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CIRCUMSCRIBED AND INSCRIBED HYPERBOLAS OF A TRIANGLE IN THE LORENTZ-MINKOWSKI PLANE

ABDÜLAZIZ AÇIKGÖZ[1] and NILGÜN SÖNMEZ[2]

[1,2]Department of Mathematics
Faculty of Science and Arts
Afyon Kocatepe University
ANS Campus 03200, Afyonkarahisar, Turkey

Abstract

In Lorentz-Minkowski plane, the shape of circle changes to equilateral hyperbola. So, it is different from the circle in the Euclidean plane. Therefore, in this paper, it is investigated whether Euclidean incircle and circumcircle theorems for triangles hold in the Lorentz-Minkowski plane. Also, we give the functional relationship between them in terms of slopes of sides of the triangle.

Introduction

Lorentz geometry is created by taking the Lorentz distance function instead of the Euclidean distance function. The basic notions, inner product, metric and vector classification in Lorentz space are given in [1] and [2].

Lorentz inner product $\langle , \rangle_L$ given by

$$\langle x, y \rangle_L = x_1 y_1 - x_2 y_2,$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$. With this inner product, the affine plane $\mathbb{R}^2$ is called the Lorentz-Minkowski plane and is denoted by L^2 or $\mathbb{R}_1^2$. This inner product is symmetrical, bi-linear, and non-degenerate. The arbitrary vector $x = (x_1, x_2) \in L^2$ is classified according to the sign of $\langle x, y \rangle_L$ as follows:

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(i) x is timelike vector if $\langle x, y\rangle_L < 0$,

(ii) x is spacelike vector if $\langle x, y\rangle_L > 0$ and $x = 0$,

(iii) x is lightlike vector if $\langle x, y\rangle_L = 0$ ve $x \neq 0$.

The norm $\|\cdot\|$ of any $x = (x_1, x_2) \in L^2$ is defined by

$$\| x \|_L = \sqrt{|\langle x, x\rangle|}$$

and Lorentz distance function is defined by

$$d_L(x, y) = \| x - y \|_L = \sqrt{|(x_1 - y_1)^2 - (x_2 - y_2)^2|}.$$

The Lorentz-Minkowski plane is almost the same as the Euclidean plane since the points and the lines are the same. The angles are measured in the same way. But, the distance function is different. For this reason, in Lorentz-Minkowski plane, the shape of circle changes to equilateral hyperbola and hyperbolas are called respect to their tangent straight lines. So, it is different from the circle in the Euclidean plane.

In the Lorentz-Minkowski plane, the angles, the side relations and the area of the triangles as well as hyperbolic sine and hyperbolic cosine rules are discussed in [3] and [4]. In this study, it was investigated whether Euclidean incircle and circumcircle theorems for triangles hold in the Lorentz-Minkowski plane.

Materials and Methods

Definition 2.1. Let be a line with slope in the Lorentz-Minkowski plane. The line is called a spacelike line, a timelike line, a lightlike (null) line if $|m| < 1$, $|m| > 1$, $|m| = 1$, respectively [8].

Theorem 2.1. *In the Lorentz-Minkowski plane, the distance from a point* $P = (x_1, x_2)$ *to the line* $\gamma : y = mx + q$ *is defined by* [5],

$$d_L(P, \gamma) = \frac{|y_1 - mx_1 - q|}{\sqrt{|m^2 - 1|}}.$$

Definition 2.2. For a vector $x \in L^2$, if there exists a vector $y \in L^2$, which satisfies $\langle x, y \rangle_L = 0$, we say that y is pseudo-orthogonal to x [7].

Theorem 2.2. *The axis of a segment in the Lorentz-Minkowski plane, as in the Euclidean plane, is pseudo-orthogonal to a segment in its middle point* [5].

Proof. Let $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ be two points in the Lorentz-Minkowski plane. The points that have the same distance from these two points are determined by

$$d_L(P_1, Q) = d_L(P_2, Q).$$

For $Q = (x, y)$.

If $\overrightarrow{P_1P_2}$ is spacelike line, we get

$$(x - p_1)^2 - (y - q_1)^2 = (x - p_2)^2 - (y - q_2)^2,$$

$$(p_1 - p_2)(2x - p_1 - p_2) = (q_1 - q_2)(2y - q_1 - q_2),$$

$$y = \frac{(p_1 - p_2)}{(q_1 - p_2)} x + \frac{(q_1^2 - q_2^2) - (p_1^2 - p_2^2)}{2(q_1 - q_2)},$$

$$y = \frac{1}{m} x + \frac{(q_1^2 - q_2^2) - (p_1^2 - p_2^2)}{2(q_1 - q_2)}.$$

Then from (2) that the axis is pseudo-orthogonal to a segment $[P_1P_2]$ and from (1) that it is passes through middle point $P_M = ((p_1 + p_2)/2, (q_1 + q_2)/2)$.

