Advances and Applications in Mathematical Sciences
Volume 19, Issue 8, June 2020, Pages 729-752

PICTURE FUZZY MULTIPLE CRITERIA DECISION MAKING BASED ON TODIM WITH TSALLIS ENTROPY WEIGHTED METHOD

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Abstract

Picture fuzzy sets proposed by Cuong and Kreinovich have been proved an effective tool in handling the uncertainty of the multiple criteria decision making (MCDM) problems. In this study, we have proposed a new picture fuzzy entropy based on well-known Tsallis entropy and the properties are investigated in a mathematical framework. An example is employed to show the validity of the proposed entropy measure. In addition, we developed an algorithm based on TODIM (An acronym in Portuguess for interactive multi-criteria decision making) and proposed entropy for solving MCDM problems under the picture fuzzy environment and the criteria weights are completely known. The criteria weights play a vital role in the solution of MCDM problems. Finally, a numerical example of selecting the governor of Bajaj Finance Limited is made to illustrate the applicability and feasibility of the proposed approach.

1. Introduction

Atanassov [1] developed the concept of intuitionistic fuzzy sets (IFSs) theory based on the notion of fuzzy set by Zadeh [2]. The application of IFSs have investigated by many authors. A characterization of IFS namely picture fuzzy set (PFS) developed by Cuong [3] with positive (υ), neutral (ν), negative (η), and refusal membership degree/grade (ϕ), respectively. He [3] studied basic operations laws and properties of PFS. Wei [4] proposed several processes to measure similarity between PFS. Peng and Dai [5] suggested an

2010 Mathematics Subject Classification: 94A15, 94A24, 26D15.
Keywords: picture fuzzy set, Tsallis entropy, Hamming distance, picture fuzzy number, TODIM.
Received November 4, 2019; Accepted November 22, 2019

algorithm for PFS based on new distance measure for the decision making process. Son [6] introduced some clustering algorithms while describing the benefits and reasons of using PFSs. Many previous studies have been used in the MCDM problems with picture fuzzy information [4, 7]. TODIM model was initially proposed by Gomes and Lima [8] based on the prospect theory. TODIM is a powerful tool widely used to handle with decision making problems under uncertainty and risk. Many scholars have suggested some TODIM approaches [9-14]. Recently, PFS is a powerful method to deal with ambiguous, uncertainty and imprecise information in real-world situations. But until now, no research work has been done by many researchers for picture fuzzy numbers (PFNs) with entropy and TODIM method. In this field, a very less research is being done on the picture fuzzy entropy from probabilistic view-point like Hung and Yang [15] for IFSs. Thus it is necessary to pay abundant attention to overcome this problem. Various researchers expand TODIM method to MCDM with PFNs to overcome this limitation [17, 18].

The prime objective of this contribution is as: (a) To develop a new entropy measure based on Tsallis entropy in picture fuzzy environment, (b) The proposed picture fuzzy information measure have been tested in a MCDM problem with TODIM model, (c) Finally, we made a practical example and a comparative study to verify the effectiveness of the proposed method.

So as to achieve the aim of the paper is set out as follows. The work done by earlier researchers in the field, source of inspiration and aims to be achieved through this contribution are depicted in Section 1. Section 2 relates to present some introduction of basic concepts, definitions and a new family of picture fuzzy information measure corresponding to Tsallis entropy is proposed and validated. Further, to verify the feasibility of proposed picture fuzzy entropy measure on the basic of linguistic variables a practical example is used in section 3. At last, the paper is summed up with conclusions and future scope in Section 4.

2. Preliminaries (Concepts and Methods)

In this section, some needed basic definitions and concepts have been reviewed related to IFS and PFS over $Z = \{p_1, p_2, \ldots, p_n\}$.

Definition 2.1 [1]**.** An IFS G in Z is defined as:

$$G = \{(p_i, \upsilon_G(p_i), \eta_G(p_i)) : p_i \in Z\}, \tag{2.1}$$

where

$$\upsilon_G : Z \to [0, 1] \text{ and } \eta_G : Z \to [0, 1],$$

with $0 \leq \upsilon_G(p_i) + \eta_G(p_i) \leq 1$, for all $p_i \in Z$. The numbers $\upsilon_G(p_i)$ and $\eta_G(p_i)$, respectively, denote the membership and negative membership degree of set G.

For each IFS G in Z, the number $\phi_G(p_i) = 1 - \upsilon_G(p_i) - \eta_G(p_i)$, $p_i \in Z$ represents hesitancy degree of p_i in Z. Also, $\phi_G(p_i)$ is known as hesitancy degree or intuitionistic index. Obviously, when $\phi_G(p_i) = 0$, that is $\eta_G(p_i) = 1 - \upsilon_G(p_i)$ for all $p_i \in Z$, IFS G alters an ordinary FS.

Definition 2.2 [3]**.** A PFS G on set Z is an object of the form as:

$$G = \{(p_i, \upsilon_G(p_i), \nu_G(p_i), \eta_G(p_i)) : p_i \in Z\} \tag{2.2}$$

where

$$\upsilon_G : Z \to [0, 1], \nu_G : Z \to [0, 1], \eta_G : Z \to [0, 1],$$

and $\upsilon_G(p_i), \nu_G(p_i), \eta_G(p_i) \in [0, 1]$, respectively, denote the positive, neutral and negative membership degrees of set G with the condition $0 \leq \upsilon_G(p_i) + \nu_G(p_i) + \eta_G(p_i) \leq 1$, for all $p_i \in Z$. Moreover, a degree of refusal membership $\phi_G(p_i)$ of p_i in G can be estimated accordingly as:

$$\phi_G(p_i) = 1 - \upsilon_G(p_i) - \nu_G(p_i) - \eta_G(p_i) \tag{2.3}$$

The neutral membership $\nu_G(p_i)$ of p_i in G can be considered of positive membership as well as degree of negative membership whereas as refusal membership $\phi_G(p_i)$ can be explained as not to take care of the system. Obviously, when $\nu_G(p_i) = 0$, then the PFSs reduce into IFS, while if $\nu_G(p_i), \eta_G(p_i) = 0$, then the PFS becomes FS.

For convenience, the pair $G = (\upsilon_G(p_i), \nu_G(p_i), \eta_G(p_i), \phi_G(p_i))$ is called a PFN and every PFN is denoted by $\beta = (\upsilon_\beta, \nu_\beta, \eta_\beta, \phi_\beta)$, where $\upsilon_\beta \in [0, 1]$, $\nu_\beta \in [0, 1]$, $\eta_\beta \in [0, 1]$, $\phi_\beta \in [0, 1]$ and $\upsilon_\beta + \nu_\beta + \eta_\beta + \phi_\beta = 1$. Sometimes, we omit ϕ_β and in short, we denote a PFN as $\beta = (\upsilon_\beta, \nu_\beta, \eta_\beta)$.

