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SOME RESULTS ON k-PRIME LABELING OF GRAPHS ON GRAPH OPERATIONS

G. VIJAYALAKSHMI and S. TERESA AROCKIAMARY

Department of Mathematics
Stella Maris College (Autonomous)
Affiliated to University of Madras
Chennai-600086, Tamil Nadu, India
E-mail: viji.gsekar@gmail.com
teresa_aroc@yahoo.co.in

Abstract

A k-prime labeling of a graph G is an injective function $f: V \to \{k, k+1, k+2, \ldots, k+|V|-1\}$ for some positive integer k that induces a function $f^{+}: E(G) \to N$ of the edges of G defined by $f^{+}(uv) = \gcd(f(u), f(v)), \forall e = uv \in E(G)$ such that $\gcd(f(u), f(v)) = 1$. A graph G that admits k-prime labeling is called a k-prime graph. In this paper, we apply the definition of k-prime labeling to certain classes of graphs $C_{2n} \cup C_{2n}$, $G \cup tP_m$, $G \cup K_{1,n}$, $G\hat{o}P_n$, $G\hat{o}K_{1,n}$ and $G\hat{o}F_{1,n}$ obtained through graph operations and have proved that they are k-prime. We have further investigated the existence of such a labeling by discussion through various cases.

1. Introduction

A simple graph G of order p is said to be k-prime for some positive integer k, if the vertices of the graph are labeled from k to $k+p-1$ such that the labels of every adjacent vertices are relatively prime. Such a graph is called a k-prime graph. Two integers a and b are said to be relatively prime, if their greatest common divisor $\gcd(a, b)$ is 1.

S. K. Vaidya and U. M. Prajapati [5] introduced the idea of k-prime labeling and proved that every path graph P_m, $m \geq 1$ is k-prime. We have

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studied the behaviour of certain cycle related graphs and proved that every cycle graph C_n, $n \geq 3$, tadpole graph $T_{n, m}$, barycentric subdivision $C_n(C_n)$ of cycle C_n and friendship graph F_n admit k-prime labeling [3]. Furthermore, we investigated the results on tree related graphs such as Y-tree, X-tree and extended to one point union of path graphs and proved that they admit k-prime labeling [4].

In this paper, we concentrate our study on special families of graphs obtained through certain graph operations.

2. Preliminaries

We now begin with few definitions.

Definition 2.1 [2]. Let $G_1(p_1, q_1)$ be a graph with vertex set V_1 and edge set E_1 respectively. Let $G_2(p_2, q_2)$ be another graph with vertex set V_2 and edge set E_2 respectively. The union of G_1 and G_2 is a graph $G = G_1 \cup G_2$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

Definition 2.2 [1]. If $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are two connected graphs then the graph obtained by superimposing any selected vertex of G_2 on any selected vertex of G_1 is denoted by $G_1 \hat{o} G_2$.

3. Main Results

3.1. Union of Graphs

Theorem 3.1.1. *Union of two copies of even cycle C_{2n} is k-prime for all k and $n > 1$.*

Proof. Let $G(V, E) = C_{2n} \cup C_{2n}$. Let the vertex and edge set of $C_{2n} \cup C_{2n}$ be defined as $V(G) = \{v_1, v_2, \ldots, v_{2n}, v_{2n+1}, v_{4n}\}$ and $E(G) = \{E_1 \cup E_2\}$ where $E_1 = \{v_i v_{i+1} : 1 \leq i \leq 2n-1\} \cup \{v_1 v_{2n}\}$ and $E_2 = \{v_i v_{i+1} : 2n+1 \leq i \leq 4n-1\}$ $\cup \{v_{2n+1} v_{4n}\}$ From the above definition, it is clear that the graph $C_{2n} \cup C_{2n}$ has $4n$ vertices and $4n$ edges. See Figure 1. Define an injective function $f : V \to \{k, k+1, \ldots, k+4n-1\}$ as follows:

Case 1. When $2n-1$ is prime for $k \not\equiv 0(\bmod\,(2n-1))$ and $k \not\equiv (2n-2)$ $(\bmod(2n-1))$.

$$f(v_i) = k + i - 1,\ 1 \le i \le 4n$$

Case 2. When $2n-1$ is not prime for k not a multiple factor of a and b of $(2n-1)$, $k \not\equiv 0(\bmod\,(a-1))$ and $k \not\equiv 0(\bmod\,(b-1))$.

$$f(v_i) = k + i - 1 \le i \le 4n$$

From the labeling pattern defined, it is easy to observe that the greatest common divisor for every adjacent vertices are 1.