If $\overrightarrow{P_1P_2}$ is timelike line, we get similar results.

We shall now consider a triangle ΔPQR with sides $BC \equiv a$, $CA \equiv b$, $AB \equiv c$ in the Lorentz-Minkowski plane. The following theorems can be given [6].

Theorem 2.3. *If all the sides of the triangle ΔPQR are same kind, then the lengths of segments QR, RP, PQ can be compared.*

Theorem 2.4. *If all the sides of the triangle* ΔPQR *are same kind, then the largest side of the triangle is larger than sum of the other two sides. That is* $q + r < p$ *(the inequality* $q + r < p$ *rules out the existence of equilateral triangles).*

Theorem 2.5. *Base of an isosceles triangle* ΔPQR *is a segment of the opposite kind from its sides.*

Theorem 2.6. *If* ΔPQR *is a right triangle, then the sides of* ΔPQR *are necessarily of different kinds.*

Theorem 2.7. *Just as Euclidean geometry, in the Lorentz-Minkowski plane the perpendicular bisectors of the sides of the triangle* ΔPQR *are also concurrent; they meet center O of its circumcircle.*

Theorem 2.8. *The triangle* ΔPQR *has three angle bisectors only if all of its sides are of the same kind.*

Theorem 2.9. *Just as Euclidean geometry, in the Lorentz-Minkowski plane angle bisectors the triangle* ΔPQR *are also concurrent; they meet center O of its incircle.*

We note that in Lorentz-Minkowski plane two perpendicular lines are always of different kinds. (The lightlike lines are the exception, because they are perpendicular to each other). So, from Theorem 2.7 and Theorem 2.9, in the Lorentz-Minkowski plane, if a triangle has circumcirle and incircle, its sides must be same kinds lines. The largest side of triangle is larger than sum of the other two sides.

Conclusion and Discussion

Now we can give the following theorems.

Theorem 3.1. *Let* $P = (p_1, p_2)$, $Q = (q_1, q_2)$, $R = (r_1, r_2)$ *be non-collinear points and vertices of the triangle* ΔPQR *and slopes of sides* PQ, QR, RP *of the triangle be* m_r, m_p, m_q, *respectively in the Lorentz-Minkowski plane. Then the center coordinates of the circumcircle of this triangle is given by*

$$M = \left(\frac{m_p m_r (c_2 - c_1)}{m_p - m_r}, \frac{m_p m_r (c_2 - c_1) + m_q (m_p - m_r) c_3}{m_q (m_p - m_r)} \right),$$

where

$$c_1 = \frac{(q_2^2 - p_2^2) - (q_1^2 - p_1^2)}{2(q_2 - p_2)},$$

$$c_2 = \frac{(r_2^2 - q_2^2) - (r_1^2 - q_1^2)}{2(rq_2 - q_2)},$$

$$c_3 = \frac{(p_2^2 - r_2^2) - (p_1^2 - r_1^2)}{2(p_2 - r_2)}.$$

Here, sides of the triangle ΔPQR *must be same kinds lines. That is, three sides of the triangle* ΔPQR *are either spacelike lines or timelike lines.*

Proof. Due to Theorem 2.2 and Theorem 2.7, the equations of the perpendicular bisectors of the sides of the triangle ΔPQR are given by,

$$h_{PQ} : y = \frac{1}{m_r} x + \frac{(q_2^2 - p_2^2) - (q_1^2 - p_1^2)}{2(q_2 - p_2)},$$

$$h_{QR} : y = \frac{1}{m_p} x + \frac{(r_2^2 - q_2^2) - (r_1^2 - q_1^2)}{2(rq_2 - q_2)},$$

$$h_{RP} : y = \frac{1}{m_q} x + \frac{(p_2^2 - r_2^2) - (p_1^2 - r_1^2)}{2(p_2 - r_2)}.$$