Definition 2.3 [3, 6]. Suppose $\beta_1 = (\upsilon_{\beta_1}, \nu_{\beta_1}, \eta_{\beta_1})$ and $\beta_2 = (\upsilon_{\beta_2}, \nu_{\beta_2}, \eta_{\beta_2})$ be two PFNs. Then the Hamming distance measures of β_1 and β_2 proposed by is computed as follows:

$$d_H(\beta_1, \beta_2) = \frac{1}{3}\left[(|\upsilon_{\beta_1} - \upsilon_{\beta_2}|) + (|\nu_{\beta_1} - \nu_{\beta_2}|) + (|\eta_{\beta_1} - \eta_{\beta_2}|)\right] \tag{2.4}$$

Definition 2.4. For every two PFSs G and H, Cuong et al. [3, 17] defined some operations in the universe Z as following.

1. $G \subseteq H$ iff $\forall\, p_i \in Z, \upsilon_G(p_i) \le \nu_H(p_i), \nu_G(p_i) \le \nu_H(p_i), \eta_G(p_i) \ge \eta_H(p_i)$;

2. $G = H$ iff $\forall\, p_i \in Z$, $G \subseteq H$ and $H \subseteq G$;

3. $G \cap H = \{\upsilon_G(p_i) \wedge \upsilon_H(p_i), \nu_G(p_i) \wedge \upsilon_H(p_i)$, and $\eta_G(p_i) \vee \eta_H(p_i) | p_i \in Z\}$;

4. $G \cup H = \{\upsilon_G(p_i) \vee \upsilon_H(p_i), \nu_G(p_i) \wedge \upsilon_H(p_i)$, and $\eta_G(p_i) \wedge \eta_H(p_i) | p_i \in Z\}$;

5. If $G \subseteq H$ and $H \subseteq P$ then $G \subseteq P$;

6. $(G^c)^c = G$;

7. $\text{co}\, G = G^c = \{(p_i, \eta_G(p_i) \nu_G(p_i), \upsilon_G(p_i) | p_i \in Z)\}$.

In order to compare two PFNs, we introduce the following comparison laws.

Definition 2.5 [19]. Let $\beta_1 = (\upsilon_{\beta_1}, \nu_{\beta_1}, \eta_{\beta_1})$ and $\beta_2 = (\upsilon_{\beta_2}, \nu_{\beta_2}, \eta_{\beta_2})$ be two PFNs. $H(\beta_i)(i = 1, 2)$ be the accuracy degree and score $(\beta_i)(i = 1, 2)$ be the score function values of β_1 and β_2, respectively then:

- If score $(\beta_1) <$ *score* (β_2) then $\beta_1 < \beta_2$;
- If score $(\beta_1) =$ *score* (β_2), then

(a) If $H(\beta_1) < H(\beta_2)$, implies that β_2 is superior to β_1, denoted by $\beta_1 < \beta_2$,

(b) If $H(\beta_1) = H(\beta_2)$, implies that β_1 is equivalent to β_2, denoted by $\beta_1 \equiv \beta_2$.

Definition 2.6. Wang et al. [19] introduced some laws for any PFNs $\beta_1 = (\upsilon_{\beta_1}, \nu_{\beta_1}, \eta_{\beta_1})$, $\beta_2 = (\upsilon_{\beta_2}, \nu_{\beta_2}, \eta_{\beta_2})$.

(1) $\beta_1 \otimes \beta_2 = (\upsilon_{\beta_1} + \nu_{\beta_1})(\upsilon_{\beta_2} + \nu_{\beta_2}) - \upsilon_{\beta_1}\nu_{\beta_2}, \nu_{\beta_1}\nu_{\beta_2}, 1 - (1 - \eta_{\beta_1})(1 - \eta_{\beta_2})$;

(2) $\beta_1^n = (\upsilon_{\beta_1} + \nu_{\beta_1}) - \upsilon_{\beta_1}^n, \upsilon_{\beta_1}^n 1 - (1 - \eta_{\beta_1})^n$ for $n > 0$.

2.1. Fuzzy Entropy for Picture Fuzzy Sets

In the context of FS-theory, the fuzzy entropy is a measure of uncertainty/fuzziness of a FS and denote the degree of fuzziness of a FS. De Luca and Termini [20] proposed the fuzzy entropy and a set of axioms. A measure of fuzziness in a FS should have the following four axioms:

Definition 2.7. A real function $\hat{E} \to [0, \infty)$ is called fuzzy entropy if it satisfies the following properties:

A1 (Sharpness): For all $G \in FS(Z)$, $\hat{E}(G) = 0$ is minimum iff G is crisp set.

A2 (Maximality): The value of $\hat{E}(G)$ is maximum iff G is the most fuzzy set.

A3 (Resolution): $\hat{E}(G) \geq \hat{E}(G^*)$, where G^* is the sharpened version of G.

A4 (Symmetry): $\hat{E}(G) = \hat{E}(G^c)$, where G^c is the complement set of G. Since for an IFS, $\upsilon + \nu + \eta = 1$, therefore, considering (υ, ν, η) as probability distribution, Hung and Yang [15] extended the concept of Luca and Termini [20] to introduce a new entropy for IFSs given by

Definition 2.8 [15]**.** A real valued function $\Theta : IFS(Z) \to [0, \infty)$ is called an entropy on IFS if it holds the following four properties:

(I1) Sharpness: $\Theta(G) = 0 \Leftrightarrow \Theta$ is a crisp set.

(I2) Maximality: $\Theta(G) = 1$, that is, attains maximum value $\Leftrightarrow \upsilon_G(p_i) = \upsilon_G(p_i)$ $= \phi_G(p_i) = \frac{1}{3}$, for all $p_i \in Z$

(I3) Symmetry: $\Theta(G) = \Theta(G^c)$, where G^c is the complement of set G.

(I4) Resolution: $\Theta(G) \leq \Theta(H) \Leftrightarrow G \subseteq H$, *i.e.*, $\upsilon_G \leq \upsilon_H$ and $\nu_G \leq \upsilon_H$ for max $(\upsilon_H, \nu_H) \leq \frac{1}{3}$ and $\upsilon_G \geq \upsilon_H$ and $\nu_G \geq \nu_H$ for min $(\upsilon_H, \nu_G) \geq \frac{1}{3}$.

Since, PFSs defeat over the IFSs having four components in its complete representation, that is $(\upsilon, \nu, \eta, \phi)$ satisfying the conditions $0 \leq \upsilon, \nu, \eta, \phi \leq 1$ and $\upsilon + \nu + \eta + \phi = 1$. Thus by considering the four parametric characterization of PFSs, $(\upsilon, \nu, \eta, \phi)$ can be regarded as a probability measure. This implies that value of entropy should be maximum of all components are equally abundent and should be zero if only one component is present.

Keeping these concepts in mind, we now demonstrate a new axiomatic definition of entropy for PFSs as an extension proposed by Hung and Yang [15] as follows:

Definition 2.9. A real function $en : PFSs(Z) \to [0, \infty)$ is an entropy on PFS if En holds the following four axiomatic requirements:

(P1) Sharpness: $en(G) = 0 \Leftrightarrow G$ is a crisp set.

(P2) Maximality: $en(G) = 1$, that is, attains maximum value $\Leftrightarrow \upsilon_{en}(p_i) = \nu_{en}(p_i) = \eta_{en} \leq (p_i) = \phi_{en}(p_i) = \frac{1}{4}$, for all $p_i \in Z$.

(P3) Symmetry: $en(G) = en(G^c)$, where G^c is the complement of G.

