposition 3.16. *If* λ *is a fuzzy nowhere dense set in a fuzzy fraction dense space* (X, T), *then there exists a fuzzy cs dense set* δ *in* (X, T) *such that* $\delta \leq cl(\lambda)$.

Proof. Let λ be a fuzzy nowhere dense set in (X, T) and then $cl(\lambda)$ is a fuzzy closed set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.14, there exists a fuzzy regular closed set η in (X, T) such that $\eta \leq cl(\lambda)$. By Corollary 3.2, there exists a fuzzy cs dense set δ in (X, T) such that $\delta \leq \eta$ and then $\delta \leq cl(\lambda)$.

Proposition 3.17. *If* λ *is a fuzzy nowhere dense set in a fuzzy fraction dense space* (X, T), *then there exists no non-zero fuzzy regular closed set* η *in* (X, T) *such that* $\lambda \leq cl(\lambda)$.

Proof. Let λ be a fuzzy nowhere dense set in (X, T) and then $\operatorname{int} cl(\lambda) = 0$. Since (X, T) is a fuzzy fraction dense space, by Proposition 3.14, there exists a fuzzy regular closed set η in (X, T) such that $\eta \leq cl(\lambda)$. Then, $\operatorname{int}(\eta) \leq \operatorname{int} cl(\lambda) = 0$ and $\operatorname{int}(\eta) = 0$. This implies that $cl \operatorname{int}(\eta) = cl(0) = 0$ and $\eta = 0$. Thus, there exists no non-zero fuzzy regular closed set λ in (X, T) such that $\eta \leq cl(\lambda)$.

Proposition 3.18. *If* λ *is a fuzzy regular open set in a fuzzy fraction dense space* (X, T), *then there exists a fuzzy* G_δ*-set* θ *in* (X, T) *such that* $\operatorname{int}(\theta) \leq \lambda$.

Proof. Let λ be a fuzzy regular open set in (X, T). Then, $\operatorname{int} cl(\lambda) = \lambda$, in (X, T). Now $cl(\lambda)$ is a fuzzy closed set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.7, there exists a fuzzy G_δ-set θ in (X, T) such that $\operatorname{int}(\theta) \leq cl(\lambda)$. Then, $\operatorname{int}[\operatorname{int}(\theta)] \leq \operatorname{int} cl(\lambda)$ and $\operatorname{int}(\theta) \leq \lambda$, in (X, T).

Proposition 3.19. *If* λ *is a fuzzy regular open set in a fuzzy fraction dense space* (X, T), *then there exists a fuzzy* G_δ*-sets* θ *and* δ *in* (X, T) *such that* $\operatorname{int}(\delta) \leq \lambda \leq \theta$.

Proof. Let λ be a fuzzy regular open set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.9, there exists a fuzzy G_δ-set θ in (X, T) such that $\lambda \leq \theta$. Also, by Proposition 3.18, there exists a fuzzy G_δ-set δ in (X, T) such that $\operatorname{int}(\delta) \leq \lambda$. Then, $\operatorname{int}(\delta) \leq \lambda \leq \theta$, in (X, T).

Proposition 3.20. *If* λ *is a fuzzy simply open set in a fuzzy fraction dense space* (X, T), *then there exists no non-zero fuzzy regular closed set* λ *in* (X, T) *such that* $\lambda \leq bd(\lambda)$.

Proof. Let λ be a fuzzy simply open set in (X, T). Then, by Theorem 2.1, $\lambda \wedge (1-\lambda)$ is a fuzzy nowhere dense set in (X, T) and $\operatorname{int} cl[\lambda \wedge (1-\lambda)] = 0$. Since (X, T) is a fuzzy fraction dense space, by Proposition 3.14, there exists a fuzzy regular closed set η in (X, T) such that $\eta \leq cl(\lambda \wedge (1-\lambda))$. This implies that $\eta \leq cl(\lambda) \wedge cl(1-\lambda)$ and hence $\eta \leq bd(\lambda)$. Then, $\operatorname{int} cl(\eta) \leq \operatorname{int} cl[bd(\lambda)]$. Since λ is a fuzzy simply open set, $\operatorname{int} cl[bd(\lambda)] = 0$ and then $\operatorname{int} cl(\lambda) \leq 0$. That is, $\operatorname{int} cl(\eta) = 0$ and $\operatorname{int}(\eta) \leq \operatorname{int} cl(\eta)$ implies that $\operatorname{int}(\eta) = 0$, in (X, T). This implies that $cl \operatorname{int}(\eta) = cl(0) = 0$ and $\lambda = 0$. Thus, there exists no non-zero fuzzy regular closed set η in (X, T) such that $\eta \leq bd(\lambda)$.

