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APPLICATION OF JACOBI POLYNOMIALS TO SOLVING NON-LINEAR DIFFERENTIAL EQUATIONS INVOLVING I-FUNCTION OF SEVERAL VARIABLES

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Abstract

In the present article an attempt have been made to solve the non-linear differential equation $w_0^2 x^{\rho} I[z_1 x^{\sigma_1}, \ldots, z_r x^{\sigma_r}] = 0$ (under the initial conditions $x = A$ and $\dot{x} = 0$ at $t = 0$), by the method of approximation using Jacobi polynomial. Here ρ and $\sigma_i\ (i = 1, 2, \ldots, r)$ are positive integer.

Introduction

In 1959, Denman [3] proved that an amplitude dependent approximation may be obtained to the frequency of the simple pendulum by using the technique of a linear Tchebycheff polynomial approximation to sinθ in an interval $(-A, A)$ where A the amplitude of the motion is. The same problem was later on in 1964, considered by Denman and Haward [2] and Denman and Liu [1]. Some forced oscillation problems were discussed by Grade [4]. Some technique was used by Saxena and Kushwaha [6] to solve a non-linear differential equation associated with Kummer's confluent hyper geometric function. In 1997, Rathie [7] introduced a new function in the literature namely the "I-function" which play important role in Physics, Mathematics and other branches of Applied Mathematics.

Saxena and Kushwaha [6], Khan and Verma [8], Shrivastava [12], Nigam [11] and several other authors have studied application of orthogonal

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polynomials to obtain the linear amplitude dependent approximate solution of the nonlinear differential equation of general type. Recently good development has been made in field of I-function by Shantha Kumari K. et al. [13] and by Mohan and Kumar [14].

In this paper we will study the application of Jacobi polynomials to non-linear differential equation associated with I-function of several variables.

1. Preliminaries

In 1986, Prasad [5], introduced a new function in the field of special functions, known as I-function of several variables,

$$I[z_1, \ldots, z_r] = I^{0,\, n_2 : 0,\, n_3 : \ldots : 0,\, n_r : (m', n'); \ldots ; (m^{(r)}, n^{(r)})}_{p_2,\, q_2 : p_3,\, q_3 : \ldots : p_r,\, q_r : [p', q']; \ldots ; [p^{(r)}, q^{(r)}]}$$

$$\left[z_1, \ldots, z_r \middle| \begin{matrix} (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1, p_2} : (a_{3j}; \alpha'_{3j}, \alpha''_{3j}, \alpha'''_{3j})_{1, p_3} : \ldots : (a_{rj}; \alpha'_{rj} \ldots \alpha^{(r)}_{rj})_{1, p_r} : (a'_j, a'_j)_{1, p'}; \ldots ; (a^{(r)}_j, \alpha^{(r)}_j)_{1, p^r} \\ (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1, q_2} : (b_{3j}; \beta'_{3j}, \beta''_{3j}, \beta'''_{3j})_{1, q_3} : \ldots : (b_{rj}; \beta'_{rj}, \beta^{(r)}_{rj})_{1, q_r} : (b'_j, \beta'_j)_{1, q'}; \ldots ; (b^{(r)}_j, \beta^{(r)}_j)_{1, q^r} \end{matrix} \right]$$

$$= \frac{1}{(2\pi i)^r} \int_{L_1}, \ldots, \int_{L_r} \phi_1(S_1), \ldots, \phi_r(S_r) \varphi(S_1, S_r) Z_1^{s_1}, \ldots, Z_r^{s_r} dS_1, \ldots, dS_r \quad (1.1)$$

Where $i = \sqrt{(-1)}$

$$\phi_i(S_i) = \frac{\coprod_{j=1}^{m(i)} \left[(b_j^{(i)} - \beta_j^{(i)} \cdot s_i) \right] \coprod_{j=1}^{n^{(i)}} \left[(1 - a_j^{(i)} + \alpha_j^{(i)} \cdot s_j \right]}{\coprod_{j=m_{+1}^{(i)}}^{q^{(i)}} \left[(1 - b_j^{(i)} + \beta_j^{(i)} \cdot s_j) \right] \coprod_{j=n_{+1}^{(i)}}^{p^i} \left[(a_j^{(i)} - \alpha_j^{(i)} \cdot s_i) \right]} \forall\, i \in \{1, \ldots, r\}$$