(P4) Resolution: $en(G) \leq en(H)$ if G is less fuzzy than F, that is $\upsilon_G \leq \upsilon_H$, $\nu_G \leq \nu_H$ and $\eta_G \leq \eta_H$ for max $(\upsilon_H, \nu_H, \eta_H) \leq \frac{1}{4}$ and $\upsilon_G \geq \upsilon_H$, $\nu_G \geq \nu_H$ and $\eta_G \geq \eta_H$ for min $(\upsilon_H, \nu_H, \eta_H) \geq \frac{1}{4}$.

In the next section, keeping these concepts in mind, we proposed a new picture fuzzy information measure based on Tsallis entropy.

2.2. A New Parametric Mearure for PFSs

Let $\Delta_n = \{Z = (p_1, p_2, \ldots, p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}, n \geq 2$ be a finite set of complete probability distribution. For some $Z \in \Delta_n$, Tsallis [21] entropy given by

$$M_{\omega}(Z) = \frac{1}{(\omega - 1)}\left[\sum_{i=1}^{n}(p_i - p_i^{\omega})\right] \tag{2.5}$$

where $\omega > 0 (\neq 1)$.

Particular Cases:

1. If $\omega \to 1$, (2.5) recovers the Shannon [22] entropy,

i.e., $\lim_{\omega \to 1} M_{\omega}(Z) = -\sum_{i=1}^{n}(p_i)\log(p_i)$.

2. If $p_1 = p_2 = \ldots = p_n = \frac{1}{n}, M_{\omega}(Z) = \frac{1}{1-\omega}[n^{1-\omega} - 1]$, which is an upper bound for $M_{\omega}(Z)$.

3. If $\omega = 2$, (2.5) becomes

$$M_{\omega=2}(Z) = \left[\sum_{i=1}^{n}(\sqrt{p_1} - p_i^2)\right],$$

which is an interesting entropy for probability distribution $Z \in \Delta_n$.

With the characterization of the theory of FS by Zadeh [2], a new way of quantifying the uncertainty/fuzziness came into existence. Zadeh [23] proposed a new fuzzy information measure but this measure failed to serve the purpose. Following on, Luca and Termini [20] proposed a set of axioms as criteria for fuzzy entropy and proposed a new fuzzy information measure based on Shannon entropy given by:

$$M_{LT}(G) = -\frac{1}{n}\left[\sum_{i=1}^{n}[v_G(p_i)\log_2(v_G(p_i)) + (1 - v_G(p_i))\log_2(1 - v_G(p_i))\right], \tag{2.6}$$

where $G \in FS(Z)$ and $p_i \in Z$. This development investigated the authors to propose fuzzy information measure according to their view points. Analogous to (2.6), Bhandari and Pal [24] extended the idea of Renyi [25] entropy from probabilistic setting to fuzzy settings to introduce a new fuzzy information measure given by

$$M_{BP}(G) = \frac{1}{n(1-w)} \sum_{i=1}^{n} \log_2 [v_G(p_i)^w + (1 - v_G(p_i)^w]. \tag{2.7}$$

The idea of Bhandari and Pal [24] was further generalized to IFSs by Hung and Yang [16] to introduce a new intuitionistic fuzzy entropy given by

$$M_{HY}(G) = \frac{1}{n(1-\omega)} \sum_{i=1}^{n} \log_2 [v_G(p_i)^\omega + \eta_G(p_i)^\omega + \varphi_G(p_i)^\omega]; \text{ where } \omega \in (0, 1). \tag{2.8}$$

Keeping these concepts in mind, we introduced a new information measure for PFSs by extending the idea of Vlachos and Sergiadis [16] from IF setting to picture fuzzy setting in next subsection.

Definition 2.3. For any $G \in PFSs$, we define

$$M_\omega(G) = \frac{1}{n(\omega - 1)} \sum_{i=1}^{n} [1 - (v_G(p_i)^\omega + \nu_G(p_i)^\omega + \eta_G(p_i)^\omega + \phi_G(p_i)^\omega)]. \tag{2.9}$$

Particular Cases:

1. If $\omega = 1$, then (2.9) becomes a new picture fuzzy entropy, which is an extension of Vlachos and Sergiadis [16] IF entropy as:

$$M_\omega = -\frac{1}{n} \sum_{i=1}^{n} [v_G(p_i) \log_2 (v_G(p_i)) + \nu_G(p_i) \log_2 (\nu_G(p_i)) + \eta_G(p_i) \log_2 (\eta_G(p_i))$$
$$+ \phi_G(p_i) \log_2 (\phi_G(p_i))]. \tag{2.10}$$

2. If $\omega = 1$ and $\nu_G(p_i) = 0$, then (2.9) becomes Vlachos and Sergiadis [16] entropy as:

$$M_\omega = -\frac{1}{n} \sum_{i=1}^{n} [v_G(p_i) \log_2 (v_G(p_i)) + \eta_G(p_i) \log_2 (\eta_G(p_i))$$
$$+ \phi_G(p_i) \log_2 (\phi_G(p_i))]. \tag{2.11}$$

3. If $\nu_G(p_i) = 0$ (neutral membership), then PF entropy reduces to IF entropy.

$$\text{i.e., } M_{\omega}(G) = \frac{1}{n(\omega - 1)} \sum_{i=1}^{n} [1 - (\upsilon_G(p_i)^{\omega} + \eta_G(p_i)^{\omega} + \phi_G(p_i)^{\omega})]. \tag{2.12}$$

2.3.1. Justification

Before proving the validity of proposed measure, firstly, we will prove the following property.

Property 2.1. Under the condition P_4 we have

$$| \upsilon_G(p_i) - \frac{1}{4} | + | \nu_G(p_i) - \frac{1}{4} | + | \eta_G(p_i) - \frac{1}{4} | + | \phi_G(p_i) - \frac{1}{4} |$$

$$\geq | \upsilon_H(p_i) - \frac{1}{4} | + | \nu_H(p_i) - \frac{1}{4} | + | \eta_H(p_i) - \frac{1}{4} | + | \phi_H(p_i) - \frac{1}{4} | \tag{2.13}$$

and

$$\left[\upsilon_G(p_i) - \frac{1}{4}\right]^2 + \left[\nu_G(p_i) - \frac{1}{4}\right]^2 + \left[\eta_G(p_i) - \frac{1}{4}\right]^2 + \left[\phi_G(p_i) - \frac{1}{4}\right]^2$$

$$\geq \left[\upsilon_H(p_i) - \frac{1}{4}\right]^2 + \left[\nu_H(p_i) - \frac{1}{4}\right]^2 + \left[\eta_H(p_i) - \frac{1}{4}\right]^2 + \left[\phi_H(p_i) - \frac{1}{4}\right]^2. \tag{2.14}$$

