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UP AND DOWN SHIFTED PROPORTIONAL FUZZY STOCHASTIC ORDERINGS BY USING EXPONENTIAL TRIANGULAR FUZZY RANDOM VARIABLES

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Abstract

L. A. Zadeh generated fuzzy set theory from the classical set theory, every element of the fuzzy set have degrees of membership function, At this point, the fundamentals of classical probability theory are used to motivate fuzzy probability theory. J. E. L. Priyakumar et al., (2001), introduced the definition of stochastic ordering of fuzzy random variables under the probability with fuzzy state, D. Rajan et al. (2015) presents the stochastic ordering of triangular fuzzy random variables by following J. E. L. Priyakumar.

The exponential distribution correlated to the triangular fuzzy random variables which form a novel idea Exponential triangular fuzzy random variables.

The present paper explores the innovated definitions of stochastic orderings such as up shifted (increasing) proportional, down shifted (increasing) proportional and up and down shifted proportional fuzzy stochastic ordering of exponential triangular fuzzy random variables. Few theorems have been derived with the relationship among the given definitions.

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1. Introduction

In 1965, Prof L. A. Zadeh [18], introduced fuzzy set theory. Cai et al. [5], introduced two assumption states, (1) Fuzzy state assumption: when the data is inaccuracy and uncertainty, the system behaviour is fuzzy characterized in the context of probability measure (2) Binary state assumption: the system accurately defined as functioning or failure.

Fuzzy random variable has been the object of study from the year 1978, it generated from the L. A. Zadeh [18], fuzzy set theory. M. L. Puri et al., [13] defines the concept of fuzzy random variable, the expectation of fuzzy random variable is generalize integral of a set valued function. Erich Peter Klement [6], proved a strong law of large numbers and a central limit theorem by using independent and identically distributed fuzzy random variables. Milo Stojakovic et al., [11] presents the definition and investigation of some properties of expectation (or integration) of fuzzy random variables with values in a separable Banach space. Miguel Lopez-Diaz et al., [10] present constructive definitions of fuzzy random variables and integrably bounded fuzzy random variables.

The fuzzy random variables subdued both fuzzy and probability uncertainty and it is defined as 'A mapping from a probability space to a collection of fuzzy variables is called a fuzzy random variable. A mapping from a pattern space onto the real line is known as fuzzy variable'. The further clarification for notions and fuzzy random variables may be referred to Kwakernaak [7, 8].

J. E. L. Priyakumar et al., [3] briefly talked about the concept of Kwakernaak fuzzy random variables. Arnold F. Shapiro [2], discussed about the definition of fuzzy random variables in the view of Kwakernaak (1978), Puri and Ralescu (1986) and Liu and Liu (2003). Zadeh [19] defined the probability of a fuzzy set as the expectation of its membership function. D. Singpurwalla et al., [15] presents the benefit of making probability theory work in concert with fuzzy set theory is an ability to deal with different kinds of uncertainties that may arise within the same problem.

Gudder., [4], presents the fundamentals of classical probability theory are used to motivate fuzzy probability theory and observables (fuzzy random variables) and their distributions are defined. Youssef Prince Abed et al., [17] present the strictly define membership function and consequently redefining fuzzy probability such that, each element has a degree of belongness to the set and still satisfying such desirable laws.

Y. S. Yun et al., [16] study the computation of the exponential fuzzy probabilities for fuzzy numbers driven by operations.

The most widely used stochastic orders are likelihood ratio ordering, hazard rate ordering and stochastic ordering. Stochastic orderings play a major role in probability theory, epidemic model, statistics, reliability theory, etc.

Cai et al., [5] presented a stochastic ordering application to the probability with fuzzy state (profust) life time having a monotone profust hazard rate function. Nakai [12], discusses a partially observable sequential decision problem under a shifted likelihood ratio ordering.

The stochastic ordering of one fuzzy random variables by invoking the measurability condition on the fuzzy perception function and verify assured properties and notions of “probability with fuzzy state” (profust) life time followed by Cai et al., are discussed by J. E. L. Piriyakumar et al. [3].

Lillo et al., [9] have studied in detail four shifted stochastic orders, namely the up likelihood ratio order, the down likelihood ratio order, the up hazard rate order and the down hazard rate order, that have been obtained starting from the well-known likelihood ratio order and hazard rate order.

D. Rajan et al., [14], presented a few stochastic orderings under triangular fuzzy random variables. R. Zarei et al., [20] discussed about the study of stochastic orderings for triangular fuzzy random variables and introduce the concept of *c*-fuzzy random variables. Then, the traditional definition of usual stochastic orderings were extended for c-fuzzy random variables.

Aboukalam et al., [1] have explained some new concepts about shifted hazard and shifted likelihood ratio orders. On the basis of mentioned papers, I have got an idea to derive new up shifted (increasing) proportional, down shifted (increasing) proportional and up and down shifted proportional fuzzy stochastic orderings under the new concepts exponential triangular fuzzy random variable, especially based on J. E. L. Priyakumar followed by Kwakernaak's [7, 8] fuzzy random variable.

This paper have the following sections: Section 2, presents the new definitions such as proportional fuzzy stochastic ordering, fuzzy stochastic ordering based on log-concave and log-convex function and increasing (decreasing) proportional fuzzy stochastic orderings of the exponential triangular fuzzy random variables. Section 3, presents some kinds of stochastic orderings. Section 4, presents the new definitions of up and down shift proportional fuzzy stochastic orderings of the exponential triangular fuzzy random variables and related theorems. Section 5, presents new definitions of up and down shift increasing proportional fuzzy stochastic orderings of exponential triangular fuzzy random variables and Verifies the relation among the proportional fuzzy stochastic, up shift (increasing) proportional fuzzy stochastic orderings of the exponential triangular fuzzy random variables and related theorems.