Then we can write,

$$h_{PQ} : y = \frac{1}{m_r} x + c_1,$$

$$h_{QR} : y = \frac{1}{m_p} x + c_2,$$

$$h_{PQ} : y = \frac{1}{m_q} x + c_3,$$

$$c_1 = \frac{(q_2^2 - p_2^2) - (q_1^2 - p_1^2)}{2(q_2 - p_2)},$$

$$c_2 = \frac{(r_2^2 - q_2^2) - (r_1^2 - q_1^2)}{2(rq_2 - q_2)},$$

$$c_3 = \frac{(p_2^2 - r_2^2) - (p_1^2 - r_1^2)}{2(p_2 - r_2)}.$$

Since these lines intersect at the center of the circumcircle, the coordinates of the center of the circumcircle are given by,

$$\frac{1}{m_r} x + c_1 = \frac{1}{m_p} x + c_2,$$

$$\frac{1}{m_r} x - \frac{1}{m_p} x = c_2 - c_1,$$

$$\frac{(m_p - m_r)}{m_p m_r} x = c_2 - c_1,$$

$$x = \frac{m_p m_r (c_2 - c_1)}{m_p - m_r},$$

$$y = \frac{1}{m_q} \cdot \frac{m_p m_r (c_2 - c_1)}{m_p - m_r} + c_3,$$

$$y = \frac{m_p m_r (c_2 - c_1) + m_q (m_p - m_r) c_3}{m_q (m_p - m_r)}.$$

That is, we can write by,

$$M = \left(\frac{m_p m_r (c_2 - c_1)}{m_p - m_r}, \frac{m_p m_r (c_2 - c_1) + m_q (m_p - m_r) c_3}{m_q (m_p - m_r)} \right),$$

where

$$c_1 = \frac{(q_2^2 - p_2^2) - (q_1^2 - p_1^2)}{2(q_2 - p_2)},$$

$$c_2 = \frac{(r_2^2 - q_2^2) - (r_1^2 - q_1^2)}{2(rq_2 - q_2)},$$

$$c_3 = \frac{(p_2^2 - r_2^2) - (p_1^2 - r_1^2)}{2(p_2 - r_2)}.$$

Example 3.1. Given three points $P_1 = (0, 0)$, $P_2 = (2, 1)$, $P_3 = (5, -1)$, these points can be considered the vertices of a triangle which sides are spacelike lines. Using Theorem 3.1, we get circumscribed hyperbola (Figure 3.1). In this figure, as for Euclidean circumcircle, circumscribed hyperbola passes from the vertices of the triangle.

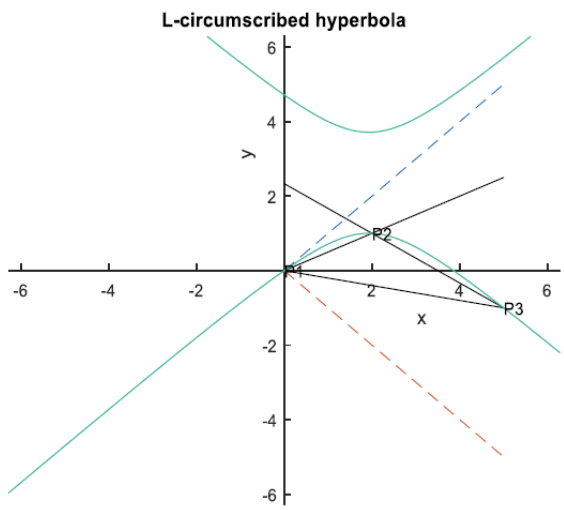

Figure 3.1. Circumscribed hyperbola of a triangle which sides are spacelike lines in the Lorentz-Minkowski plane.

Example 3.2. Given three points $P_1 = (0, 0)$, $P_2 = (1, 2)$, $P_3 = (-2, 6)$, these points can be considered the vertices of a triangle which sides are timelike lines. Using Theorem 3.1, we get circumscribed hyperbola (Figure 3.2). In this figure, as for Euclidean circumcircle, circumscribed hyperbola passes from the vertices of the triangle.

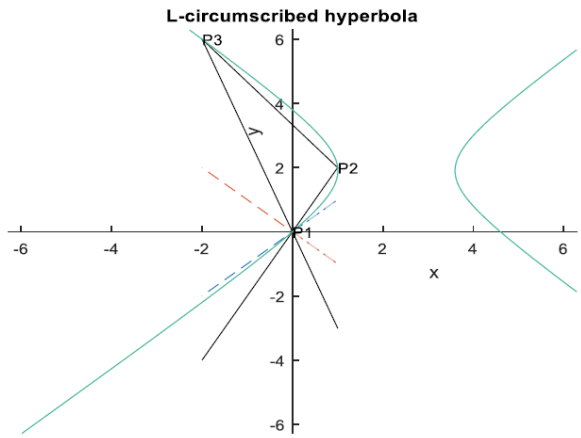

Figure 3.2. Circumscribed hyperbola of a triangle which sides are timelike lines in the Lorentz-Minkowski plane.