Proof. If $\upsilon_G(p_i) \leq \upsilon_H(p_i)$, $\nu_G(p_i) \leq \nu_H(p_i)$ and $\eta_G(p_i) \leq \eta_H(p_i)$ with $\frac{1}{4} \geq \max\{\upsilon_H(p_i), \upsilon_H(p_i), \eta_H(p_i)\}$ then $\upsilon_G(p_i) \leq \upsilon_H(p_i) \leq \frac{1}{4}$, $\nu_G(p_i) \leq \nu_H(p_i) \leq \frac{1}{4}$, $\eta_G(p_i) \leq \eta_H(p_i) \leq \frac{1}{4}$ and $\phi_G(p_i) \leq \phi_H(p_i) \geq \frac{1}{4}$ which shows that (2.13) and (2.14) hold. Similarly, if $\upsilon_G(p_i) \geq \upsilon_H(p_i)$, $\nu_G(p_i) \geq \nu_H(p_i)$, $\eta_G(p_i) \geq \eta_H(p_i) \leq \frac{1}{4}$ with $\max\{\upsilon_H(p_i), \nu_H(p_i), \nu_H(p_i) \geq \frac{1}{4}\}$ then (2.13) and (2.14) hold. Szmidt and Kacpryzk [26] developed the distance measure between two IFSs. The Euclidean distance or Hamming distance measure are two popular distance measures used to calculate the distance between two IFSs. Since, PFSs are the generalization of IFSs having four parameters $(\upsilon, \nu, \eta, \phi)$, thus, extending the idea of distance measure from IFSs to PFSs, it may be concluded from property (2.1) PFS H is closer to maximum value $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ than PFS G.

Theorem 2.1. *Proposed measure* (2.9) *is a valid picture fuzzy measure for PFSs.*

Proof. To establish (2.9) as a valid entropy measure for PFSs, we prove that it obey the axioms in definition (2.9).

P1: Let G is a crisp set having membership values either $\upsilon_G(p_i) = 1,$ and $\nu_G(p_i) = \eta_G(p_i) = \phi_G(p_i) = 0$ or $\nu_G(p_i) = 1,$ and $\upsilon_G(p_i) = \eta_G(p_i) = \phi_G(p_i) = 0$ or $\eta_G(p_i) = 1$ and $\upsilon_G(p_i) = \nu_G(p_i) = \phi_G(p_i) = 0$ or $\phi_G(p_i) = 1$ and $\upsilon_G(p_i) = \nu_G(p_i) = \eta_G(p_i) = 0.$

$$\Rightarrow 1 - (\upsilon_G(p_i)^{\omega} + \nu_G(p_i)^{\omega} + \eta_G(p_i)^{\omega} + \phi_G(p_i)^{\omega}) = 0.$$

Since $\omega > 0 (\omega \neq 1),$ $M_{\omega}(G) = 0.$

Conversely, if $M_{\omega}(G) = 0,$ we have

$$1 - (\upsilon_G(p_i)^{\omega} + \nu_G(p_i)^{\omega} + \eta_G(p_i)^{\omega} + \phi_G(p_i)^{\omega}) = 0.$$

Since $\omega > 0 (\omega \neq 1)$ this is possible only in the following cases:

1. either $\upsilon_G(p_i) = 0$ and $\nu_G(p_i) = \eta_G(p_i) = \phi_G(p_i) = 0$ or

2. $\nu_G(p_i) = 1,$ and $\upsilon_G(p_i) = \eta_G(x_I) = \phi_G(p_i) = 0$ or

3. $\eta_G(p_i) = 0$ and $\upsilon_G(p_i) = \nu_G(x_I) = \phi_G(p_i) = 0$ or

4. $\phi_G(p_i) = 1$ and $\upsilon_G(p_i) = \nu_G(p_i) = \eta_G(p_i) = 0.$

From all the above cases, G is a crisp set if and only if $M_{\omega}(G) = 0.$ This proves (P1).

P2: Since $\upsilon_G(p_i) + \nu_G(p_i) + \eta_G(p_i) + \phi_G(p_i) = 1,$ to obtain the maximim value of picture fuzzy entropy $M_{\omega}(G),$ we write $g(\upsilon_G, \nu_G, \phi_G) = \upsilon_G(p_i) + \nu_G(p_i) + \eta_G + \phi_G(p_i) - 1$ and taking the Lagrange's multipliers $\lambda,$ we construct the Lagrange's function as:

$$G^*(\upsilon_G, \nu_G, \phi_G) = M_{\omega}(\upsilon_G, \nu_G, \eta_G, \phi_G) + \lambda h(\upsilon_G, \nu_G, \eta_G, \phi_G). \tag{2.15}$$

To obtain the maximality of $M_{\omega}(G),$ differentiating (2.15) partially with respect to $\upsilon_M, \nu_M, \eta_M, \phi_M$ and λ and put them equal to zero, we get

$\upsilon_G(p_i) = \nu_G(p_i) = \eta_G(p_i) = \phi_G(p_i) = \frac{1}{4}$. It can be easy to calculate that all the first order derivative becomes zero, we get $\upsilon_G(p_i) = \nu_G(p_i) = \phi_G(p_i) = \eta_G(p_i) = \frac{1}{4}$. The stationary point of $M_\omega(G)$, is $\upsilon_G(p_i) = \nu_G(p_i) = \eta_G(p_i) = \phi_G(p_i) = \frac{1}{4}$. Now, we prove $M_\omega(G)$ is a concave function at the stationary points with the help of Hessian matrix.

Definition 2.10 (Hessian)**.** The Hessian matrix of a function $\Omega(x_1, x_2, x_3, x_4)$ of four variables is given by

$$\lceil HEN \rceil(\Omega) = \begin{bmatrix} \frac{\partial^2\Omega}{\partial x_1^2} & \frac{\partial^2\Omega}{\partial x_2 \partial x_1} & \frac{\partial^2\Omega}{\partial x_3 \partial x_1} & \frac{\partial^2\Omega}{\partial x_4 \partial x_1} \\ \frac{\partial^2\Omega}{\partial x_1 \partial x_2} & \frac{\partial^2\Omega}{\partial x_2^2} & \frac{\partial^2\Omega}{\partial x_3 \partial x_2} & \frac{\partial^2\Omega}{\partial x_4 \partial x_2} \\ \frac{\partial^2\Omega}{\partial x_1 \partial x_3} & \frac{\partial^2\Omega}{\partial x_2 \partial x_3} & \frac{\partial^2\Omega}{\partial x_3^2} & \frac{\partial^2\Omega}{\partial x_4 \partial x_3} \\ \frac{\partial^2\Omega}{\partial x_1 \partial x_4} & \frac{\partial^2\Omega}{\partial x_2 \partial x_4} & \frac{\partial^2\Omega}{\partial x_3 \partial x_4} & \frac{\partial^2\Omega}{\partial x_4^2} \end{bmatrix}. \quad (2.16)$$

The function Ω is said to be strictly convex at a point in its domain if $\lceil HEN \rceil(\Omega)$ is positive definite and strictly concave if $\lceil HEN \rceil(\Omega)$ is negative definite and The Hessian of $M_\omega(G)$ is given by

$$\lceil HEN \rceil(M_\omega(G)) \frac{l}{n(\omega - 1)} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (2.17)$$

which is negative definite for all $\omega > 0 (\neq 1)$, where $l = \omega(\omega - 1)4^{(\omega - 2)}$. Thus, $M_\omega(G)$ is strictly concave function for all $\omega > 0 (\neq 1)$ having its maximum value at stationary point $\upsilon_G(p_i) = \nu_G(p_i) = \eta_G(p_i) = \phi_G(p_i) = \frac{1}{4}$.