2. Preliminary Definitions

The basic notion such as fuzzy set, alpha-cut and membership function of fuzzy set are well-known. In this section, the discussion made about the improvised definitions related to fuzzy stochastic ordering of exponential triangular fuzzy random variables.

2.1 Definition. Let (Ω, F, P) be a probability space, where Ω-denotes the sample space, $\mathfrak{F}$ the σ-algebra on Ω and P a probability measure. A fuzzy set A on Ω is called a fuzzy event. Let $\mu_A(\cdot)$ be the membership function of the fuzzy event A. Then the probability of the fuzzy event A is defined by Zadeh [16] as

$$\widetilde{P}(A) = \int_{\Omega} \mu_A(\omega) dP(\omega), \ \mu_A(\omega) : \Omega \to [0, 1].$$

2.2 Definition. The exponential fuzzy probability $\widetilde{P}(A)$ of a fuzzy set A on R is defined by Y. C. Yun et al., [14] as

$$\widetilde{P}(A) = \int_{R} \mu_A(x) dP(X)$$

Where P_X is the exponential distribution.

2.3 Definition. A random variable X is said to have an exponential distribution with parameter $\theta > 0$, if its probability density function is given by: $f(x, \theta) = \theta e^{-\theta x}$, for $x \geq 0, \theta > 0$.

2.4 Definition. A triangular fuzzy random variable X with mean and standard deviation is said to have an exponential distribution with parameter $\theta > 0$, if its membership function is given by $f(x, \theta) = \theta e^{-\theta\left(\frac{X-\mu}{\sigma}\right)}$, for $(\mu - \sigma \leq x < \mu + \sigma), \theta \geq 0$.

2.3 Definition. A random variable X is said to have an exponential distribution with parameter $\theta > 0$, if its probability density function is given by: $f(x, \theta) = \theta e^{-\theta x}$, for $x \geq 0$.

Here, Mean $\geq$ Standard Deviation, if $\theta = 1$.

The probability density function of the exponential distribution function is $f(x) = e^{-x}$, for $x \geq 0$, if $\theta = 1$, were extended for the triangular fuzzy random variable with mean μ and standard deviation σ, then X become the exponential triangular fuzzy random variable with mean μ and standard deviation σ and it is denoted by $(X \sim ETFRV(\mu, \sigma))$.

2.4 Definition. Let $(X \sim ETFRV(\mu, \sigma))$ is a fuzzy number, its coordinate point is $X(\mu - \sigma, \mu, \mu + \sigma)$ and it is interpreted as the membership function

$$\mu_X(x) = \begin{cases} e^{-\frac{(\mu - X)}{-\sigma}}, & \mu - \sigma \leq x < \mu \\ e^{-\frac{(x-\mu)}{-\sigma}}, & \mu \leq x < \mu + \sigma \end{cases}.$$

The α-cut of the membership function of the exponential triangular fuzzy random variable is defined as follows.

First, the lower α-cut is derived as follows.

$$e^{-\frac{(\mu - X_\alpha^L)}{-\sigma}} \geq \alpha$$

$$-\frac{(\mu - X_\alpha^L)}{-\sigma} \geq \log \alpha$$

$$X_\alpha^L - \mu + \sigma \log \alpha \geq 0.$$

Which is the required lower α-cut of the exponential triangular fuzzy random variable.

Secondly, the upper α-cut is obtained as follows:

$$e^{-\frac{(X_\alpha^U - \mu)}{-\sigma}} \geq \alpha$$

$$-\frac{(X_\alpha^U - \mu)}{-\sigma} \geq \log \alpha$$

$$X_\alpha^U - \mu + \sigma \log \alpha \leq 0.$$

Which is the required upper α-cut of the exponential triangular fuzzy random variable.

Therefore, the exponential triangular fuzzy random variable is $P\{(X_\alpha^L - \mu + \sigma \log \alpha) \geq 0 \vee (X_\alpha^U - \mu + \sigma \log \alpha) \leq 0\}$. Where $(0 < \alpha \leq 1)$

If $\alpha = 0$, then $P\{X_\alpha^L \geq \mu - \sigma \vee X_\alpha^U \leq \mu + \sigma\}$, is known as the support of X.

3. Some Kinds of Fuzzy Stochastic Orderings

3.1. Definition. [J. Earnest Lazarus Piriyakumar et al. [3]]**.**

Let X and Y be fuzzy random variables. X is said to be Stochastically larger than Y. if $P\{(U_{X,\mu}^{*} - a) \vee (U_{X,\mu}^{**} - a) > 0\} \geq P\{(V_{Y,\mu}^{*} - a) \vee (V_{Y,\mu}^{**} - a) > 0\}$, for $a \in R$ and $\mu \in (0, 1]$.

Where $U_{X,\mu}^{*}(\omega) = \inf\{x \in R : X_\omega(x) \geq \mu\}$ and $U_{X,\mu}^{**}(\omega) = \sup\{x \in R : X_\omega(x) \geq \mu\}$.

3.2 Definition. [D. Rajan et al., [14]] If $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, then the fuzzy Likelihood ratio ordering $(X \leq_{FLRO} Y)$ is defined by

$$\frac{P\{(Y_\alpha^L - \mu_1 + \sigma_1 \log \alpha) \geq 0 \vee (Y_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}}{P\{(X_\alpha^L - \mu_1 + \sigma_1 \log \alpha) \geq 0 \vee (X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}}$$

$$\leq \frac{P\{(Y_\alpha^L - \mu_2 + \sigma_2 \log \alpha) \geq 0 \vee (Y_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}{P\{(X_\alpha^L - \mu_2 + \sigma_2 \log \alpha) \geq 0 \vee (X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}.$$

If whenever $s \leq t$ and $u \leq v$ and $s = \mu_1 - \sigma_1$, $t = \mu_2 - \sigma_2$, $u = \mu_1 - \sigma_1$ and $v = \mu_2 + \sigma_2$.