(1.2)

$$\varphi(S_1, \ldots, S_r) = \frac{\coprod_{j=2}^{n_2} \left[\left(1 - a_{2j} + \sum_{i=1}^{2} \alpha_{2j}^{(i)} \cdot s_i \right) \right] \coprod_{j=1}^{n_3} \left[\left(1 - a_{3j} + \sum_{i=1}^{3} \alpha_{3j}^{(i)} \cdot s_j \right) \right]}{\coprod_{j=n_{2+1}}^{p^2} \left[\left(a_{2j} - \sum_{i=1}^{2} \alpha_j^{(i)} \cdot s_j \right) \right] \coprod_{j=n_3+1}^{p^3} \left[\left(a_{3j} - \sum_{i=1}^{3} \alpha_{3j}^{(i)} \cdot s_i \right) \right]}$$

$$\frac{\ldots \coprod_{j=1}^{n_r} \left[\left(1 - a_{rj} + \sum_{i=r}^{r} \alpha_{rj}^{(i)} \cdot s_i \right) \right]}{\ldots \coprod_{j=n_r+1}^{p_r} \left[\left(a_{rj} - \sum_{i=1}^{r} \alpha_{rj}^{(i)} \cdot s_i \right) \right] \coprod_{j=1}^{q_2} \left[\left(1 - b_{2j} + \sum_{i=1}^{2} \beta_{rj}^{(i)} \cdot s_i \right) \right]}$$

$$\frac{1}{\coprod_{j=1}^{q_3}\left[\left(1-b_{3j}+\sum_{i=1}^{3}\beta_{3j}^{(i)}\cdot s_i\right)\right],\ldots,\coprod_{j=1}^{q_r}\left[\left(1-b_{rj}+\sum_{i=1}^{r}\beta_{rj}^{(i)}\cdot s_i\right)\right]} \tag{1.3}$$

Where the conditions of convergent and restrictions on parameters are defined in [5].

The following known results will be required,

$$P_0^{(\alpha,\beta)}(x)=1; \quad P_1^{(\alpha,\beta)}(x)=\frac{\alpha-\beta}{2}+\frac{\alpha+\beta+2}{2}x \tag{1.4}$$

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}[P_v^{(\alpha,\beta)}(x)]^2dx = \frac{2^{\alpha+\beta+1}\lceil(v+\alpha+1)\rceil\lceil(v+\beta+1)\rceil}{v!\,(\alpha+\beta+2v+1)\lceil(\alpha+\beta+v+1)\rceil} \tag{1.5}$$

$\text{Re}(\alpha) > -1,\ \text{Re}(\beta) > -1$

$$\int_{-1}^{1}x^{\rho}(1-x)^{\alpha}(1+x)^{\beta}P_v^{(\alpha,\beta)}(x)dx = 0, \text{ for } \rho < v \tag{1.6}$$

$$\int_{-1}^{1}x^{\rho}(1-x)^{\alpha}(1+x)^{\beta}P_v^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+1}\rho!\lceil(\alpha+v+1)\rceil\lceil(\beta+v+1)\rceil}{v!\,(\rho-v)!\lceil(v+\alpha+\beta+\rho+2)\rceil}$$

$$\cdot 2F_1[v-\rho,\ \alpha+v;-\beta-\rho;-1] \text{ for } \rho \geq v \tag{1.7}$$

$$H_{0,1}^{1,0}\left[\frac{z}{(\alpha,1)}\right] = Z^{\alpha}e^{-z} \tag{1.8}$$

$$H_{0,2}^{1,0}\left[\frac{z^2}{4}\ |_{(1/2,1),\,(0,1)}\right] = \pi^{-1/2\sin z} \tag{1.9}$$

$$\frac{1}{-(\alpha')}H_{1,1}^{1,1}\left[-Z\ |_{(0,1)}^{(1-\alpha',1)}\right] = (1-Z)^{-\alpha'} \tag{1.10}$$

and

$$H_{0,2}^{1,0}\left[\frac{z^2}{4}\ |_{\left(\frac{v'}{2},1\right),\left(\frac{-v'}{2},1\right)}\right] = J_{v'}(z) \tag{1.11}$$