P3: Since, $M_\omega(G)$ is a concave function of $G \in PFS(Z)$, with maximum value at stationary point, then if $\max\{\upsilon_G(p_i), \nu_G(p_i), \eta_G(p_i), \phi_G(x_I)\} \leq \frac{1}{4}$, then $\upsilon_G(p_i) \le \upsilon_H(p_i)$, $\nu_G(p_i) \le \nu_H(p_i)$ and $\eta_G(p_i) \le \eta_H(p_i)$ implies $\phi_G(p_i) \ge \phi_H(p_i) \ge \frac{1}{4}$. Therefore, by using property (2.1), we see that $M_\omega(G)$ satisfies the condition P4.

Similarly, if $\min\{\upsilon_G(p_i), \nu_G(p_i), \eta_G(p_i)\} \ge \frac{1}{4}$, then $\upsilon_G(p_i) \le \upsilon_H(p_i)$, $\nu_G(p_i) \ge \nu_H(p_i)$ and $\eta_G(p_i) \ge \eta_H(p_i)$.

Again, by using property (2.1), we see that $M_\omega(G)$ satisfies condition $P4$.

P4: For any PFS, $M_\omega(G) = M_\omega(G^c)$, which is clear from the definition.

Theorem 2.2. *For any two* $G, H \in PFS(Z)$, *such that for all* $p_i \in Z$ *either* $G \subseteq H$ *or* $H \subseteq G$; *then,*

$$M_\omega(G \cup H) + M_\omega(G \cap H) = M_\omega(G) + M_\omega(H). \tag{2.18}$$

Proof. To prove theorem (2.2), seperate Z into two parts say Z_1 and Z_2, such that

$$Z_1 = \{p_i \in Z : G \subseteq H\}, \text{ and } Z_2 = \{p_i \in Z : G \supseteq H\} \tag{2.19}$$

$$\upsilon_G(p_i) \le \upsilon_H(p_i),\ \nu_G(p_i) \le \nu_H(p_i),\ \eta_G(p_i) \ge \eta_H(p_i)\ \forall\ p_i \in Z_1 \tag{2.22}$$

$$\upsilon_G(p_i) \ge \upsilon_H(p_i),\ \nu_G(p_i) \ge \nu_H(p_i),\ \eta_G(p_i) \ge \eta_H(p_i)\ \forall\ p_i \in Z_2. \tag{2.21}$$

$$\text{Now } M_\omega(G \cup H) = \frac{1}{n(\omega - 1)} \sum_{i=1}^{n} [1 - (\upsilon_{(G\cup H)}(p_i)^\omega + \nu_{(G\cup H)}(p_i)^\omega$$

$$+ \eta_{(G\cup H)}(p_i)^\omega + \phi_{(G\cup H)}(p_i)^\omega] \tag{2.22}$$

$$= \frac{1}{n(\omega - 1)} \sum_{Z_1} [(1 - (\upsilon_H(p_i)^\omega + \nu_H(p_i)^\omega + \phi_H(p_i)^\omega))]$$

$$+ \frac{1}{n(\omega - 1)} \sum_{Z_2} [1 - (\upsilon_H(p_i)^\omega + \nu_G(p_i)^\omega + \phi_G(p_i)^\omega)]. \tag{2.23}$$

Similarly, we get

$$M_\omega(G \cap H) = \frac{1}{n(\omega - 1)} \sum_{Z_1} [(1 - (\upsilon_G(p_i)^\omega + \nu_G(p_i)^\omega + \phi_G(p_i)^\omega))]$$

$$+ \frac{1}{n(\omega - 1)} \sum_{Z_2} [1 - (\upsilon_H(p_i)^\omega + \nu_H(p_i)^\omega + \phi_H(p_i)^\omega)] \quad (2.24)$$

Now, adding (2.23) and (2.24), we have

$$M_\omega(G \cup H) + M_\omega(G \cap H) = M_\omega(G) + M_\omega(H). \quad (2.25)$$

This proves the theorem.

2.3.2. Practical example

This section presents a simple example involving linguistic variables to demonstrate the evaluation of proposed entropy measure M_ω for PFSs.

Example 1. Let us consider a PFS G_1 in $Z = \{3, 4, 5, 6, 7\}$ is defined as

$$G_1\{(3, 0.1, 0.4, 0.4), (4, 0.3, 0.5, 0.2), (5, 0.6, 0.2, 0.1),$$

$$(6, 0.8, 0.1, 0.1), (7, 0.5, 0.2, 0.1)\}.$$

Then the modifier for the PFS

$$G = \{p, \upsilon_M(p), \nu_M(p), \eta_M(p) |\ p \in Z\ |)\}$$

in Z is given by

$$G^n = \{(p, (\upsilon_M(p) + \mu_M(p))^n - \mu_M(p)^n), \mu_M(x)^n, 1 - (1 - \eta_M(x))^n | p \in Z|\}. \quad (2.26)$$

Based on the operations Wang et al. [19] in equation (2.26), we have:

$$G_1^{\frac{1}{2}} = \{(3, 0.0747, 0.6325, 0.\ 2254), (4, 0.1873, 0.7071, 0.1056),$$

$$(5, 0.4472, 0.4472, 0.0513), (6, 0.6325, 0.3162, 0.0513),$$

$$(7, 0.3894, 0.4472, 0.0513)\},$$

$$G_1^2 = \{(3, 0.09, 0.16, 0.64), (4, 0.39, 0.25, 0.36), (5, 0.6, 0.04, 19), (6, 0.8, 0.01, 0.19),$$

$$(7, 0.45, 0.04, 0.19\})$$

$$G_1^3 = \{(3, 0.0610, 0.0640, 0.7840), (4, 0.3870, 0.1250, 0.4880),$$

$(5, 0.5040, 0.0080, 0.2710), (6, 0.7280, 0.001, 0.2710),$

$(7, 0.3350, 0.0080, 0.2710)\},$

$$G_1^4 = \{(3, 0.0369, 0.0256, 0.8704), (4, 0.3471, 0.0625, 0.5904),$$

$(5, 0.4080, 0.0016, 0.3439), (6, 0.6560, 0.0001, 0.3439),$

$(7, 0.2385, 0.0016, 0.3439)\}.$

By considering the classfication of linguistics variables are given by Wang et al. [19], we can regard the PFS G_1 a "LARGE" on Z. Correspondingly, to PFSs $G_1^{\frac{1}{2}}, G_1^2, G_1^3$ and G_1^4 can be treated as "Less or More High", "Very High", "Quite Very High", "Very Very High", respectively.

Intuitively, from $G_1^{\frac{1}{2}}$ to G_1^4, the loss of information hidden in them become less. The entropy conveyed by them increasing. So the following relation holds for good performance:

$$M(G_1^{\frac{1}{2}}) > M(G_1) > M(G_1^2) > M(G_1^3) > M(G_1^4). \tag{2.27}$$

Table 1. Picture Fuzziness values with proposed Entropy at different values of ω.