The definition of stochastic ordering of exponential triangular fuzzy random variables is derived from the above definition, put $s = \mu_1 - \sigma_1 \to -\infty$, $t = \mu_2 - \sigma_2 \to -\infty$.

3.3 Definition. If $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, then the fuzzy stochastic ordering $(X \leq_{FSO} Y)$ is defined by

$$\frac{P\{(Y_\alpha^L - \mu_1 - \sigma_1 \log \alpha) \geq 0\}}{P\{(X_\alpha^L - \mu_1 - \sigma_1 \log \alpha) \geq 0\}} \leq \frac{P\{(Y_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

if whenever $s \leq t$ and $u \leq v$ and $s = \mu_1 - \sigma_1 \to -\infty$, $t = \mu_2 - \sigma_2 \to -\infty$, $u = \mu_1 + \sigma_1$ and $v = \mu_2 + \sigma_2$.

3.4 Definition. If $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, then the proportional fuzzy stochastic ordering $(X \leq_{PFSO} Y)$ is defined by

$$\Leftrightarrow \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

Here, the RHS is increasing in exponential triangular fuzzy random variable for all λ in $(0, 1)$.

3.5 Definition. If $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, and Y has a log-concave function, then the fuzzy stochastic ordering based on log-concave function $(X \leq_{FSOLCV} Y)$ is defined by

$$\frac{P\{(bY_\alpha^U - b\mu_1 - b\sigma_1 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(bY_\alpha^U - b\mu_2 - b\sigma_2 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}} \quad (3.1)$$

And it can also be written as

$$\Leftrightarrow \frac{P\{((Y_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0)^b\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{((Y_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0)^b\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}} \quad (3.2)$$

If inequality (3.1) $\geq$ inequality (3.2), then it is log-concave function (FSOLCV), otherwise it is log-convex (FSOLCX) and the constants b be the oefficient of convex for each alpha-cut of the exponential triangular fuzzy number, it is denoted by $X \leq_{FSOLCV} Y$ and $X \leq_{FSOLCX} Y$ respectively.

All kinds of IFSO defined itself, so that the parameter of the exponential triangular fuzzy random variables are $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$ respectively.

3.6 Definition. Let $X \sim ETFRV(\mu_1, \sigma_1)$, then the increasing proportional fuzzy stochastic ordering $(X \leq_{IPFSO} X)$ is defined by

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

Here, the RHS is increasing in exponential triangular fuzzy random variables for all λ in $(0, 1)$.

3.7 Definition. Let $X \sim ETFRV(\mu_1, \sigma_1)$, then the decreasing proportional fuzzy stochastic ordering $(R \leq_{DPFSO} R)$ is defined by

$$\frac{P\{(\lambda X_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda X_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

Here, the RHS is decreasing in exponential triangular fuzzy random variables for all λ in $(0, 1)$.

4. Up and Down Shifted Proportional Fuzzy Stochastic Orderings

In this section, the relations among PFSO, USFSO, DSFSO, USPFSO, DSPFSO, UDSPFSO are discussed.

4.1 Definition. If $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, then up shift fuzzy stochastic ordering $(X \leq^{USFSO} Y)$ is defined by

$$\frac{P\{((Y_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0)\}}{P\{((X - r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{((Y_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0)\}}{P\{((X - r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

then RHS is increasing in the exponential triangular fuzzy number, $r \in (0, \mu_2 + \sigma_2)$.

4.2 Definition. If $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, then the down shift fuzzy stochastic ordering $(X \leq_{DSFSO} Y)$ is defined by

$$\frac{P\{(((Y - r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0)\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(((Y - r)_\alpha^U - (\mu_2 - r)\sigma_2 \log \alpha) \leq 0)\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

RHS is decreasing in the exponential triangular fuzzy number, $r \in (0, \mu_2 + \sigma_2), \lambda < 1$.

4.3 Definition. If $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, then the up Shift Proportional fuzzy stochastic ordering $(X \leq_{USPFSO} Y)$ is defined by

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{((X - r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{((X - r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

the RHS is increasing in the exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right), \lambda < 1$.

4.4 Definition. If $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, then the down shift proportional fuzzy stochastic ordering $(X \leq_{DSFSO} Y)$ is defined by

$$\frac{P\{(\lambda(Y - r)_\alpha^U - \lambda(\mu_1 - r) - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}}$$

$$\leq \frac{P\{(\lambda(Y - r)_\alpha^U - \lambda(\mu_2 - r) - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

then, the RHS is decreasing in the exponential triangular fuzzy number,

$r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right), \lambda < 1.$

4.5 Definition. If $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, then the up and down shift proportional fuzzy stochastic ordering $(X \leq^{UDSPFSO} Y)$ is defined by

$$\frac{P\{(\lambda(Y - r)_\alpha^U - \lambda(\mu_1 - r) - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{((X - r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}}$$

$$\leq \frac{P\{(\lambda(Y - r)_\alpha^U - \lambda(\mu_2 - r) - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{((X - r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

Here, the RHS is increasing in the exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right), \lambda < 1.$

In the following theorems, prove the characteristics of up shift proportional fuzzy stochastic ordering.

4.6 Theorem. *If* $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, *and* $X \leq_{USPFSO} Y$, *then* $\mu_1 - \sigma_1 \leq \mu_2 - \sigma_2$ *and* $\mu_1 + \sigma_1 \leq \mu_2 + \sigma_2$.

Proof. Suppose $\mu_1 - \sigma_1 > \mu_2 - \sigma_2$.