Therefore $C_{2n} \cup C_{2n}$ is k-prime for $k \ge 1$. □

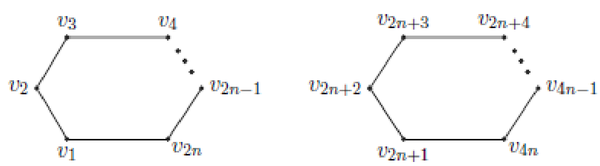

Figure 1. $C_{2n} \cup C_{2n}$.

Lemma 3.1.1. *Union of cycle graph C_n and t copies of Path graph P_m is k-prime for $n > 2$, $m > 1$ and $k, t \ge 1$.*

Proof. Let $G(V, E) = C_n \cup tP_m$. Let the vertex and edge set of $C_n \cup tP_m$ be defined as $V(G) = \{v_1, v_2, \ldots, v_n\} \cup \{u_a^b : 1 \le a \le m, 1 \le b \le t\}$ and $E(G) = E_1 \cup E_2$ where $E_1 = \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{v_1 v_n\}$ and $E_2 = \{u_a^b u_{a+1}^b : 1 \le a \le m-1, 1 \le b \le t\}$. From the above definition, it is clear that the graph $C_n \cup tP_m$ has $n + tm$ vertices and $n + m(t-1)$ edges. Define an injective function $f : V \to \{k, k+1, \ldots, k+n+tm-1\}$ as follows:

Case 1. When $n = 3, 5$ and for odd k

$$f(v_i) = k + i - 1,\ 1 \le i \le n$$

$$f(u_a^b) = k + n + (b-1)m + a - 1,\ 1 \le a \le m,\ 1 \le b \le t.$$

Case 2. When $n \equiv 0 (\bmod 2)$ and for $n \geq 4$.

Subcase 1. $n-1$ is prime

$f(v_i) = k + i - 1, 1 \leq i \leq n$ and $k \not\equiv 0 (\bmod (n-1))$

$f(u_a^b) = k + n + (b-1)m + a - 1, 1 \leq a \leq m, 1 \leq b \leq t.$

Subcase 2. $n-1$ is not prime

$f(v_i) = k + i - 1, 1 \leq i \leq n$ and k not a multiple factor of $(n-1)$

$f(u_a^b) = k + n + (b-1)m + a - 1, 1 \leq a \leq m, 1 \leq b \leq t.$

From the labeling pattern defined, it is easy to observe that the greatest common divisor for every adjacent vertices are 1.

Therefore $C_n \cup tP_m$ is k-prime for $k \geq 1$. □

Theorem 3.1.2. *Let G be k-prime graph. Then there exists a class* $G \cup tP_m$ *of graphs that is k-prime for* $k \geq 1$.

Proof. *Let* $G(p, q)$ *be a k-prime graph. Consider t copies of Path graph* P_m *with vertex set* $\{u_i^j : 1 \leq i \leq m, 1 \leq j \leq t\}$ *and edge set* $\{u_i^j u_{i+1}^j : 1 \leq i \leq m-1, 1 \leq j \leq t\}$. *The union of G and t copies of Path graph* P_m *is* $G_1 = G \cup tP_m$ *with vertex and edge set as* $V_1(G_1) = V(G) \cup \{u_i^j : 1 \leq i \leq m, 1 \leq j \leq t\}$ *and* $E_1(G_1) = E(G) \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq m-1, 1 \leq j \leq t\}$ *respectively. From the above definition, it is clear that the graph* G_1 *has* $p + tm$ *vertices and* $q + m(t-1)$ *edges. Define an injective function* $g : V_1(G_1) \to \{k, k+1, k+2, \ldots, k+p-1, k+p, k+p+1, \ldots, k+p+tm-1\}$ *as follows:*