Proposition 3.21. *If λ is a fuzzy simply open set in a fuzzy fraction dense space (X, T), then there exists a fuzzy F_σ-set δ in (X, T) such that $\delta \leq bd(\lambda)$.*

Proof. Let λ be a fuzzy simply open set in (X, T). Then, by Theorem 2.1, $\lambda \wedge (1-\lambda)$ is a fuzzy nowhere dense set in (X, T) and $\operatorname{int} cl[\lambda \wedge (1-\lambda)] = 0$. Since (X, T) is a fuzzy fraction dense space, by Proposition 3.14, there exists a fuzzy regular closed set η in (X, T) such that $\eta \leq cl(\lambda \wedge (1-\lambda))$. Then, $1-\eta$ is a fuzzy regular open set in (X, T). By Proposition 3.9, there exists a fuzzy G_δ-set θ in (X, T) such that $1-\eta \leq \theta$. Then, $1-\theta \leq \eta$. Let $\delta = 1-\eta$. Then, δ is a fuzzy F_σ-set in (X, T). Thus, $\delta \leq \eta\lambda \leq cl(\lambda \wedge (1-\lambda)) \leq cl(\lambda)$ $\wedge cl(1-\lambda) = bd(\lambda)$. Hence there exists a fuzzy F_σ-set δ in (X, T) such that $\delta \leq bd(\lambda)$.

Proposition 3.22. *If λ is a fuzzy simply open set in a fuzzy fraction dense space (X, T), then there exists a fuzzy σ-nowhere dense set δ in (X, T) such that $\delta \leq bd(\lambda)$.*

Proof. Let λ be a fuzzy simply open set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.21, there exists a fuzzy F_σ-set δ in (X, T) such that $\delta \leq bd(\lambda)$. This implies that $\operatorname{int}(\delta) \leq \operatorname{int}(bd(\lambda))$ $\leq \operatorname{int} cl(bd(\lambda)) = 0$. Then, $\operatorname{int}(\delta) = 0$. Thus, δ is a fuzzy F_σ-set in (X, T) such that $\operatorname{int}(\delta) = 0$ and then δ is a fuzzy σ-nowhere dense set in (X, T). Hence there exists a fuzzy σ-nowhere dense set δ in (X, T) such that

$\delta \leq bd(\lambda)$.

4. Fuzzy Fraction Dense Spaces and Fuzzy P-Spaces

The following propositions show that fuzzy regular closed sets are fuzzy F_σ-sets and fuzzy regular open sets are fuzzy G_δ-sets in fuzzy fraction dense and fuzzy P-spaces.

Proposition 4.1. *If* μ *is a fuzzy regular closed set in a fuzzy fraction dense and fuzzy P-space* (X, T), *then* μ *is a fuzzy* F_σ*-set in* (X, T).

Proof. Let μ be a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.1, $\mu = cl(\lambda)$, where λ is a fuzzy F_σ-set in (X, T). Since (X, T) is a fuzzy P-space, the fuzzy F_σ-set λ is a fuzzy closed set and then $cl(\lambda) = \lambda$, in (X, T). Hence the fuzzy regular closed set μ is a fuzzy F_σ-set in (X, T).

Corollary 4.1. *If* λ *is a fuzzy regular open set in a fuzzy fraction dense and fuzzy P-space* (X, T), *then* λ *is a fuzzy* G_δ*-set in* (X, T).

Proof. Let λ be a fuzzy regular open set in (X, T). Then, $1 - \lambda$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 4.1, $1 - \lambda$ is a fuzzy F_σ-set in (X, T) and thus λ is a fuzzy G_δ-set in (X, T).

Proposition 4.2. *If* μ *is a fuzzy* F_σ*-set in a fuzzy fraction dense and fuzzy P-space* (X, T), *then* μ *is a fuzzy somewhere dense set in* (X, T).