2. Main Problem

We solve the differential equation

$$\frac{d^2x}{dt^2} + f(x) = 0, \tag{2.1}$$

where,

$$f(x) = w_0^2 x^\rho I[z_1 x^{\sigma_1}, \ldots, z_r x^{\sigma_r}]. \tag{2.2}$$

The linear Jacobi polynomial approximation of $f(x)$ yields

$$f^* = w_0^2 x^\rho I[z_1 x^{\sigma_1}, \ldots, z_r x^{\sigma_r}] = a_0^{(\alpha, \beta)} P_0^{(\alpha, \beta)}(x/A) + a_1^{(\alpha, \beta)} P_1^{(\alpha, \beta)}(x/A) \tag{2.3}$$

where

$$a_0^{(\alpha, \beta)} = \frac{\lceil(\alpha + \beta + 2) w_0^2 A\rho\rceil}{2^{\alpha+\beta+1}\lceil(\alpha + 1)\rceil\lceil(\beta + 1)\rceil}$$

$$\left\{\int_{-1}^{1} x^\rho (1 - x)^\alpha (1 + x)^\beta P_0^{(\alpha, \beta)}(x) I[z_1(Ax)^{\sigma_1}, \ldots, z_r(Ax)^{\sigma_r}]dx\right\} \tag{2.4}$$

and

$$a_1^{(\alpha, \beta)} = \frac{(\alpha + \beta + 3)\lceil(\alpha + \beta + 2) w_0^2 A^\rho\rceil}{2^{\alpha+\beta^{+1}}\lceil(\alpha + 2)\rceil\lceil(\beta + 2)\rceil}$$

$$\cdot\left\{\int_{-1}^{1} x^\rho (1 - x)^\alpha (1 + x)^\beta P_1^{(\alpha, \beta)}(x) I[z_1(Ax)^{\sigma_1}, \ldots, z_r(Ax)^{\sigma_r}]dx\right\} \tag{2.5}$$

Evaluating the integrals in (2.4) and (2.5) with an appeal to the result (1.7) and following the well-known technique of change of order of integrations, we arrive at,

$$a_0^{(\alpha, \beta)} = w_0^2 A^\rho \lceil(\alpha + \beta + 2)\rceil \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha)k}{k!} G_1(k) \tag{2.6}$$

$$\alpha_1^{(\alpha, \beta)} = 2 w_0^2 A^\rho (\alpha + \beta + 3)\lceil(\alpha + \beta + 2)\rceil \sum_{k=0}^{\infty} \frac{(-1)^k (1 + \alpha)k}{k!} G_2(k) \tag{2.7}$$

Provided that $\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\beta) > -1$,

$$\mathrm{Re}\left[\rho + \sum_{i=1}^{r} \alpha_i \sigma_i + 1\right] > 0, \mid \arg z_i \mid < \frac{1}{2}\pi u_i, \; u_i > 0 (i = 1, \ldots, r),$$

Where α_i and u_i are

$$\alpha_i = \min \operatorname{Re}\{(b_j^{(i)}/\beta_j^{(i)}\} \quad \begin{matrix} j = 1, \ldots, m^{(i)} \\ i = 1, \ldots, r \end{matrix} \tag{2.8}$$

and

$$u_i = \sum_{j=1}^{n^{(i)}} \alpha_j^{(i)} - \sum_{j=1}^{p^{(i)}} \alpha_j^{(i)} + \sum_{j=1}^{m^{(i)}} \beta_j^{(i)} - \sum_{j=m^{(i)}+1}^{q^{(i)}} \beta_j^{(i)}$$