Let $\in_1$ and $\in_2$ be such that $\mu_2 - \sigma_2 < \in_1 < \mu_1 - \sigma_1 < \in_2 < \min\{\mu_1 + \sigma_1, \mu_2 + \sigma_2\}$ and

Let $\lambda \in (0, 1)$ such that $\mu_2 - \sigma_2 < \lambda\in_1 < \mu_1 - \sigma_1 < \lambda\in_2 < \min\{\mu_1 + \sigma_1, \mu_2 + \sigma_2\}$. By definition of USPFSO under the given condition, we get,

$$\frac{P\{((\lambda Y_\alpha^U - \lambda(\mu_2 - \sigma_2) - \lambda\sigma_2 \log \alpha) \leq 0)\}}{P\{((X - r)_\alpha^U - (\mu_1 - \sigma_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

$$\leq \frac{P\{((\lambda Y_\alpha^U - \lambda(\mu_1 - \sigma_1) - \lambda\sigma_1 \log \alpha) \leq 0)\}}{P\{((X - r)_\alpha^U - (\mu_1 - \sigma_1 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

$$\therefore \frac{P\{((\lambda Y_\alpha^U - \lambda^2 \in_1 - \lambda\sigma_2 \log \alpha) \leq 0)\}}{P\{((X - r)_\alpha^U - (\mu_1 - \sigma_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

$$\leq \frac{P\{((\lambda Y_\alpha^U - \lambda^2 \in_2 - \lambda\sigma_1 \log \alpha) \leq 0)\}}{P\{((X - r)_\alpha^U - (\mu_1 - \sigma_1 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

Which is a contradiction to the definition of USPFSO.

Therefore, we must have $\mu_1 - \sigma_1 \leq \mu_2 - \sigma_2$. Similarly, it can be shown that $\mu_1 + \sigma_1 \leq \mu_2 + \sigma_2$.

4.7 Theorem. *If* $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, *and* $X \leq_{USPFSO} Y$, *then* $\mu_X \leq \mu_Y$.

Proof. If $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, whenever,

$$P\{\mu_1 - \sigma_1 \leq (X - r) \leq \mu_1 + \sigma_1\} = P\{((X - r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0)\}$$

$$P\{\mu_1 - \sigma_1 \leq (Y - r) \leq \mu_1 + \sigma_1\} = P\{((Y - r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0)\}$$

$$P\{\mu_2 - \sigma_2 \leq (X - r) \leq \mu_2 + \sigma_2\} = P\{((X - r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0)\}$$

$$P\{\mu_2 - \sigma_2 \leq (Y - r) \leq \mu_2 + \sigma_2\} = P\{((Y - r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0)\}$$

$$P\{\mu_1 - \sigma_1 \leq \lambda(Y - r) \leq \mu_1\} = P\{(\lambda(Y - r)_\alpha^U - \lambda(\mu_1 - r) - \lambda\sigma_1 \log \alpha) \leq 0)\}$$

$$P\left\{\mu_2 - \sigma_2 \leq \frac{(Y-r)}{\lambda} \leq \mu_2 + \sigma_2\right\} = P\left\{\left(\frac{(Y-r)_\alpha^U}{\lambda} - \frac{(\mu_2 - r)}{\lambda} - \frac{\sigma_2 \log \alpha}{\lambda}\right) \geq 0\right\}$$

Let Y_λ be the exponential triangular fuzzy number of $\frac{Y}{\lambda}$.

Suppose, by contradiction that $\mu_X > \mu_Y$.

Since, $P\{\lambda(\mu - r) - \lambda\sigma \leq \lambda(Y - r) \leq \lambda(\mu - r) + \lambda\sigma\}$

$$= \frac{1}{\lambda} P\left\{\frac{(\mu - r) - \sigma}{\lambda} \leq \frac{(Y - r)}{\lambda} \leq \frac{(\mu - r) + \sigma}{\lambda}\right\}$$

Where $\lambda = \frac{1}{a}\langle 1, a\rangle 1$. In exponential triangular fuzzy random variable, this equation can be written as.

(1) If Y is an exponential triangular fuzzy random variable, then

$$P\{(\lambda(Y-r)_{\alpha}^{L} - (\alpha-1)\lambda\sigma_2 - \lambda(\mu_2 - r)) \geq \frac{1}{\lambda} P\left\{\left(\frac{(Y-r)_{\alpha}^{L}}{\lambda} - \frac{(\mu_2 - r)}{\lambda} - \frac{\sigma_2 \log \alpha}{\lambda}\right) \geq 0\right\}$$

(2) If X is an exponential triangular fuzzy random variable, then

$$P\{(\lambda(X-r)_{\alpha}^{L} - (\alpha-1)\lambda\sigma_2 - \lambda(\mu_2 - r)) \geq 0\}$$

$$= \frac{1}{\lambda} P\left\{\left(\frac{(X-r)_{\alpha}^{L}}{\lambda} - \frac{(\mu_2 - r)}{\lambda} - \frac{\sigma_2 \log \alpha}{\lambda}\right) \geq 0\right\}$$

Where $\lambda = \frac{1}{a}\langle 1, a\rangle 1$. It follows from the assumption that

$$\frac{P\{\lambda(\mu - r) - \lambda\sigma \leq \lambda(Y - r) \leq \lambda(\mu - r) + \lambda\sigma\}}{P\{\mu - \sigma \leq X \leq \mu + \sigma\}}$$

is increasing in exponential triangular fuzzy random variable for all λ in $(0, 1)$. Hence, $S(Y_{\lambda} - X) = 1$ for each λ in $(0, 1)$. Here, $S(Y_{\lambda} - X)$ means that the number of sign changes of the functions Y_{λ} and X. (i.e.,) X and Y_{λ} are stochastically ordered for each λ in $(0, 1)$. In particular, by taking $\lambda = \frac{\mu_Y}{\mu_X} < 1$. It follows that the exponential triangular fuzzy random variables X and $\frac{\mu_X Y}{\mu_Y}$ are stochastically ordered. Since X and $\frac{\mu_X Y}{\mu_Y}$ have the same mean, ordinary stochastic order is possible. If they have the same distribution, which is a contradiction to our assumption. Hence $\mu_X \leq \mu_Y$ holds. Now, the relations among PFSO, IPFSO, USFSO, DSFSO USIFSO, DSIFSO and USPFSO are discussed.