$g(v_a) = f(v_a), \forall v_a \in V$

$g(u_i^j) = k + p + (j-1)m + i - 1, 1 \leq i \leq m, 1 \leq j \leq t$

Now we have to prove that G_1 is k-prime. Given that G is k-prime, it is enough to prove that for any edge $u_i^j u_{i+1}^j \in E_1$, which is not in

G, gcd $(g(u_i^j), g(u_{i+1}^j)) = 1$.

The above labeling function induces a edge function $g^{+} : E(G_1) \to N$ satisfying the condition of k-prime labeling as $g^{+}(u_i^j u_{i+1}^j) = \gcd(g(u_i^j), g(u_{i+1}^j))$ $= \gcd(k+p+(j-1)m+i-1, k+p+(j-1)m+i) = 1$ since $k+p+(j-1)m+i-1$ and $k+p+(j-1)m+i$ are consecutive integers.

Hence there exist a class $G \cup tP_m$ of graphs that admit k-prime labeling.

Therefore $G \cup tP_m$ is k-prime for $k \geq 1$. □

Lemma 3.1.2. *$K_{1,n} \cup K_{1,m}$ is k-prime for $k \geq 1$ and $n, m > 1$.*

Proof. Let $K_{1,n} : \{x, u_1, u_2, \ldots, u_n\}$ be a star of order $n+1$ with x as central vertex and let $K_{1,m} : \{y, v_1, v_2, \ldots, v_m\}$ be a star of order $m+1$ with y as central vertex respectively. Let $G(V, E) = K_{1,n} \cup K_{1,m}$. Let the vertex set and edge set of $K_{1,n} \cup K_{1,m}$ be defined as $V(G) = \{x, u_i : 1 \leq i \leq n\}$ $\cup \{y, v_j : 1 \leq j \leq m\}$ and $E(G) = \{xu_i : 1 \leq i \leq n\} \cup \{yv_j : 1 \leq j \leq m\}$. From the above definition, it is clear that the graph $K_{1,n} \cup K_{1,m}$ has $n+m+2$ vertices and $n+m$ edges. Let p be the largest prime from $k \leq p \leq k+n$ and let q be the largest prime from $k+n+1 \leq q \leq k+n+m+1$ respectively. Define an injective function $f : V \to \{k, k+1, \ldots, k+n+m+1\}$ as follows:

Case 1. $p = k$.

Subcase 1. When $q = k+n+1$.

$f(x) = p$,

$f(u_i) = p+i, 1 \leq i \leq n$

$f(y) = q$,

$f(v_j) = p+n+j+1, 1 \leq j \leq m$.

Subcase 2. When $q = k+n+m+1$.

$f(x) = p,$

$f(u_i) = p + i, 1 \leq i \leq n$

$f(y) = q,$

$f(v_j) = p + n + j, 1 \leq j \leq m.$

Subcase 3. When $k + n + m + 1 = q + s,\ s \geq 1.$

$f(x) = p,$

$f(u_i) = p + i, 1 \leq i \leq n$

$f(y) = q,$

$$f(v_j) = \begin{cases} p + n + j & if\ 1 \leq j \leq m - s \\ p + n + j + 1 & if\ m - s + 1 \leq j \leq m. \end{cases}$$

Case 2. $p = k + n.$

Subcase 1. When $q = k + n + m + 1.$

$f(x) = k + n = p,$

$f(u_i) = k + i - 1, 1 \leq i \leq n$

$f(y) = q = k + n + m + 1,$

$f(v_j) = k + n + j, 1 \leq j \leq m.$

Subcase 2. When $k + n + m + 1 = q + s,\ s \geq 1.$

$f(x) = k + n = p,$

$f(u_i) = k + i - 1, 1 \leq i \leq n$

$f(y) = q = k + n + m + 1 - s,$

$$f(v_j) = \begin{cases} k + n + j & if\ 1 \leq j \leq m - s \\ k + n + j + 1 & if\ m - s + 1 \leq j \leq m. \end{cases}$$

Case 3. $k + n = p + r,\ r \geq 1$

Subcase 1. When $q = k + n + 1$.