Proof. Let μ be a fuzzy F_σ-set in (X, T). Since (X, T) is a fuzzy P-space, the fuzzy F_σ-set μ is a fuzzy closed set and then by Proposition 3.12, μ is not a fuzzy nowhere dense set in (X, T). Thus, $\operatorname{int} cl(\mu) \neq 0$, in (X, T). Hence μ is a fuzzy somewhere dense set in (X, T).

Corollary 4.2. *If* λ *is a fuzzy* G_δ*-set in a fuzzy fraction dense and fuzzy P-space* (X, T), *then* λ *is a fuzzy cs dense set in* (X, T).

Proof. Let λ be a fuzzy G_δ-set in (X, T). Then, $1-\lambda$ is a fuzzy F_σ-set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 4.2, $1-\lambda$ is a fuzzy somewhere dense set in (X, T) and thus λ is a fuzzy cs dense set in (X, T).

Proposition 4.3. *If μ is a fuzzy F_σ-set in a fuzzy fraction dense and fuzzy P-space (X, T), then there exists a fuzzy regular closed set η in (X, T) such that $\eta \leq cl(\mu)$.*

Proof. Let μ be a fuzzy F_σ-set in (X, T). Since (X, T) is a fuzzy fraction dense and fuzzy P-space, by Proposition 4.2, μ is a fuzzy somewhere dense set in (X, T). By Theorem 2.3, there exists a fuzzy regular closed set η in (X, T) such that $\eta \leq cl(\mu)$.

Corollary 4.3. *If λ is a fuzzy G_δ-set in a fuzzy fraction dense and fuzzy P-space (X, T), there exists a fuzzy regular open set δ in (X, T) such that $\text{int}(\lambda) \leq \delta$.*

Proof. Let λ be a fuzzy G_δ-set in (X, T). Then, $1-\lambda$ is a fuzzy F_σ-set in (X, T). Since (X, T) is a fuzzy fraction dense and fuzzy P-space, by Proposition 4.3, there exists a fuzzy regular closed set η in (X, T) such that $\eta \leq cl(1-\lambda)$. This implies that $\eta \leq 1-\text{int}(\lambda)$ and $\text{int}(\lambda) \leq 1-\eta$. Let $\delta = 1-\eta$. Then, δ is a fuzzy regular open set in (X, T) and $\text{int}(\lambda) \leq \delta$.

Proposition 4.4. *If λ is a fuzzy G_δ-set in a fuzzy fraction dense and fuzzy P-space (X, T), then*

(i) $cl\,\text{int}(\lambda) \neq 1$, *in* (X, T).

(ii) λ *is not a fuzzy dense set in* (X, T).

Proof. (i) Let λ be a fuzzy G_δ-set in (X, T). Then, $1-\lambda$ is a fuzzy F_σ-set in (X, T). Since (X, T) is a fuzzy fraction dense and fuzzy P-space, by Proposition 4.2, $1-\lambda$ is a fuzzy somewhere dense set and $\text{int}\,cl(1-\lambda) \neq 0$, in (X, T). Then, $1-cl\,\text{int}(\lambda) \neq 0$ and $cl\,\text{int}(\lambda) \neq 1$, in (X, T).

(ii) Since (X, T) is a fuzzy P-space, the fuzzy G_δ-set λ is a fuzzy open set in (X, T). Also since (X, T) is a fuzzy fraction dense space, by Proposition 3.4, there exists a fuzzy F_σ-set η in (X, T) such that $\lambda \leq cl(\eta)$ and $cl(\eta)$ is a fuzzy closed set in (X, T), implies that $cl(\lambda) \neq 1$, in (X, T). Hence λ is not a fuzzy dense set in (X, T).

Proposition 4.5. *If λ is a fuzzy open set in a fuzzy fraction dense and fuzzy P-space (X, T), then $cl(\lambda)$ is a fuzzy F_σ-set in (X, T).*

Proof. Let λ be a fuzzy open set in (X, T). Since (X, T) is a fuzzy fraction dense space, $cl(\lambda) = cl(\mu)$, where μ is a fuzzy F_σ-set in (X, T). Also since (X, T) is a fuzzy P-space, the fuzzy F_σ-set μ is a fuzzy closed set in (X, T) and then $cl(\mu) = \mu$. Then, $cl(\lambda) = \mu$, where μ is a fuzzy F_σ-set in (X, T). Hence $cl(\lambda)$ is a fuzzy F_σ-set in (X, T).