4.8 Theorem. *If* $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, $X \leq_{PFSO} Y$ *and* $X \leq_{DSIFSO} X$, then $X \leq_{USPFSO} Y$.

Proof. Since $X \leq^{PFSO} Y \Leftrightarrow$

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

the RHS is increasing in exponential triangular fuzzy random variable for all λ in $(0, 1)$.

Since $R \leq^{DSIFSO} R$, then

$$\frac{P\{((X-r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

the RHS is decreasing in exponential triangular fuzzy random variable, $r \in (0, \mu_2 + \sigma_2)$.

Divide the above two inequalities, we get

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

Here RHS is increasing in the exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right)$, $\lambda < 1$. Which implies that $X \leq_{USPFSO} Y$.

4.9 Theorem. *If* $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, $X \leq^{USFSO} Y$ *and* $Y \leq^{IPFSO} Y$, *then* $X \leq^{USPFSO} Y$.

Proof. Since $X \leq_{USFSO} Y$, Then

$$\frac{P\{(Y_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(Y_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

the RHS is increasing in the exponential triangular fuzzy number, $r \in (0, \mu_2 + \sigma_2)$ and Since $Y \leq_{IPFSO} Y$, then

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(Y_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{(Y_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

Here, the RHS is increasing in exponential triangular fuzzy random variable, for all λ in $(0, 1)$.

Multiply the above two inequalities, we get

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \le 0\}}{P\{((X - r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \le 0\}} \le \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \le 0\}}{P\{((X - r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \le 0\}}$$

Here, the RHS is increasing in the exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right), \lambda < 1.$ Which implies that $X \le_{USPFSO} Y$.

4.10 Theorem. *Let* $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2)), X \le_{PFSO} Y$ *and* $X \le_{USIFSO} X$, *then* $X \le_{USPFSO} Y$.

Proof. Since $X \le^{PFSO} Y$, then

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \le 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \le 0\}} \le \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \le 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \le 0\}}$$

the RHS is increasing in exponential triangular fuzzy random variable for all λ in $(0, 1)$.

Since $X \le_{USPFSO} X$, then

$$\frac{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \le 0\}}{P\{((X - r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \le 0\}} \le \frac{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \le 0\}}{P\{((X - r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \le 0\}}$$

the RHS is increasing in the exponential triangular fuzzy number, $r \in (0, \mu_2 + \sigma_2), \lambda < 1$.

Multiply the above two inequalities, we get

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \le 0\}}{P\{((X - r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \le 0\}} \le \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \le 0\}}{P\{((X - r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \le 0\}}$$

the RHS is increasing in the exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right), \lambda < 1$.

Which implies that $X \leq_{USPFSO} Y$.

This below theorem shows that the relation between USFSO, USPFSO and the exponential triangular fuzzy random variable Y is log - concave.

4.11 Theorem. *Let* $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, $X \leq_{USFSO} Y$ *and* Y *is log-concave*, *then* $X \leq_{USPFSO} Y$.

Proof. Since $X \leq^{USPFSO} Y$, Then

$$\frac{P\{(Y_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(Y_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

the RHS is increasing in the exponential triangular fuzzy number, $r \in (0, \mu_2 + \sigma_2)$.

Since Y is log-concave function, then

$$P\{(bY_\alpha^U - b\mu_1 - b\sigma_1 \log \alpha) \leq 0\} \geq P\{(Y_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0)^b\}, \; 0 \leq b \leq 1.$$

Now, the above inequality become,

$$\frac{P\{(bY_\alpha^U - b\mu_1 - b\sigma_1 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(bY_\alpha^U - b\mu_2 - b\sigma_2 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

Here, we use the relationship between λ and $b(\lambda < b, \lambda \neq 0, \lambda < 1$ and $0 \leq b \leq 1)$.

Then we get

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

the RHS is increasing in the exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right)$, and $\lambda < 1$. Which implies that $X \leq_{USPFSO} Y$.

The following theorem shows that the relationship among USPFSO, DSPFSO, PFSO and UDSPFSO.

Theorem. *If* $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, $X \leq_{USPFSO} Y$ and

$X \leq_{DSPFSO} Y$, *then* $X \leq_{UDSPFSO} Y$ *and* $X \leq_{PFSO} Y$.

Proof. Since, $X \leq^{USPFSO} Y$, then

$$\frac{P\{(\lambda Y_{\alpha}^{U} - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{((X - r)_{\alpha}^{U} - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Y_{\alpha}^{U} - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{((X - r)_{\alpha}^{U} - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

the RHS is increasing in the triangular fuzzy number, $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right), \lambda < 1$.

Since $X \leq^{DSPFSO} Y$, then

$$\frac{P\{(\lambda(Y - r)_{\alpha}^{U} - \lambda(\mu_1 - r) - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(X_{\alpha}^{U} - \mu_1 - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda(Y - r)_{\alpha}^{U} - \lambda(\mu_2 - r) - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{(X_{\alpha}^{U} - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

the RHS is decreasing in the exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right), \lambda < 1$.