$f(x) = p = k + n - r,$

$$f(u_i) = \begin{cases} k + i - 1 & if\ 1 \leq i \leq n - r \\ k + i & if\ n - r + 1 \leq i \leq n \end{cases}$$

$f(y) = q = k + n + 1,$

$f(v_j) = k + n + j + 1,\ 1 \leq j \leq m.$

Subcase 2. When $q = k + n + m + 1$.

$f(x) = p = k + n - r,$

$$f(u_i) = \begin{cases} k + i - 1 & if\ 1 \leq i \leq n - r \\ k + i & if\ n - r + 1 \leq i \leq n \end{cases}$$

$f(y) = q = k + n + m + 1,$

$f(v_j) = k + n + j,\ 1 \leq j \leq m.$

Subcase 3. When $k + n + m + 1 = q + s,\ s \geq 1$.

$f(x) = p = k + n - r,$

$$f(u_i) = \begin{cases} k + i - 1 & if\ 1 \leq i \leq n - r \\ k + i & if\ n - r + 1 \leq i \leq n \end{cases}$$

$f(y) = q = k + n + m + 1 - s,$

$$f(v_j) = \begin{cases} k + n + j & if\ 1 \leq j \leq m - s \\ k + n + j + 1 & if\ m - s + 1 \leq j \leq m. \end{cases}$$

From the labeling pattern defined in all the above cases, it is easy to observe that the greatest common divisor for every adjacent vertices are 1.

Therefore $K_{1,n} \cup K_{1,m}$ is k-prime for $k \geq 1$.

Observation: For case $2, q = k + n + 1$ is not possible because $p = k + n$ and $q = k + n + 1$ are consecutive integers. The two consecutive integers which are prime are 2 and 3 only which is a contradiction since $n > 1$.

Remark 3.1.1. For larger values of k and for smaller $n, m,$ there may occur only one prime number from k to $k + |V| - 1$. In such cases, the graph does not satisfy k-prime labeling. □

Theorem 3.1.3. *Let G be k-prime graph. Then there exists a class $G \cup K_{1,n}$ of graphs that is k-prime for $k, n \geq 1$.*

Proof. Let $G(p, q)$ be a k-prime graph. Consider the star graph $K_{1,n}$ with vertex set $\{u, u_i : 1 \leq i \leq n\}$ and edge set $\{uu_i : 1 \leq i \leq n\}$. The union of G and $K_{1,n}$ is $G_1 = G \cup K_{1,n}$ with vertex and edge set as $V_1(G_1) = V(G) \cup \{u, u_i : 1 \leq i \leq n\}$ and $E_1(G_1) = E(G) \cup \{uu_i : 1 \leq i \leq n\}$ respectively. From the above definition, it is clear that the graph G_1 has $p + n + 1$ vertices and $q + n$ edges. Let l be the largest prime number from $k + p \leq l \leq k + p + n$. Define an injective function $g : V_1(G_1) \to \{k, k+1, k+2, \ldots, k+p-1, k+p, k+p+1, \ldots, k+p+n\}$ as follows:

$g(v_j) = f(v_j)\ \forall_{v_j} \in V$

Case 1. When $k + p = l$.

$g(u) = k + p = l$

$g(v_j) = f(v_j), \forall v_j \in V$

$g(u_i) = k + p + i, 1 \leq j \leq n.$

Case 2. When $l = k + p + n$.

$g(u) = k + p + n = l$

$g(v_j) = f(v_j), \forall v_j \in V$

$g(u_i) = k + p + i - 1, 1 \leq i \leq n.$

Case 3. When $k + p + n = l + s, s \geq 1$.