Corollary 4.4. *If μ is a fuzzy closed set in a fuzzy fraction dense and fuzzy P-space (X, T), then $\text{int}(\mu)$ is a fuzzy G_δ-set in (X, T).*

Proof. Let μ be a fuzzy closed set in (X, T). Then, $1 - \mu$ is a fuzzy open set in the fuzzy fraction dense and fuzzy P-space (X, T) and by proposition 4.5, $cl(1 - \mu)$ is a fuzzy F_σ-set in (X, T). By Lemma 2.1, $cl(1 - \mu) = 1 - \text{int}(\mu)$ and hence $\text{int}(\mu)$ is a fuzzy G_δ-set in (X, T).

5. Fuzzy Fraction Dense Spaces and Other Fuzzy Topological Spaces

The following proposition shows that fuzzy fraction dense spaces are not fuzzy hyper connected spaces.

Proposition 5.1. *If (X, T) is a fuzzy fraction dense space, then (X, T) is not a fuzzy hyperconnected space.*

Proof. Let λ be a fuzzy open set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.4, there exists a fuzzy F_σ-set η in (X, T) such that $\lambda \leq cl(\eta)$. Now $cl(\lambda)$ is a fuzzy closed set in (X, T) and $\lambda \leq cl(\eta)$, implies that $cl(\lambda) \neq 1$, in (X, T). Hence λ is not a fuzzy dense set in (X, T).

Proposition 5.2. *If* λ *is a fuzzy nowhere dense set in a fuzzy fraction dense and fuzzy* DG_{δ}*-space* (X, T), *then* λ *is a fuzzy* F_{σ}*-set in* (X, T).

Proof. Let λ be a fuzzy nowhere dense set in (X, T). By Proposition 3.12, fuzzy closed sets in a fuzzy fraction dense space are not fuzzy nowhere dense sets in (X, T) and thus λ is a fuzzy nowhere dense set but not a fuzzy closed set in (X, T). By Theorem 2.4, λ is a fuzzy F_{σ}-set in the fuzzy DG_{δ}-space (X, T).

Fuzzy nowhere dense sets become fuzzy σ-nowhere dense sets in fuzzy fraction dense and fuzzy DG_{δ}-spaces. For, consider the following proposition.

Proposition 5.3. *If* λ *is a fuzzy nowhere dense set in a fuzzy fraction dense and fuzzy* G_{δ}*-space* (X, T), *then* λ *is a fuzzy* σ*-nowhere dense set in* (X, T).

Proof. Let λ be a fuzzy nowhere dense set in (X, T). Then, $\operatorname{int} cl(\lambda) = 0$, in (X, T). Now $\operatorname{int}(\lambda) \leq \operatorname{int} cl(\lambda)$, implies that $\operatorname{int}(\lambda) = 0$ and by Proposition 5.2, λ is a fuzzy F_{σ}-set and thus λ is a fuzzy F_{σ}-set with $\operatorname{int}(\lambda) = 0$, in (X, T). Hence λ is a fuzzy σ-nowhere dense set in (X, T).

The following propositions shows that fuzzy fraction dense and fuzzy DG_{δ}-spaces are fuzzy Volterra spaces and fuzzy Baire spaces.

Proposition 5.4. *If* (X, T) *is a fuzzy fraction dense and fuzzy* DG_{δ}*-space*, *then* (X, T) *is a fuzzy Volterra space*.

Proof. Let λ be a fuzzy nowhere dense set in (X, T). Since (X, T) is a fuzzy fraction dense and fuzzy DG_{δ}-space, by Proposition 5.2, λ is a fuzzy F_{σ}-set in (X, T). By Theorem 2.5, (X, T) is a fuzzy Volterra space.

Proposition 5.5. *If* (X, T) *is a fuzzy fraction dense and fuzzy* G_{δ}*-space*, *then* (X, T) *is a fuzzy Baire space*.

Proof. Let λ be a fuzzy nowhere dense set in (X, T). Since (X, T) is a fuzzy fraction dense and fuzzy DG_{δ}-space, by Proposition 5.2, λ is a fuzzy F_{σ}-set in (X, T). By Theorem 2.6, (X, T) is a fuzzy Baire space.