Multiplying of the above two inequalities, we get

$$\frac{P\{(\lambda Y_{\alpha}^{U} - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{((X - r)_{\alpha}^{U} - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda(Y - r)_{\alpha}^{U} - \lambda(\mu_1 - r) - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(X_{\alpha}^{U} - \mu_1 - \sigma_1 \log \alpha) \leq 0\}}$$

$$\frac{P\{(\lambda Y_{\alpha}^{U} - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{((X - r)_{\alpha}^{U} - (\mu_2 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda(Y - r)_{\alpha}^{U} - \lambda(\mu_2 - r) - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{(X_{\alpha}^{U} - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

$$\frac{P\{(\lambda(Y - r)_{\alpha}^{U} - \lambda(\mu_1 - r) - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{((X - r)_{\alpha}^{U} - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Y_{\alpha}^{U} - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(X_{\alpha}^{U} - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

$$\leq \frac{P\{(\lambda(Y - r)_{\alpha}^{U} - \lambda(\mu_2 - r) - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{((X - r)_{\alpha}^{U} - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Y_{\alpha}^{U} - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{(X_{\alpha}^{U} - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

We get,

$$\frac{P\{(\lambda(Y - r)_{\alpha}^{U} - \lambda(\mu_1 - r) - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{((X - r)_{\alpha}^{U} - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda(Y - r)_{\alpha}^{U} - \lambda(\mu_2 - r) - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{((X - r)_{\alpha}^{U} - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

and

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \le 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \le 0\}} \le \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_1 \log \alpha) \le 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \le 0\}}$$

Which implies that $X \le^{UDSPFSO} Y$ and $X \le^{PFSO} Y$, where $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right)$, $\lambda < 1$.

5. Up and Down Shifted Increasing Proportional Fuzzy Stochastic Orderings

In this section, among the relationship IPFSO, DPFSO, USIFSO, DSIFSO, USIPFSO, DSIPFSO are discussed.

5.1 Definition. If $X \sim ETFRV(\mu_1, \sigma_1)$, then the up shift increasing proportional fuzzy stochastic ordering $(X \le_{USIPFSO} X)$ is defined by

$$\frac{P\{(\lambda X_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \le 0\}}{P\{((X - r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \le 0\}} \le \frac{P\{(\lambda X_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \le 0\}}{P\{((X - r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \le 0\}}$$

the RHS is increasing in exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right)$, $\lambda < 1$.

5.2 Definition. If $X \sim ETFRV(\mu_1, \sigma_1)$, then the down shift increasing proportional fuzzy stochastic ordering $(X \le^{DSIPFSO} X)$ is defined by

$$\frac{P\{(\lambda (X - r)_\alpha^U - \lambda(\mu_1 - r) - \lambda\sigma_1 \log \alpha) \le 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \le 0\}} \le \frac{P\{(\lambda (X - r)_\alpha^U - \lambda(\mu_2 - r) - \lambda\sigma_2 \log \alpha) \le 0\}}{P\{(X_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \le 0\}}$$

the RHS is decreasing in the exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right)$, $\lambda < 1$.

5.3 Theorem. *If* $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, $X \le^{USIFSO} X$ and $X \le^{IPFSO} X$, then $X \le^{USIPFSO} X$.

Proof. Since, $X \leq^{USIPFSO} X$, then

$$\frac{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

Where RHS is increasing in the exponential triangular fuzzy number, $r \in (0, \mu_2 + \sigma_2)$ and $\lambda < 1$.

Since, $X \leq^{IPFSO} X$, then

$$\frac{P\{(\lambda X_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda X_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}$$

Where RHS is increasing in exponential triangular fuzzy random variable for all λ in $(0, 1)$.

Multiply the above two inequalities, we get

$$\frac{P\{(\lambda X_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda X_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

the RHS is increasing in the exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right)$, $\lambda < 1$. Which implies that $X \leq_{USIPFSO} X$.

5.4 Theorem. *If* $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, $X \leq^{USFSO} Y$ *and* $X \leq^{IPFSO} X$, *then* $X \leq^{USIPFSO} X$ *and* $X \leq^{FSO} Y$.

Proof. Since $X \leq^{USFSO} Y$, Then

$$\frac{P\{(Y_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(Y_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

Where RHS is increasing in the exponential triangular fuzzy number, $r \in (0, \mu_2 + \sigma_2)$.

Since $X \leq^{IPFSO} X$, then

$$\frac{P\{(\lambda X_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log\alpha) \le 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log\alpha) \le 0\}} \le \frac{P\{(\lambda X_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log\alpha) \le 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log\alpha) \le 0\}}$$

RHS is increasing in the exponential triangular fuzzy random variable for all λ in $(0, 1)$.

Multiply the above two inequalities, we get

$$\frac{P\{(Y_\alpha^U - \mu_1 - \sigma_1 \log\alpha) \le 0\}}{P\{((X-r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log\alpha) \le 0\}} \le \frac{P\{(\lambda X_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log\alpha) \le 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log\alpha) \le 0\}}$$

$$\le \frac{P\{(Y_\alpha^U - \mu_2 - \sigma_2 \log\alpha) \le 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_1 \log\alpha) \le 0\}} \le \frac{P\{(\lambda X_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log\alpha) \le 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log\alpha) \le 0\}}$$

$$\frac{P\{(\lambda X_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log\alpha) \le 0\}}{P\{((X-r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log\alpha) \le 0\}} \le \frac{P\{(Y_\alpha^U - \mu_1 - \sigma_1 \log\alpha) \le 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log\alpha) \le 0\}}$$

$$\le \frac{P\{(\lambda X_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log\alpha) \le 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_1 \log\alpha) \le 0\}} \le \frac{P\{(Y_\alpha^U - \mu_2 - \sigma_2 \log\alpha) \le 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log\alpha) \le 0\}}$$

Then

$$\frac{P\{(\lambda X_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log\alpha) \le 0\}}{P\{((X-r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log\alpha) \le 0\}} \le \frac{P\{(\lambda X_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log\alpha) \le 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log\alpha) \le 0\}}$$

and

$$\frac{P\{(Y_\alpha^U - \mu_1 - \sigma_1 \log\alpha) \le 0\}}{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log\alpha) \le 0\}} \le \frac{P\{(Y_\alpha^U - \mu_2 - \sigma_2 \log\alpha) \le 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log\alpha) \le 0\}}$$

Which implies that $X \le^{USIPFSO} X$ and $X \le^{FSO} Y$, whenever $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right)$ and $\lambda < 1$.