Entropy	$G_1^{\frac{1}{2}}$	G_1	G_1^2	G_1^3	G_1^4
$M_{\omega=2}$	0.3624	0.3813	0.3613	0.3537	0.3361
$M_{\omega=3}$	0.2820	0.2871	0.2799	0.2746	0.2640
$M_{\omega=4}$	0.2279	0.2277	0.2236	0.2210	0.2132
$M_{\omega=5}$	0.1898	0.1878	0.1853	0.1839	0.1776
$M_{\omega=6}$	0.1618	0.1595	0.1580	0.1570	0.1518
$M_{\omega=7}$	0.1405	0.1384	0.1374	0.1368	0.1324

$M_{\omega=8}$	0.1240	0.1221	0.1215	0.1211	0.1173
$M_{\omega=9}$	0.1107	0.1092	0.1088	0.1086	0.1053
$M_{\omega=10}$	0.0999	0.0987	0.0985	0.0984	0.0956

For $0 < \omega \leq 4(\neq 1)$, the proposed PF entropy measure satisfies $M_{\omega}(G_1) > M_{\omega}(GE_1^2) > M_{\omega}(G_1^{\frac{1}{2}}) > M_{\omega}(G_1^3) > M_{\omega}(G_1^4)$. That is, PFS G_1 will be assigned more entropy than the PFS $G_1^{\frac{1}{2}}$ when entropy measures $M_{\omega}(G_1)$ is applied. But for $\omega > 4$, the proposed PF entropy follows the ranking orders is listed below.

$$M_{\omega}(G_1^{\frac{1}{2}}) > M_{\omega}(G_1) > M_{\omega}(G_1^2) > M_{\omega}(G_1^3) > M_{\omega}(F_1^4).$$

From table 1, we observe that proposed entropy measure with linguistic variables for values of ω greater than 4 is more reliable and suitable to exhibit the degree of fuzziness of PFS.

2.4. Picture Fuzzy Multicriterion Decision Making Based on TODIM Method

The MCDM method with picture fuzzy environment is to verify the best choice from a set of alternatives which are calculated based on a collection of criteria, where the evaluation terms are PFN given by DMs. We try to introduce a Picture fuzzy TODIM approach to solve the MCDM problem, based on the proposed entropy measure for PFSs. To show validity and practical reasonability, we apply proposed measure in a MCDM problem involving partially known information for criteria weights for alternatives in picture fuzzy information.

TODIM Method

The essential steps for the new Picture fuzzy TODIM approach are briefly as follows: Consider $B = \{f_1, f_2, \ldots, f_m\}$ and $G' = \{g_1, g_2, \ldots, g_n\}$ is a set of alternatives and criterion, respectively. Let $D = d_{ij}$ be a picture fuzzy decision matrix, where d_{ij} denotes the preference information of the alternatives f_i with respect to the criterio g_j is expressed in terms of PFN $d_{ij} = (\upsilon_{ij}, \nu_{ij}, \eta_{ij}); 1 \le i \le m, 1 \le j \le n.$ The MCDM problem with PFNs depicted in picture fuzzy matrix as:

$D = [d_{ij}]_{m \times n} =$

$$\begin{array}{c} \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{array} \begin{bmatrix} g_1 & g_2 & \cdots & g_n \\ (\upsilon_{\beta_{11}}, \nu_{\beta_{11}}, \eta_{\beta_{11}}) & (\upsilon_{\beta_{12}}, \nu_{\beta_{12}}, \eta_{\beta_{12}}) & \cdots & (\upsilon_{\beta_{1n}}, \nu_{\beta_{1n}}, \eta_{\beta_{1n}}) \\ (\upsilon_{\beta_{21}}, \nu_{\beta_{21}}, \eta_{\beta_{21}}) & (\upsilon_{\beta_{22}}, \nu_{\beta_{22}}, \eta_{\beta_{22}}) & \cdots & (\upsilon_{\beta_{2n}}, \nu_{\beta_{2n}}, \eta_{\beta_{2n}}) \\ \vdots & \vdots & \ddots & \vdots \\ (\upsilon_{\beta_{m1}}, \nu_{\beta_{m1}}, \eta_{\beta_{m1}}) & (\upsilon_{\beta_{m2}}, \nu_{\beta_{m2}}, \eta_{\beta_{m2}}) & \cdots & (\upsilon_{\beta_{mn}}, \nu_{\beta_{mn}}, \eta_{\beta_{mn}}) \end{bmatrix}. \quad (2.28)$$

Step 1. Transform the decision matrix $D = (d_{ij})_{m \times n}$ into a normalized Picture fuzzy decision matrix is symbolized by q_{ij} as follows:

$$q_{ij} = \begin{cases} neg(d_{ij})^c, & \text{for cost criteria} \\ d_{ij} & \text{for cost benefit criteria} \end{cases} \quad (2.29)$$

where $neg(d_{ij}) = (\eta_{ij}, \nu_{ij}, \upsilon_{ij})$ denotes the complement of d_{ij}. Then, we obtain a new Picture fuzzy decision matrix $D^* = (q_{ij})_{m \times n}$.

Step 2. Determination of Completely known criteria weights

To calculate the criteria weights, we developed a technique based on the proposed entropy measure. By minimizing the sum of all information amount under all attributes, we can construct the following models.

$$Min\, \hat{T} = \sum_{j=1}^{n} w_j \sum_{i=1}^{m} M_{\omega}(\upsilon_{ij})$$

$$= \frac{1}{n(\omega - 1)} \sum_{i=1}^{m} \sum_{i=1}^{n} w_j \times [1 - (\upsilon_G(p_i)^{\omega} + \nu_G(p_i)^{\omega} + \eta_G(p_i)^{\omega} + \phi_G(p_i)^{\omega})]. \quad (2.30)$$

such that $w_j \ge 0,\ j = 1, 2, \ldots, n; \sum_{j=1}^{n} w_j = 1$ where T denotes the set of all whole information about criteria weights and $M_{\omega}(\upsilon_{ij})$ is the information measure calculated by our proposed measure.

Now, we determine the relative weight of each criterion G_j based on the weights of criteria $w = (w_1, w_2, \ldots, w_n)^T$ as:

$$w_{jr} = \frac{w_j}{w_r}, \; j, r = 1, 2, \ldots, n, \tag{2.31}$$

where w_j is the weight of the criterion $G_j w_r = \max\{w_1, w_2, \ldots, w_n\}$ and $0 \le w_{jr} \le 1$.

Step 3. With equation (2.31), compute the measurement of dominance degree of f_i over each alternative f_j concerning each criterion g_j by the following mathematical model:

$$Z_j(f_i, f_{t1}) = \begin{cases} \sqrt{\dfrac{w_{jr} d_H(q_{ij}, q_{t_1 j})}{\sum_{j=1}^{n} w_{jr}}} & \text{if} \quad q_{ij} - q_{t_{1j}} > 0 \\ 0, & \text{if} \quad q_{ij} - q_{t_1 j} > 0 \\ -\dfrac{1}{\gamma}\sqrt{\dfrac{\left(\sum_{j=1}^{n} w_{jr}\right) d_H(q_{ij}, q_{t_1 j})}{w_{jr}}} & \text{if} \quad q_{ij} - q_{t_1 j} > 0, \end{cases} \tag{2.32}$$

where $d_H(q_{ij}, q_{t_{1j}})$ is to measure the distance between the two PFNs. q_{ij} and $q_{t_{1j}}$ and the parameter $\gamma > 0$ denotes the attenuation factor of the losses. By definition if $q_{ij} > q_{t_{1j}}$, then $Z_j(f_i, f_{t_1})$ signifies a gain; $q_{ij} < q_{t_{1j}}$, then $Z_j(f_i, f_{t_1})$ represents a loss.