Proposition 5.6. *If* (X, T) *is a fuzzy fraction dense and fuzzy* G_δ *-space, then* (X, T) *is a fuzzy* σ*-Baire space.*

Proof. Let (X, T) be a fuzzy fraction dense and fuzzy DG_δ -space. By Proposition 5.5, (X, T) is a fuzzy Baire space and then $\operatorname{int}(\vee_{i=1}^{\infty} (\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). By Proposition 5.3, (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T). Thus, $\operatorname{int}(\vee_{i=1}^{\infty} (\lambda_i)) = 0$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T), implies that (X, T) is a fuzzy σ-Baire space.

The following proposition gives a condition for fuzzy fraction dense spaces to become fuzzy σ-Baire spaces.

Proposition 5.7. *If* $\operatorname{int}(\vee_{i=1}^{\infty} bd(\lambda_i)) = 0$, *where* (λ_i)*'s are fuzzy simply open sets in a fuzzy fraction dense space* (X, T), *then* (X, T) *is a fuzzy* σ*-Baire space.*

Proof. Let (λ_i)'s $(i = 1 \text{ to } \infty)$ be fuzzy simply open sets in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.22, there exist fuzzy σ-nowhere dense sets δ_i in (X, T) such that $\delta_i \leq bd(\lambda_i)$. Then, $\operatorname{int}(\vee_{i=1}^{\infty} (\delta_i)) \leq \operatorname{int}(\vee_{i=1}^{\infty} bd(\lambda_i))$. By hypothesis $\operatorname{int}(\vee_{i=1}^{\infty} bd(\lambda_i)) = 0$ and then $\operatorname{int}(\vee_{i=1}^{\infty} (\delta_i)) = 0$, where (δ_i)'s are fuzzy σ-nowhere dense sets in (X, T). Hence (X, T) is a fuzzy σ-Baire space.

It is established in [16] that "If λ is a fuzzy first category set in a fuzzy globally disconnected, fuzzy Baire and fuzzy P-space (X, T), then λ is a fuzzy nowhere set in (X, T)." The following proposition shows that fuzzy nowhere dense sets in fuzzy fraction dense and fuzzy DG_δ -spaces are fuzzy first category sets.

Proposition 5.8. *If* λ *is a fuzzy nowhere dense set in a fuzzy fraction dense and fuzzy* DG_δ *-space* (X, T), *then* λ *is a fuzzy first category set in* (X, T).

Proof. Let λ be a fuzzy nowhere dense set in (X, T). Since (X, T) is a fuzzy fraction dense and fuzzy G_δ -space, by Proposition 5.3, λ is a fuzzy

σ-nowhere dense set in (X, T) and by Theorem 2.7, $1-\lambda$ is a fuzzy residual set in (X, T). Hence λ is a fuzzy first category set in (X, T).

Proposition 5.9. *If λ is a fuzzy G_δ-set in a fuzzy fraction dense and fuzzy ∂-space (X, T), then there exists a fuzzy F_σ-set δ in (X, T) such that $\delta \leq bd(\lambda)$.*

Proof. Let λ be a fuzzy G_δ-set in (X, T). Since (X, T) is a fuzzy ∂-space, the fuzzy G_δ-set λ is a fuzzy simply open set in (X, T). Since (X, T) is a fuzzy fraction dense space, by Proposition 3.21, there exists a fuzzy F_σ-set δ in (X, T) such that $\delta \leq bd(\lambda)$.

Proposition 5.10. *If λ is a fuzzy G_δ-set in a fuzzy fraction dense and fuzzy ∂-space (X, T), then there exists a fuzzy σ-nowhere dense set δ in (X, T) such that $\delta \leq bd(\lambda)$.*

Proof. Let λ be a fuzzy G_δ-set in (X, T). Since (X, T) is a fuzzy fraction dense and fuzzy ∂-space, by Proposition 5.9, there exists a fuzzy F_σ-set δ in (X, T) such that $\delta \leq bd(\lambda)$. Then $\operatorname{int}(\delta) \leq \operatorname{int}(bd(\lambda)) \leq \operatorname{int} cl(bd(\lambda))$. Since (X, T) is a fuzzy ∂-space, the fuzzy G_δ-set λ is a fuzzy simply open set in (X, T) and $\operatorname{int} cl(bd(\lambda)) = 0$, in (X, T). This implies that $\operatorname{int}(\delta) = 0$. Thus δ is a fuzzy F_σ-set with $\operatorname{int}(\delta) = 0$, in (X, T) and then δ is a fuzzy σ-nowhere dense set in (X, T). Hence there exists a fuzzy σ-nowhere dense set δ in (X, T) such that $\delta \leq bd(\lambda)$.