5.5 Theorem. *If* $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, $X \le^{DSPFSO} Y$ and $X \le^{FSO} Y$, then $Y \le^{DSIPFSO} Y$.

Proof. Since $X \le^{DSIPFSO} Y$, then

$$\frac{P\{(\lambda(Y-r)_{\alpha}^{U}-\lambda(\mu_1-r)-\lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(X_{\alpha}^{U}-\mu_1-\sigma_1 \log \alpha) \leq 0\}}$$

$$\leq \frac{P\{(\lambda(Y-r)_{\alpha}^{U}-\lambda(\mu_2-r)-\lambda\sigma_2 \log \alpha) \leq 0\}}{P\{(X_{\alpha}^{U}-\mu_2-\sigma_2 \log \alpha) \leq 0\}}$$

the RHS is decreasing in the exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2+\sigma_2}{\lambda}\right), \lambda < 1$.

Since $X \leq_{FSO} Y$, then

$$\frac{P\{(Y_{\alpha}^{U}-\mu_1-\sigma_1 \log \alpha) \leq 0\}}{P\{(X_{\alpha}^{U}-\mu_1-\sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(Y_{\alpha}^{U}-\mu_2-\sigma_2 \log \alpha) \leq 0\}}{P\{(X_{\alpha}^{U}-\mu_2-\sigma_2 \log \alpha) \leq 0\}}$$

Divide the above two inequalities, we get

$$\frac{P\{(\lambda(Y-r)_{\alpha}^{U}-\lambda(\mu_1-r)-\lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(Y_{\alpha}^{U}-\mu_1-\sigma_1 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda(Y-r)_{\alpha}^{U}-\lambda(\mu_2-r)-\lambda\sigma_2 \log \alpha) \leq 0\}}{P\{(Y_{\alpha}^{U}-\mu_2-\sigma_2 \log \alpha) \leq 0\}}$$

the RHS is decreasing in exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2+\sigma_2}{\lambda}\right), \lambda < 1$.

Which implies that $Y \leq^{DSIPFSO} Y$.

In this section, the relations among PFSO, USPFSO and USIPFSO using parameters μ and σ are discussed.

5.6 Theorem. *Let* $X, Y \sim ETFRV((\mu_1, \sigma_1), (\mu_2, \sigma_2))$, *and* $X \leq_{USIPFSO} Y$, *then there exists an exponential triangular fuzzy random variable Z is USIPFSO with mean* $\frac{\mu_1+\mu_2}{2}$ *and standard deviation* $\left|\frac{\sigma_1-\sigma_2}{2}\right|$ *such that* $X \leq_{PFSO} Z \leq_{PFSO} Y$.

Proof. Since, $X \leq^{USIPFSO} Y$, then

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log\alpha) \le 0\}}{P\{((X-r)_\alpha^U - (\mu_1 - r) - \sigma_1 \log\alpha) \le 0\}} \le \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log\alpha) \le 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log\alpha) \le 0\}}$$

the RHS is increasing in the exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right), \lambda < 1$.

It can also be written as

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log\alpha) \le 0\}}{P\{(\lambda Y_\alpha^U - \mu_2 - \sigma_2 \log\alpha) \le 0\}} \le \frac{P\{(\lambda Y_\alpha^U - \lambda(\mu_1 - r) - \lambda\sigma_1 \log\alpha) \le 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log\alpha) \le 0\}}$$

Now, By definition of (USIPFSO) Z have the parameters mean $\frac{\mu_1 + \mu_2}{2} = \mu_3 = \mu_4$ and standard deviation $\left|\frac{\sigma_1 - \sigma_2}{2}\right| = \sigma_3 = \sigma_4$ and it properly defined on $(\mu_2 - \sigma_2 \le \mu_1 + \sigma_1)$.

Then $Z \le_{USIPFSO} Z$

$$\frac{P\{(\lambda Z_\alpha^U - \lambda\mu_3 - \lambda\sigma_3 \log\alpha) \le 0\}}{P\{((Z-r)_\alpha^U - (\mu_3 - r) - \sigma_3 \log\alpha) \le 0\}} \le \frac{P\{(\lambda Z_\alpha^U - \lambda\mu_4 - \lambda\sigma_4 \log\alpha) \le 0\}}{P\{((X-r)_\alpha^U - (\mu_4 - r) - \sigma_4 \log\alpha) \le 0\}}$$

The RHS is increasing in the exponential triangular fuzzy number, $r \in \left(0, \frac{\mu_2 + \sigma_2}{\lambda}\right), \lambda = \frac{1}{a}, a > 1$.