$$+ \left(\sum_{j=1}^{n_2} \alpha_{2j}^{(i)} - \sum_{j=n_{2+1}}^{p_2} \alpha_{2j}^{(i)}\right) + \ldots + \left(\sum_{j=1}^{n_r} \alpha_{rj}^{(i)} - \sum_{j=n_r+1}^{p_r} \alpha_{rj}^{(i)}\right)$$

$$- \left(\sum_{j=1}^{q_2} \beta_{2j}^{(i)} + \ldots + \sum_{j=1}^{qr} \beta_{rj}^{(i)}\right) (i = 1, \ldots, r) \tag{2.9}$$

and

$G_1(k)$ and $G_2(k)$ stand for

$$G_1(k) = I^{0,\, 0:0,\, 0:\ldots:2,\, 0:(m',\, n');\ldots;(m^{(r)},\, n^{(r)}),}_{p_2,\, q_2:p_3,\, q_3:\ldots:\, p_r+2,\, q_r+3:[p',\, q'];\ldots;[p^{(r)},\, q(r)],}$$

$$\left[z_1 A^{\sigma_1}, \ldots, z_r A^{\sigma r} \left| \begin{matrix} (a_{2j};\alpha'_{2j}, \alpha''_{2j})_{1, p_2}:\ldots:(a_{rj};\alpha'_{rj}, \ldots, \alpha_{rj}^{(r)})_{1 p_r}, (-\rho;\sigma_1, \ldots, \sigma_r), (k-\beta-\rho;\sigma_1, \ldots, \sigma_r) \\ (b_{2j};\beta'_{2j}, \beta''_{2j})_{1, q_2}:\ldots:(k-\rho;\sigma_1, \ldots, \sigma_r), (-\beta-\rho;\sigma_1, \ldots, \sigma_r), (b_{rj};\beta'_{rj}, \ldots, \beta_{rj}^{(r)})_{1, q_r} \end{matrix}\right.\right.$$

$$\left. \begin{matrix} (\alpha'_j, \alpha'_j)_{1, p_1}:\ldots:(\alpha_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)}, \ldots} \\ (-\alpha-\beta-\rho-1;\sigma_1, \ldots, \sigma_r):(b'_j, \beta'_j)_{1, q'};\ldots;(b_j^{(r)}, \beta_j^{(r)})_{1, q^r} \end{matrix}\right] \tag{2.10}$$

$$G_2(k) = I^{0, 0:0, 0:\ldots:2,\, 1:(m',\, n');\ldots;(m^{(r)},\, n^{(r)})}_{p_2,\, q_2:p_3,\, q_3:\ldots:p_r+3,\, q_r+3:[p',\, q'];\ldots;[p^{(r)},\, q(r)]}$$

$$\left[z_1 A^{\sigma_1}, \ldots, z_r A^{\sigma_r} \left| \begin{matrix} (a_{2j};\alpha'_{2j}, \alpha''_{2j})_{1, p_2}:\ldots:(1-\rho;\sigma_1, \ldots, \sigma_r), (a_{rj};\alpha'_{rj};, \ldots, \alpha_{rj}^{(r)})_{1, p_r}, (1-\rho;\sigma_1, \sigma_r), \\ (b_{2j};\beta'_{2j}, \beta''_{2j})_{1, q_2}:\ldots:(1+k-\rho;\sigma_1, \ldots, \sigma_r), (-\beta-\rho:\sigma_1, \ldots, \sigma_r), (b_{rj}, \beta'_{rj}, \ldots, \beta_{rj}^{(r)})_{1, qr} \end{matrix}\right.\right.,$$

$$\left. \begin{matrix} (k-\beta-\rho;\sigma_1, \ldots, \sigma_r):(\alpha'_j, \alpha'_j)_{1, p'};\ldots;(a_j^{(r)}, a_j^{(r)})_{1, p^{(r)}} \\ (-\alpha-\beta-\rho-2;\sigma_1, \ldots, \sigma_r):(b'_j, \beta'_j)_{1, q'};\ldots;(b_j^{(r)}, b_j^{(r)})_{1, q^{(r)}} \end{matrix}\right] \tag{2.11}$$