The following proposition gives a condition for fuzzy fraction dense and fuzzy ∂-spaces to become fuzzy σ-Baire spaces.

Proposition 5.11. *If $\operatorname{int}(\vee_{i=1}^{\infty} bd(\lambda_i)) = 0$, where (λ_i)'s are fuzzy G_δ-sets in a fuzzy fraction dense and fuzzy ∂-space (X, T), then (X, T) is a fuzzy σ-Baire space.*

Proof. Let (λ_i)'s (i = 1 to ∞) be fuzzy G_δ-sets in (X, T). Since (X, T) is a fuzzy fraction dense and fuzzy ∂-space, by Proposition 5.10, there exist fuzzy σ-nowhere dense sets δ_i in (X, T) such that $\delta_i \leq bd(\lambda_i)$. Then, $\operatorname{int}(\vee_{i=1}^{\infty} (\delta_i)) \leq \operatorname{int}(\vee_{i=1}^{\infty} bd(\lambda_i))$. By hypothesis $\operatorname{int}(\vee_{i=1}^{\infty} bd(\lambda_i)) = 0$ and then

$\operatorname{int}(\vee_{i=1}^{\infty}(\delta_i)) = 0$, where (δ_i)'s are fuzzy σ-nowhere dense sets in (X, T). Hence (X, T) is a fuzzy σ-Baire space.

Proposition 5.12. *If (X, T) is a fuzzy fraction dense and fuzzy DG_δ-space, then (X, T) is a fuzzy nodef space.*

Proof. Let λ be a fuzzy nowhere dense set in (X, T). Since (X, T) is a fuzzy fraction dense and fuzzy DG_δ-space, by Proposition 5.2, λ is a fuzzy F_σ-set in (X, T). Hence (X, T) is a fuzzy nodef space.

The following propositions show that fuzzy fraction dense spaces are not fuzzy nodec spaces and fuzzy fraction dense and fuzzy DG_δ-spaces, are not fuzzy P-spaces.

Proposition 5.13. *If (X, T) is a fuzzy fraction dense space, then (X, T) is not a fuzzy nodec space.*

Proof. Let λ be a fuzzy nowhere dense set in (X, T). Then, the Proposition 3.12, implies that λ is not a fuzzy closed set in (X, T) and hence (X, T) is not a fuzzy nodec space.

Proposition 5.14. *If (X, T) is a fuzzy fraction dense and fuzzy DG_δ-space, then (X, T) is not a fuzzy P-space.*

Proof. Suppose that (X, T) is a fuzzy P-space. By Proposition 5.12, the fuzzy fraction dense and fuzzy DG_δ-space (X, T) is a fuzzy nodef space. Then, the fuzzy nodef and fuzzy P-space (X, T), by Theorem 2.9, will be a fuzzy nodec space, a contradiction [by Proposition 5.13]. Hence (X, T) is not a fuzzy P-space.

Proposition 5.15. *If for each fuzzy open set λ in (X, T), $cl(\lambda) = cl(\mu)$, where μ is a fuzzy first category set in a fuzzy globally disconnected space (X, T), then (X, T) is a fuzzy fraction dense space.*

Proof. Let λ be a fuzzy open set in (X, T) such that $cl(\lambda) = cl(\mu)$, where μ is a fuzzy first category set in (X, T). Since (X, T) is a fuzzy globally disconnected space, by Theorem 2.10, the fuzzy first category set μ in (X, T)

is a fuzzy F_σ-set in (X, T). Hence, for each fuzzy open set λ in (X, T), $cl(\lambda) = cl(\mu)$, where μ is a fuzzy F_σ-set in (X, T), implies that (X, T) is a fuzzy fraction dense space.

Conclusion

In this paper, the notion of fuzzy fraction dense space is introduced. It is established that each fuzzy regular closed set is the closure of a fuzzy F_σ-set and each fuzzy regular open set is the interior of a fuzzy G_δ-set in fuzzy fraction dense spaces. Also it is obtained that fuzzy closed sets are not fuzzy nowhere dense sets and fuzzy open sets are not fuzzy dense sets in fuzzy fraction dense spaces. It is obtained that fuzzy regular closed sets are fuzzy F_σ-sets and fuzzy regular open sets are fuzzy G_δ-sets in fuzzy fraction dense and fuzzy P-spaces. Also it is established that fuzzy F_σ-sets are fuzzy somewhere dense sets and fuzzy G_δ-sets are fuzzy cs dense sets in fuzzy fraction dense and fuzzy P-spaces.