The exponential triangular fuzzy number Z lies between X and Y. we get

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log\alpha) \le 0\}}{P\{(\lambda Y_\alpha^U - \mu_2 - \sigma_2 \log\alpha) \le 0\}} \le \frac{P\{(\lambda Z_\alpha^U - \lambda\mu_3 - \lambda\sigma_3 \log\alpha) \le 0\}}{P\{((Z-r)_\alpha^U - (\mu_3 - r) - \sigma_3 \log\alpha) \le 0\}}$$

$$\le \frac{P\{(\lambda Z_\alpha^U - \lambda\mu_4 - \lambda\sigma_4 \log\alpha) \le 0\}}{P\{((Z-r)_\alpha^U - (\mu_4 - r) - \sigma_4 \log\alpha) \le 0\}} \le \frac{P\{((X-r)_\alpha^U - \lambda(\mu_1 - r) - \lambda\sigma_1 \log\alpha) \le 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log\alpha) \le 0\}}$$

$$\Rightarrow \frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log\alpha) \le 0\}}{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log\alpha) \le 0\}} \le \frac{P\{(\lambda Z_\alpha^U - \lambda\mu_3 - \lambda\sigma_3 \log\alpha) \le 0\}}{P\{(Z_\alpha^U - \lambda\mu_4 - \lambda\sigma_4 \log\alpha) \le 0\}}$$

$$\leq \frac{P\{((Z-r)_\alpha^U - (\mu_3 - r) - \sigma_3 \log \alpha) \leq 0\}}{P\{((Z-r)_\alpha^U - (\mu_4 - r) - \sigma_4 \log \alpha) \leq 0\}} \leq \frac{P\{((X-r)_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{((X-r)_\alpha^U - (\mu_2 - r) - \sigma_2 \log \alpha) \leq 0\}}$$

Put $(Z-r)_\alpha^U = Z_\alpha^U - r$ and $(X-r)_\alpha^U = X_\alpha^U - r$, we get

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Z_\alpha^U - \lambda\mu_3 - \lambda\sigma_3 \log \alpha) \leq 0\}}{P\{(\lambda Z_\alpha^U - \lambda\mu_4 - \lambda\sigma_4 \log \alpha) \leq 0\}}$$

$$\leq \left(\frac{P\{(Z_\alpha^U - \mu_3 - \sigma_3 \log \alpha) \leq 0\}}{P\{(Z_\alpha^U - \mu_4 - \sigma_4 \log \alpha) \leq 0\}} \leq \frac{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\} - r}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\} - r} \right)$$

$$\Rightarrow \left(\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Z_\alpha^U - \lambda\mu_3 - \lambda\sigma_3 \log \alpha) \leq 0\}}{P\{(\lambda Z_\alpha^U - \lambda\mu_4 - \lambda\sigma_4 \log \alpha) \leq 0\}} \right)$$

$$\leq \begin{pmatrix} (P\{(Z_\alpha^U - \mu_3 - \sigma_3 \log \alpha) \leq 0\} - r)(P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\} - r) \\ \leq (P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\} - r)(P\{(Z_\alpha^U - \mu_4 - \sigma_4 \log \alpha) \leq 0\} - r) \end{pmatrix}$$

After simplification, we get

$$\left(\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Z_\alpha^U - \lambda\mu_3 - \lambda\sigma_3 \log \alpha) \leq 0\}}{P\{(\lambda Z_\alpha^U - \lambda\mu_4 - \lambda\sigma_4 \log \alpha) \leq 0\}} \right)$$

$$\leq \left(\frac{P\{(Z_\alpha^U - \mu_3 - \sigma_3 \log \alpha) \leq 0\}}{P\{(Z_\alpha^U - \mu_4 - \sigma_4 \log \alpha) \leq 0\}} \leq \frac{P\{(X_\alpha^U - \mu_1 - \sigma_1 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_2 - \sigma_2 \log \alpha) \leq 0\}} \right)$$

Taking second and third terms,

$$\frac{P\{(\lambda Z_\alpha^U - \lambda\mu_3 - \lambda\sigma_3 \log \alpha) \leq 0\}}{P\{(\lambda Z_\alpha^U - \lambda\mu_4 - \lambda\sigma_4 \log \alpha) \leq 0\}} \leq \frac{P\{(Z_\alpha^U - \mu_3 - \sigma_3 \log \alpha) \leq 0\}}{P\{(Z_\alpha^U - \mu_4 - \sigma_4 \log \alpha) \leq 0\}}$$

It can be written as

$$\frac{P\{(Z_\alpha^U - \mu_4 - \sigma_4 \log \alpha) \leq 0\}}{P\{(X_\alpha^U - \mu_3 - \sigma_3 \log \alpha) \leq 0\}} \leq \frac{P\{(\lambda Z_\alpha^U - \lambda\mu_4 - \lambda\sigma_4 \log \alpha) \leq 0\}}{P\{(\lambda Z_\alpha^U - \lambda\mu_3 - \lambda\sigma_3 \log \alpha) \leq 0\}}$$

Use this terms, we get

$$\frac{P\{(\lambda Y_\alpha^U - \lambda\mu_1 - \lambda\sigma_1 \log \alpha) \leq 0\}}{P\{(\lambda Y_\alpha^U - \lambda\mu_2 - \lambda\sigma_2 \log \alpha) \leq 0\}} \leq \frac{P\{(Z_\alpha^U - \mu_4 - \sigma_4 \log \alpha) \leq 0\}}{P\{(Z_\alpha^U - \mu_3 - \sigma_3 \log \alpha) \leq 0\}}$$

$$\leq \frac{P\{(\lambda Z_{\alpha}^{U} - \lambda\mu_{4} - \lambda\sigma_{4}\log\alpha) \leq 0\}}{P\{(\lambda Z_{\alpha}^{U} - \lambda\mu_{3} - \lambda\sigma_{3}\log\alpha) \leq 0\}} \leq \frac{P\{(X_{\alpha}^{U} - \mu_{1} - \sigma_{1}\log\alpha) \leq 0\}}{P\{(X_{\alpha}^{U} - \mu_{2} - \sigma_{2}\log\alpha) \leq 0\}}$$

Here, the RHS is increasing in exponential triangular fuzzy random variable for all λ in (0, 1).

This implies that $Z \leq^{PFSO} Y$ and $X \leq^{PFSO} Z$.

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