An appeal to the results (1.4), thus gives

$$f^*(x) = k_1^2 x + \frac{(\alpha - \beta)A}{\alpha + \beta + 2}[k_1^2 + k_2^2] \tag{2.12}$$

Where

$$k_1^2 = w_0^2 A^{\rho-1} \lceil (\alpha + \beta + 4) \rceil \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha+1)_k}{k!} G_2(k) \tag{2.13}$$

and

$$k_2^2 = \frac{w_0^2 A^{\rho-1} \lceil (\alpha + \beta + 3) \rceil}{(\alpha - \beta)} \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha)_k}{k!} G_1(k) \tag{2.14}$$

Replacing $f(x)$ by its approximation $f^*(x)$; (2.1) transforms to

$$\frac{d^2 x}{dt^2} + k_1^2 x = \frac{(\beta - \alpha) A}{(\alpha + \beta + 2)} (k_1^2 + k_2^2) \tag{2.15}$$

An approximate solution of (2.1) under the condition given in (1.4), is thus

$$x^* = \left[A - \frac{(\beta - \alpha) A}{(\alpha + \beta + 2)} \left(1 + \frac{k_2^2}{k_1^2} \right) \right] \cos(k_1 t) + \frac{(\beta + \alpha) A}{(\alpha + \beta + 2)} \left(1 + \frac{k_2^2}{k_1^2} \right) \tag{2.16}$$

and the approximate period is given by

$$T = 2\pi / k_1 \tag{2.17}$$

3. Particular Cases

(i) When we take $p_2 = q_2 = p_3 = q_3 =, \ldots, = p_r = q_r = 0\ p^{(i)} = 0 = q^{(i)} (i = 2, \ldots, r), m' = 1, n' = 0, p' = 0, q' = 1, b_1' = 0, \beta_1' = 1, z_i \to 0 (i = 2, \ldots, r) = 1, b_1' \to 0 (i = 2, , \ldots, r)$ in (2.16) and equation (1.8), then we give at the general exponential non-linear differential equation given by $\ddot{x} + w_0^2 x^{\rho} \exp(-z_1 x_1^{\sigma_1}) = 0$ and its corresponding solution.

(ii) When $p_2 = q_2 = p_3 = q_3 =, \ldots, = p_r = q_r = 0,\ p^{(i)} = 0 = q^{(i)} (i = 2, r), m' = 1, n' = 1, p' = 0, q' = 1, b_1' = 0, \beta_1' = 1, \alpha_1' = 1 - \alpha', \alpha_1' = 1, z_i \to 0 (i = 2, \ldots, r)$ and using equation (1.10), then we find a binomial non-linear differential equation $\ddot{x} w_0^2 x^{\rho} (1 - z_1 x_1^{\sigma_1})^{-\alpha'} = 0,$ and its corresponding solution.

(iii) When $p_2 = q_2 = p_3 = q_3 =, \ldots, = p_r = q_r = 0, p^{(i)} = 0 = q^{(i)} (i = 2, \ldots, r)$, $m' = 1, n' = 0, p' = 0, q' = 2, b_1' = v'/2, \beta_1' = 1, b_2' = -v'/2, \beta_2' = 1, z_i \to 0 (i = 2, \ldots r)$ and using equation (1.11), then we find a Bessel non-linear differential equation $\ddot{x} + w_0^2 x^\rho J_{v'}(z_1 x^{\sigma_1}) = 0$, and its corresponding solution.

(iv) When $p_2 = q_2 = p_3 = q_3 =, \ldots, = p_r = q_r = 0, p^{(i)} = 0 = q^{(i)} (i = 2, \ldots, r)$, $m' = 1, n' = 0, p' = 0, q' = 2, b_1' = 1/2, \beta_1' = 1, b_2' = 0, \beta_2' = 1, z_i \to 0 (i = 2, \ldots r)$ and using equation (1.9), then we find a Trigonometric non-linear differential equation $\pi^{1/2} \ddot{x} + w_0^2 x^\rho \sin(z_1 x) = 0$, and its corresponding solution.

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