It is established that fuzzy fraction dense spaces are not fuzzy hyper connected and fuzzy nodec spaces. In fuzzy fraction dense and fuzzy DG_δ-spaces, fuzzy nowhere dense sets become fuzzy F_σ-sets and fuzzy first category sets. Fuzzy fraction dense and fuzzy DG_δ-spaces are found to be fuzzy Baire, fuzzy σ-Baire and fuzzy Volterra spaces. A condition which ensures the existence of fuzzy σ-Baireness in fuzzy fraction dense spaces is obtained by means of fuzzy boundary of fuzzy simply open sets. Also a condition which ensures the existence of fuzzy σ-Baireness in fuzzy fraction dense and fuzzy ∂-spaces is obtained by means of the fuzzy boundary of fuzzy G_δ-sets. It is established that fuzzy fraction dense and fuzzy DG_δ-spaces are fuzzy nodef spaces and are not fuzzy P-spaces. It is found that fuzzy fraction dense spaces are not fuzzy nodec spaces. A condition under which fuzzy globally disconnected spaces become fuzzy fraction dense spaces is also obtained in this paper.

References

[1] K. K. Azad, On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly contiunity, J. Math. Anal. Appl. 82 (1981), 14-32.

[2] G. Balasubramanian, Maximal Fuzzy Topologies, Kybernetika 31(5) (1995), 459-464.

[3] C. L. Chang, Fuzzy Topological Spaces, J. Math. Anal. Appl. 24 (1968), 182-190.

[4] Gary Gruenhage, Products of co-zero complemented spaces, Houston J. Math. 32(3) (2006), 757-773.

[5] B. Ghosh, Fuzzy extremally disconnected spaces, Fuzzy Sets and Sys. 46(2) (1992), 245-250.

[6] A. W. Hager and J. Martinez, Fraction-dense algebras and spaces, Can. J. Math. 45(5) (1993), 977-996.

[7] Miguel Caldas, Govindappa Navalagi and Ratnesh Saraf, On fuzzy weakly semi-open functions, Proyecciones, Universidad Catolica del Norte, Antofagasta-Chile 21(1) (2002), 51-63.

[8] G. Thangaraj and G. Balasubramanian, On fuzzy basically disconnected spaces, J. Fuzzy Math. 9(1) (2001), 103-110.

[9] G. Thangaraj and G. Balasubramanian, On somewhat fuzzy continuous functions, J. Fuzzy Math 11(2) (2003), 725-736.

[10] G. Thangaraj, Resolvability and irresolvability in fuzzy topological spaces, News Bull. Cal. Math. Soc. 31(4-6) (2008), 11-14.

[11] G. Thangaraj and S. Anjalmose, On fuzzy baire spaces, J. Fuzzy Math. 21(3) (2013), 667-676.

[12] G. Thangaraj and S. Anjalmose, Some remarks on fuzzy baire spaces, Scientia Magna 9(1) (2013), 1-6.

[13] G. Thangaraj and E. Poongothai, On fuzzy σ-baire spaces, Int. J. Fuzzy Math. Sys. 3(4) (2013), 275-283.

[14] G. Thangaraj and S. Soundararajan, On fuzzy Volterra spaces, J. Fuzzy Math. 21(4) (2013), 895-904.

[15] G. Thangaraj and K. Dinakaran, On fuzzy simply continuous functions, J. Fuzzy Math. 25(1) (2017), 99-124.

[16] G. Thangaraj and S. Muruganantham, Fuzzy globally disconnected spaces, Jour. Tri. Math. Soc. 21 (2019), 37-46.

[17] G. Thangaraj and J. Premkumar, On fuzzy DG_{δ}-spaces, Adv. Fuzzy Math. 14(1) (2019), 27-40.

[18] G. Thangaraj and J. Premkumar, On Fuzzy ∂-Spaces, Inter. J. Applied Engin. Research 13(15) (2018), 12417-12421.

[19] G. Thangaraj and S. Senthil, On somewhere fuzzy continuous functions, Annl. Fuzzy Math. Inform. 15(2) (2018), 181-198.

[20] L. A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965), 338-353.