$g(u) = l = k + p + n - r,\ k + p \le l \le k + p + n$

$g(v_j) = f(v_j),\ \forall v_j \in V$

$$g(u_i) = \begin{cases} k + p + i - 1 & \text{if } 1 \le i \le n - s \\ k + p + i + 1 & \text{if } n - s + 1 \le i \le n. \end{cases}$$

Now we have to prove that G_1 is k-prime. Given that G is k-prime, it is enough to prove that for any edge $uu_i \in E_1$, which is not in G, $\gcd(g(u), g(u_i)) = 1$.

The above labeling function induces a edge function $g^+ : E(G_1) \to N$ satisfying the condition of k-prime labeling as $g^+(uu_i) = \gcd(g(u), g(u_i)) = \gcd(l, k + p + i - 1) = 1, \forall uu_i \in E_1 (1 \le i \le n - s)$ and $g^+(u_i))$ $= \gcd(g(u), g(u_i)) = \gcd(l, k + p + i + 1) = 1, \forall uu_i \in E_1(n - s + 1 \le i \le n)$ since l is the largest prime integer.

Hence there exist a class $G \cup K_{1,n}$ of graphs that admit k-prime labeling.

Therefore $G \cup K_{1,n}$ is k-prime for $k \ge 1$. □

3.2. Superimposing of Graphs

Lemma 3.2.1. *$K_{1,n}\ \hat{o}\ P_m$ is k-prime for $k \ge 1$ and $n, m > 1$.*

Proof. Let $K_{1,n} : \{u, v_1, v_2, \ldots, v_n\}$ be a star of order $n + 1$ with u as central vertex and $P_m : \{y_1, y_2, \ldots, y_m\}$ be a path of order m respectively. Let $G(V, E) = K_{1,n}\ \hat{o}\ P_m$ be the graph obtained by superimposing the central vertex u of $K_{1,n}$ with pendant vertex y_1 of P_m. See Figure 2. Let the vertex and edge set of $K_{1,n}\ \hat{o}\ P_m$ be defined as

$V(G) = \{u, v_i : 1 \le i \le n\} \cup \{y_j : 2 \le j \le m\}$

$E(G) = \{uv_i\} \cup \{uy_i\} \cup \{y_j y_{j+1} : 2 \le j \le m - 1\}.$

From the above definition, it is clear that $K_{1,n}\ \hat{o}\ P_m$ has $n + m$ vertices and $n + m - 1$ edges.

Let p be the largest prime such that $k \leq p \leq k+n$. Define an injective function $f: V \to \{k, k+1, k+2, \ldots, k+n+m-1\}$ as follows:

Case 1. When k is p

$f(u) = p$,

$f(v_i) = p + i, 1 \leq i \leq n$

$f(y_j) = p + n + j - 1, 2 \leq j \leq m.$

Case 2. When $k+n$ is p

$f(u) = p$,

$f(v_j) = k + i - 1, 1 \leq i \leq n$

$f(y_j) = k + n + j - 1, 2 \leq j \leq m.$

Case 3. When $k+n$ is $p+s$, $s \geq 1$

$f(u) = p, k \leq p \leq k+n$

$f(v_i) = k + i - 1, 1 \leq i \leq n - s$

$f(v_i) = k + i, n - s + 1 \leq i \leq n$

$f(y_j) = k + n + j - 1, 2 \leq j \leq m.$

From the labeling pattern defined, it is easy to observe that the greatest common divisor for every adjacent vertices are 1.

Therefore $K_{1,n}$ $\hat{o}$ P_m is k-prime for $k \geq 1$. □

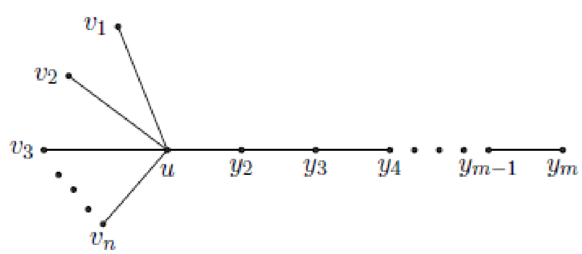

Figure 2. $K_{1,n}$ $\hat{o}$ P_m.