Step 4. Compute the dominance degree matrix of each alternative f_i, with respect to the each criteria g_j is shown below:

$$Z_j = [Z_j(f_i, f_{t_1})]_{m \times m} = \begin{array}{c} \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{array} \begin{bmatrix} g_1 & g_2 & \cdots & g_m \\ 0 & Z_j(f_1, f_2) & \cdots & Z_j(f_1, f_m) \\ Z_j(f_2, f_1) & 0 & \ldots & Z_j(f_2, f_m) \\ \vdots & \vdots & \ddots & \vdots \\ Z_j(f_m, f_1) & Z_j(f_m, f_2) & \ldots & 0 \end{bmatrix} \tag{2.33}$$

Step 5. Compute the totally dominance degree of each alternative f_i under the criterion g_j with respect to another alternatives $f_{t_1}(t_1 = 1, 2, \ldots, m)$

as follows:

$$\Delta_j(f_i, f_{t_1}) = \sum_{t_1=1}^{m} Z_j(f_i, f_{t_1}) \tag{2.34}$$

Therefore, by equation (2.34), the overall dominance degree matrix can be obtained as:

$$Z = [Z_j(f_i, f_{t_1})]_{m\times m} = \begin{array}{c} \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{array} \begin{bmatrix} g_1 & g_2 & \cdots & g_m \\ 0 & Z_j(f_1, f_2) & \ldots & Z_j(f_1, f_m) \\ Z_j(f_2, f_1) & 0 & \cdots & Z_j(f_2, f_m) \\ \vdots & \vdots & \ddots & \vdots \\ Z_j(f_m, f_1) & Z_j(f_m, f_2) & \cdots & 0 \end{bmatrix} \tag{2.35}$$

Step 6. Finally, compute the overall value or dominance degree of each alternative f_i by the following formula:

$$\xi(f_i) = \frac{\sum_{t_1=1}^{m} Z_j(f_i, f_{t_1}) - \min_i\left(\sum_{t_1=1}^{m} Z_j(f_i, f_{t_1})\right)}{\max_i\left(\sum_{t_1=1}^{m} Z_j(f_i, f_{t_1})\right) - \min_i\left(\sum_{t_1=1}^{m} Z_j(f_i, f_{t_1})\right)}, \tag{2.36}$$

where $0 \leq \xi(f_i) \leq 1$. On the value of $\xi(f_i)$, the rank of each alternative f_i is dependent. The bigger the value of $\xi(f_i)$ is, the better the alternative f_i is. Finally, the measures computed by (2.36) permit the order ranking of all alternatives.

3. Results

3.1. The Application of Picture Fuzzy TODIM Approach

Bajaj Finance Limited (BFL) is the most profitable and broadened non-bank in India with a wide arrangement of items spread across consumer, SME (Small and Medium-sized enterprises) and wealth management as well as commercial lending. BFL which was formed in 2007 is the holding company in the area of financial services for the business managing. It serves a large number of consumers in the financial services space by providing solutions for resource obtaining through general insurance, financing, income and family protection in the form of medical and life insurance. Based on some preparatory work, BFL makes further efforts in many areas. A problem, which has obtained a lot of attention in the whole country, is the choice of the governor of BFL, which considers a MCDM problem. For this purpose, we put this problem in the MCDM environment and simulate a decision making procedure of this choice problem. The choice of governor of BFL depends on some criteria, like management skill (f_1), the ability of communicating (f_2), the ability of establishing stable financial mechanisms (f_3), the degree of familiarity of governmental issues (f_4), basic education (f_5).

Decision analysis with proposed method in Picture fuzzy environment

Step 1. Construct a Picture fuzzy (PF) decision matrix D as:

Table 2. PF-decision matrix.

Decision value	g_1	g_2	g_3	g_4	g_5
f_1	(0.4,0.2,0.1)	(0.6,0.1,0.1)	(0.1,0.2,0.6)	(0.4,0.1,0.4)	(0.1,0.4,0.2)
f_2	(0.2,0.1,0.6)	(0.4,0.3,0.1)	(0.5,0.1,0.3)	(0.4,0.3,0.2)	(0.2,0.3,0.4)
f_3	(0.3,0.1,0.6)	(0.2,0.4,0.2)	(0.8,0.1,0.1)	(0.1,0.4,0.2)	(0.4,0.4,0.1)
f_4	(0.5,0.3,0.1)	(0.5,0.2,0.2)	(0.2,0.3,0.2)	(0.2,0.1,0.6)	(0.5,0.2,0.1)
f_5	(0.2,0.3,0.3)	(0.2,0.6,0.1)	(0.4,0.2,0.3)	(0.6,0.1,0.1)	(0.6,0.1,0.3)

Step 2. Since g_1 and g_4 are cost criterion and g_2, g_3 and g_5 are benefit criterion, the normalized matrix is shown in table 3.

Table 3. PF Normalized decision matrix.

Decision value	g_1	g_2	g_3	g_4	g_5
b_1	(0.1,0.2,0.4)	(0.6,0.1,0.1)	(0.1,0.2,0.6)	(0.4,0.1,0.4)	(0.1,0.4,0.2)
b_2	(0.6,0.1,0.2)	(0.4,0.3,0.1)	(0.5,0.1,0.3)	(0.2,0.3,0.4)	(0.2,0.3,0.4)
b_3	(0.6,0.1,0.3)	(0.2,0.4,0.2)	(0.8,0.0,0.1)	(0.2,0.4,0.1)	(0.4,0.4,0.1)
b_4	(0.1,0.3,0.5)	(0.5,0.2,0.2)	(0.2,0.3,0.2)	(0.6,0.1,0.2)	(0.5,0.2,0.1)
b_5	(0.2,0.3,0.3)	(0.2,0.6,0.1)	(0.4,0.2,0.3)	(0.1,0.1,0.6)	(0.6,0.1,0.3)

Step 3. The information on the weights is complete known. The partial information on attribute weights is listed in set T,

$$T = \{0.12 \le w_1 \le 0.27,\ 0.15 \le w_2 \le 0.19,\ 0.28 \le w_3 \le 0.39,\ 0.19 \le w_4 \le 0.42,\ 0.15 \le w_5 \le 0.30\}.$$

The overall entropy of each attribute can be calculated by the equations below

$$K_1 = \sum_{i=1}^{5} v_1 j = \sum_{i=1}^{5} M_\omega(q_{1j}) = 0.9857; K_2 = \sum_{i=1}^{5} M_\omega(q_{2j}) = 0.8656;$$

$$K_3 = \sum_{i=1}^{5} v_{3j} = \sum_{i=1}^{5} M_\omega(q_{3j}) = 0.9253; K_4 = \sum_{i=1}^{5} v_{4j} = \sum_{i=1}^{5} M_\omega(q_{4j}) = 0.6086;$$

$$K_5 = \sum_{i=1}^{5} v_{5j} = \sum_{i=1}^{5} M_\omega(q_{5j}) = 0.9277.$$

The optimal model to compute the weights can be constructed as;

$$\text{Min}\, T = 0.9857 w_1 + 0.8656 w_2 + 0.9253 w_3 + 0.6086 w_4 + 0.9277 w_5$$

$$\text{such that } \begin{cases} \sum_{j=1}^{5} w_j \\ w_j \ge 0,\ j = 1, 2, 3, 4, 5. \end{cases}$$