Theorem 3.2.1. *Let G be k-prime graph. Then there exists a class* G $\hat{o}$ P_n

of graphs that is k-prime for $k \geq 1$.

Proof. Let $G(p, q)$ be a k-prime graph. Consider the path graph P_n with vertex set $\{y_i : 1 \leq i \leq n\}$ and edge set $\{y_i y_i : 1 \leq i \leq n-1\}$. Let $G_1 = G \hat{o} P_n$ be the graph obtained by superimposing one of the pendent vertex of the path P_n say y_1 on selected vertex $v \in V$ of G. See Figure 3. Let the vertex and edge set of G_1 be defined as $V_1(G_1) = V(G) \cup \{y_i : 2 \leq i \leq n\}$ and $E_1(G_1) = E(G) \cup \{y_i y_{i+1} : 1 \leq i \leq n-1\}$. From the above definition, it is clear that the graph G_1 has $p + n - 1$ vertices and $q + n - 1$ edges. Define an injective function $g : V_1 \to \{k, k+1, k+2, \ldots, k+p-1, k+p, k+1, \ldots, k +p+n-2\}$ as follows:

$$g(v_j) = f(v_j), \forall v_j \in V$$

$$g(y_i) = k + p + i - 2, 2 \leq i \leq n$$

Now we have to prove that G_1 is k-prime. Given that G is k-prime, it is enough to prove that for any edge $y_i y_{i+1} \in E_1$, which is not in G, gcd $(g(y_i), g(y_{i+1})) = 1$.

The above labeling function induces a edge function $g^+ : E(G_1) \to N$ satisfying the condition of k-prime labeling as

$$g^+(y_i y_{i+1}) = \gcd(g(y_i), g(y_{i+1})) = \gcd(k+p+i-2, k+p+i-1) = 1, \forall y_i y_{i+1} \in E_1$$

since $k+p+i-2$ and $k+p+i-1$ are consecutive integers.

Hence there exist a class $G \hat{o} P_n$ of graphs that admit k-prime labeling.

Therefore $G \hat{o} P_n$ is k-prime for all k. □

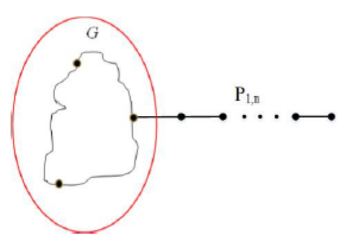

Figure 3. $G \hat{o} P_n$.

Lemma 3.2.2. *$C_n \hat{o} K_{1,n}$ is k-prime for $k \geq 1$ and $n, m > 2$.*

Proof. Let $C_n : \{a_1, a_2, \ldots, a_n\}$ be the cycle graph of order n and $K_{1,m} : \{v, b_1, b_2, \ldots, b_m\}$ be a star graph of order $m+1$ respectively. Let $G(V, E) = C_n \hat{o} K_{1,m}$ be the graph obtained by superimposing the vertex a_n of C_n with central vertex v of $K_{1,m}$. Let the vertices and edges of $C_n \hat{o} K_{1,m}$ be defined as $V(G) = \{a_i : 1 \leq i \leq n\} \cup \{b_j : 1 \leq j \leq m\}$ and $E(G) = \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{a_1 a_n\} \cup \{a_n b_j\}$. From the above definition, it is clear that the graph has $n+m$ vertices and $n+m$ edges. Let p be the largest prime number such that $k+n-1 \leq p \leq k+n+m-1$ respectively. Define an injective function $f : V \to \{k, k+1, k+2, \ldots, k+n+m-1\}$ as follows:

Case 1. When k is p

$f(a_n) = k$

$f(a_i) = k+i, 1 \leq i \leq n-1$

$f(b_j) = k+n+j-1, 1 \leq j \leq m.$

Case 2. When $k+n-1$ is p

$f(a_n) = k+n-1 = p$

$f(a_i) = k+i-1, 1 \leq i \leq n-1$

$f(b_j) = k+n+j-1, 1 \leq j \leq m.$

Case 3. When $k+n+m-1$ is p

$f(a_n) = k+n+m-1 = p$

$f(a_i) = k+i-1, 1 \leq i \leq n-1$

$f(b_j) = k+n+j-2, 1 \leq j \leq m.$

Case 4. When $k+n+m-1$ is $p+s, s \geq 1$

$f(a_n) = p = k+n+m-1,$

$f(a_i) = k+i-1, 1 \leq i \leq n-1$

$$f(b_j) = \begin{cases} k + n + j - 2 & \text{if } 1 \le j \le m - s \\ k + n + j - 1 & \text{if } m - s + 1 \le j \le m. \end{cases}$$

From the labeling pattern defined, it is easy to observe that the greatest common divisor for every adjacent vertices are 1.

Therefore $C_n \hat{o} K_{1,m}$ is k-prime for $k \ge 1$. □

Theorem 3.2.2. *Let G be k-prime graph. Then there exists a class $G \hat{o} K_{1,n}$ of graphs that is k-prime for $k \ge 1$.*

Proof. Let $G(p, q)$ be a k-prime. Let $u \in V$ be the vertex in G. Consider the star graph $K_{1,n}$ with vertex set $\{v, x_i : 1 \le i \le n\}$ and edge set $\{vx_i : 1 \le i \le n\}$. Let $G_1 = G \hat{o} K_{1,n}$ be the graph obtained by superimposing one of the vertex v of the star $K_{1,n}$ graph on selected vertex $u \in V$ of G. See Figure 4. Let the vertex set and edge set of $G_1 = G \hat{o} K_{1,n}$ be defined as $V_1(G_1) = V(G) \cup \{x_i : 1 \le i \le n\}$ and $E_1(G_1) = E(G) \cup \{vx_i : 1 \le i \le n\}$. From the above definition, it is clear that the graph G_1 has $p + n$ vertices and $q + n$ edges respectively. Define an injective function $g : V_1 \to \{k, k+1, k+2, \ldots, k+p-1, k+p, k+p+1, \ldots, k+p+n-1\}$ as follows:

$g(v_j) = f(v_j) \forall v_j \in V$

$f(u) = g(v)$ be the largest prime say l.

Case 1. When $l = k$.

$g(v) = l$

$g(v_j) = f(v_j) \ \forall v_j \in V(G)$

$g(x_i) = l + p + i, 1 \le i \le n.$

Case 2. When $l = k + p - 1$.

$g(v) = k + p - 1 = l$

$g(v_j) = f(v_j)\ \forall v_j \in V(G)$

$g(x_i) = k + p + i,\ 1 \le i \le n.$

Case 3. When $l = k + p + n - 1.$

$g(v) = k + p + n - 1 = l$

$g(v_j) = f(v_j)\ \forall v_j \in V(G)$

$g(x_i) = k + p + i - 1,\ 1 \le i \le n.$

Case 4. When $k + p + n - 1 = l + s, s \ge 1.$

$g(v) = l,\ k + p - 1 \le l \le k + p + n - 1$

$g(v_j) = f(v_j)\ \forall v_j \in V(G)$

$g(x_i) = k + p + i - 1,\ 1 \le i \le n - s$

$g(x_i) = k + p + i,\ n - s + 1 \le i \le n.$

Now we have to prove that G_1 is k-prime. Given that G is k-prime, it is enough to prove that for any edge $vx_i \in E_1$ which is not in G, gcd $(g(v), g(x_i)) = 1$.