The weighting vector of the attribute can be obtained as:

$$w = (0.23,\ 0.15,\ 0.28,\ 0.19,\ 0.15)^T..$$

Step 4. Determine the dominance of each alternative f_{t_1} over the alternative ft1 corresponding to each criterion $(i,\ t_1 = 1, 2, 3, 4, 5)$. Assume the value of $\tau = 2.5$. From the set of criteria and the decision matrix D, we can construct five dominance matrices as follows:

$$Z_1 = \begin{bmatrix} & g_1 & g_2 & g_3 & g_4 & g_5 \\ f_1 & 0.0000 & -0.4578 & -0.5164 & 0.1095 & -0.5164 \\ f_2 & 0.2173 & 0.0000 & 0.0736 & 0.2000 & 0.1776 \\ f_3 & 0.1549 & -0.2108 & 0.0000 & 0.1897* & 0.1789 \\ f_4 & -0.4652 & -0.6867 & -0.6325 & 0.0000 & -0.5578 \\ f_5 & 0.1849 & -0.5578 & -0.5963 & 0.1673 & 0.0000 \end{bmatrix}$$

$$Z_2 = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} \begin{array}{c} \begin{array}{ccccc} g_1 & g_2 & g_3 & g_4 & g_5 \end{array} \\ \begin{bmatrix} 0.0000 & 0.3259 & 0.1506 & 0.2258 & 0.2380 \\ -0.4354 & 0.0000 & 0.1643 & 0.2165 & 0.2864 \\ -0.5432 & -0.4761 & 0.0000 & -0.4168 & 0.5961 \\ -0.4314 & 0.7505 & 0.1004 & 0.0000 & 0.2129 \\ 0.0560 & 0.4339 & 0.1483 & -0.5010 & 0.0000 \end{bmatrix} \end{array}$$

$$Z_3 = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} \begin{array}{c} \begin{array}{ccccc} g_1 & g_2 & g_3 & g_4 & g_5 \end{array} \\ \begin{bmatrix} 0.0000 & -03739 & -0.6269 & -04104 & -04739 \\ 0.5251 & 0.0000 & 0.3919 & 0.9169 & 0.0000 \\ 0.4978 & -0.7114 & 0.0000 & 0.2517 & 0.1949 \\ 0.1949 & -0.4104 & -0.3298 & 0.0000 & -0.4104 \\ 0.2451 & 0.0000 & -0.4104 & 0.7949 & 0.0000 \end{bmatrix} \end{array}$$

$$Z_4 = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} \begin{array}{c} \begin{array}{ccccc} g_1 & g_2 & g_3 & g_4 & g_5 \end{array} \\ \begin{bmatrix} 0.0000 & 0.7191 & -0.7404 & -0.4434 & 0.2450 \\ 0.6431 & 0.0000 & 0.6431 & -0.3443 & 0.2450 \\ 0.3098 & 0.2191 & 0.0000 & -0.3443 & 0.3286 \\ 0.1491 & 0.1298 & -0.3098 & 0.0000 & 0.3286 \\ 0.4622 & -0.4622 & -0.3701 & -0.3701 & 0.0000 \end{bmatrix} \end{array}$$

$$Z_5 = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} \begin{array}{c} \begin{array}{ccccc} g_1 & g_2 & g_3 & g_4 & g_5 \end{array} \\ \begin{bmatrix} 0.0000 & 0.1461 & -0.3652 & -0.4831 & 0.5477 \\ -0.3652 & 0.0000 & -0.4472 & -0.4831 & -0.4831 \\ 0.3461 & 0.4789 & 0.0000 & -0.3162 & 0.0000 \\ 0.1945 & 0.1432 & 0.1465 & 0.0000 & 0.1461 \\ 0.1291 & 0.1532 & 0.0000 & -0.3652 & 0.0000 \end{bmatrix} \end{array}$$

Step 5. Totaly dominance degree obtained as:

$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} \begin{array}{c} \begin{array}{ccccc} g_1 & g_2 & g_3 & g_4 & g_5 \end{array} \\ \begin{bmatrix} 0.0000 & 0.1461 & -0.3652 & -0.4831 & 0.5477 \\ -0.3652 & 0.0000 & -0.4472 & -0.4831 & -0.4831 \\ 0.3461 & 0.4789 & 0.0000 & -0.3162 & 0.0000 \\ 0.1945 & 0.1432 & 0.1465 & 0.0000 & 0.1461 \\ 0.1291 & 0.1532 & 0.0000 & -0.3652 & 0.0000 \end{bmatrix} \end{array}$$

Step 6. Compute the overall dominance degree $\xi(f_i)$ of alternatives f_i using equation (2.36)

$$\xi(f_1) = 0.0000,\ \xi(f_2) = 1.000,\ \xi(f_3) = 0.9612,\ \xi(f_4) = 0.3397,\ \xi(f_5) = 0.2983.$$

Step 7. Ranking of the alternatives is $f_2 > f_3 > f_4 > f_5 > f_1$ and the result shows that f_2 is the most optimal choice of the governor of BFL.

3.2. Comparative Analysis

To present the comparative study, the same example was computed by using the methods in the existing literatures proposed by [4, 15, 18, 27] with same weight information as shown in table 4.

Table 4. Ranking results for different Methods.

Methods	Ranking method	Ranking
Method proposed by Wei [4]	Cross entropy	$f_2 \succ f_4 \succ f_5 \succ f_3 \succ f_1$
Method proposed by Amalendu et al. [18]	New rankig method	$f_2 \succ f_5 \succ f_4 \succ f_3 \succ f_1$
Method proposed by Chunxin and Zhang [27]	Score function	$f_2 \succ f_3 \succ f_5 \succ f_2 \succ f_4$
Method proposed by Nei [15]	Comparison rule	$f_2 \succ f_3 \succ f_5 \succ f_1 \succ f_2$
Proposed method	TODIM	$f_2 \succ f_3 \succ f_4 \succ f_5 \succ f_1$

The ranking of alternatives so obtained is given by: $f_2 \succ f_3 \succ f_4 \succ f_5 \succ f_1$ with f_2 as the most suitable alternative. In our proposed method f_2 is best choice but ranking order does not matter for other alternatives. Hence, the results of proposed approach are more reasonable and simple.

4. Conclusions

PFSs are appropriate in dealing and addressing the uncertainty and vagueness information measure that occurring in MCDM problems. In this study, we have successfully developed a new entropy measure employing the

Tsallis entropy under picture fuzzy situation. The entropy evaluation model is used to determine criteria weights. Then, a model of multiple criteria decision making is presented to compute the information of the PFSs. Additionally, the operating of the proposed method is throughly explained with the assistance of a numerical example on the basis of the TODIM method. To demonstrate the effectiveness and rationality of the proposed MCDM approach, its output is compared with other MCDM problems to make a comparison. The proposed MCDM approach can also be used to other complicated problems like risk evaluation, emerging technology, uncertain decision-making, project installation, site selection etc.

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