The above labeling function induces the edge function $g^+ : E(G_1) \to N$ satisfying the condition of k-prime labeling as $g^+(vx_i) = \gcd(g(v), g(x_i))$ $= \gcd(l, k + p + i) = 1, \forall vx_i \in E_1 (1 \le i \le n - s)$ and $g^+(vx_i) = \gcd(g(v),$ $g(x_i)) = \gcd(l, k + i - 1) = 1, \forall vx_i \in E_1 (n - s + 1 \le i \le n)$ since l is the largest prime integer.

Hence there exist a class $G\ \hat{o}\ K_{1,n}$ of graphs that admit k-prime labeling.

Therefore $G\ \hat{o}\ K_{1,n}$ is k-prime for all k. □

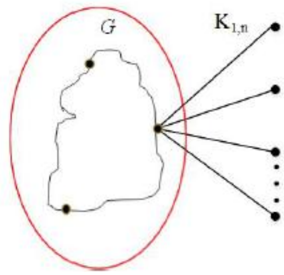

Figure 4. $G \,\hat{o}\, K_{1,n}$.

Theorem 3.2.3. *Let G be k-prime graph. Then there exists a class $G \,\hat{o}\, F_{1,n}$ of graphs that is k-prime for $k + |V_1| - 1$ as a largest prime and $k \geq 1$.*

Proof. Let $G(p, q)$ be *a* k-prime. Let $u \in V$ be the vertex in G whose label is $f(u) = l = k + p + n - 1$ where l is the largest prime number such that $K \geq l \leq k + p + n - 1$. Consider the fan graph $F_{1,n}$ with vertex set $\{x_0 x_i : 1 \leq i \leq n\}$ and edge set $\{x_0 x_i : 1 \leq i \leq n\} \cup \{x_i x_{i+1} : 1 \leq i \leq n-1\}$. Let $G_1 = G \,\hat{o}\, F_{1,n}$ be the graph obtained by superimposing the vertex x_0 of the fan $F_{1,n}$ graph on vertex $u \in V$ of G. Let the vertex set and edge set of $G_1 = G \,\hat{o}\, F_{1,n}$ be defined as $V_1(G_1) = V(G) \cup \{x_i : 1 \leq i \leq n\}$ and $E_1(G_1) = E(G) \cup \{x_0 x_i : 1 \leq i \leq n\} \cup \{x_i x_{i+1} : 1 \leq i \leq n-1\}$. From the above definition, it is clear that the graph G_1 has $p + n$ vertices and $q + 2n - 1$ edges. Define an injective function $g : V_1 \to \{k, k+1, k+2, \ldots, k+p-2, k+p-1, k+p, k+p+1, \ldots, k+p+n-2, k+p+n-1\}$ as follows:

$$f(u) = g(x_0) = l = k + p + n - 1.$$

$$g(v_j) = f(v_j) \;\forall v_j \in V$$

$$g(v_p) = g(x_0) = l = k + p + n - 1$$

$$g(x_i) = k + p + i - 2,\; 1 \leq i \leq n.$$

Now we have to prove that G_1 is k-prime. Given that G is k-prime, it is enough to prove that for any edge $x_0 x_i \in E_1$, which is not in G, gcd (gcd (x_0), $g(x_i)) = 1$.

The above labeling function induces the edge function $g^+ : E(G_1) \to N$ satisfying the condition of k-prime labeling as $g^+(x_0 x_i) = \gcd(g(x_0), g(x_i)) = \gcd(l, k + p + i - 2) = 1, \forall x_0 x_i \in E_1 (1 \leq i \leq n)$ and $g^+(x_i x_{i+1}) = \gcd(g(x_i), g(x_{i+1})) = \gcd(k + p + i - 2, k + p + i - 1) = 1, \forall x_i x_{i+1} \in E_1 (1 \leq i \leq n)$ since $k + p + i - 2$ and $k + p + i - 1$ are consecutive integers.

Hence there exist a class $G \; \hat{o} \; F_{1,n}$ of graphs that admit k-prime labeling.

Therefore $G \; \hat{o} \; F_{1,n}$ is k-prime for all k. □

4. Conclusion

To study analogous results for different families of graphs on graph operations that admits k-prime labeling is an open area of research.

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