Theorem 3.2. *Let* $P = (p_1, p_2)$, $Q = (0, 0)$, $R = (r_1, r_2)$ *be non-collinear points and vertices of the triangle* ΔPQR *and slopes of sides* PQ, QR, RP *of the triangle* ΔPQR *be* m_r, m_p, m_q, *respectively in the Lorentz-Minkowski plane. Then the center coordinates of the incircle of this triangle is given by*,

$$M = \left(\frac{c(k-t)}{m_p(k-z) + m_q(t-k) + m_r(z-t)}, - \frac{c(tm_r - km_p)}{m_p(k-z) + m_q(t-k) + m_r(z-t)} \right),$$

where

$$k = \sqrt{|\, m_r^2 - 1 \,|},$$

$$t = \sqrt{|\, m_p^2 - 1 \,|},$$

$$z = \sqrt{|\, m_q^2 - 1 \,|},$$

$$c = p_2 - m_q p_1.$$

Here, sides of the triangle must be same kinds lines. That is, three sides of the triangle are either spacelike lines or timelike lines.

Proof. For the center of the incircle is $I = (x_m, y_m)$, the shortest Lorentz distance from the center to the sides of the triangle is the radius of incircle.

Thus, it can be written by

$$\frac{|\, y_m - m_r x_m \,|}{\sqrt{|\, m_r^2 - 1 \,|}} = \frac{|\, y_m - m_p x_m \,|}{\sqrt{|\, m_p^2 - 1 \,|}} = \frac{|\, y_m - m_q x_m - c \,|}{\sqrt{|\, m_q^2 - 1 \,|}},$$

where

$$PQ : y = m_r x,$$

$$QR : y = m_p x,$$

$$RP : y = m_r x + c(c = p_2 - m_q p_1).$$

Let $k = \sqrt{|\, m_r^2 - 1 \,|}$, $t = \sqrt{|\, m_p^2 - 1 \,|}$, $z = \sqrt{|\, m_q^2 - 1 \,|}$ and $c = p_2 - m_q p_1$. From here, there are two cases, if three sides of the triangle ΔPQR are either spacelike lines or timelike lines.

Case 1. If three sides of the triangle ΔPQR are spacelike lines, we can write,

$$\frac{|\, y_m - m_r x_m \,|}{\sqrt{1 - m_r^2}} = \frac{|\, y_m - m_p x_m \,|}{\sqrt{1 - m_p^2}} = \frac{|\, y_m - m_q x_m - c \,|}{\sqrt{1 - m_q^2}}$$

$$\frac{|\, y_m - m_r x_m \,|}{t} = \frac{|\, y_m - m_p x_m \,|}{t} = \frac{|\, y_m - m_q x_m - c \,|}{z},$$

$$t y_m - t m_r x_m = k y_m - k m_q x_m,$$

$$(t - k) y_m = (t m_r - k m_q) x_m,$$

$$y_m = \frac{(t m_r - k m_p)}{t - k} x_m,$$

on the other hand,

$$z y_m - z m_r x_m = k y_m - k m_q x_m - kc,$$

$$(z - k) y_m = (z m_r - k m_q) x_m - kc,$$

$$y_m = \frac{(z m_r - k m_q)}{z - k} x_m - \frac{kc}{z - k}.$$

Thus, from (3) and (4), we obtain,

$$\frac{(tm_r - km_p)}{t-k} x_m = \frac{(zm_r - km_q)x_m - kc}{z-k},$$

$$(ztm_r - zkm_p - ktm_r + k^2 m_p)x_m = (tzm_r - tkm_q - kzm_r + k^2 m_q)x_m$$

$$- tkc + kc^2,$$

$$[m_p(k-z) + m_q(t-k) + m_r(z-t)]x_m = c(k-t),$$

$$x_m = \frac{c(k-t)}{m_q(k-z) + m_q(t-k) + m_r(z-t)}.$$

Then, from (3) and (5), we get,

$$y_m = -\frac{c(tm_r - km_p)}{m_p(k-z) + m_q(t-k) + m_r(z-t)}.$$

Case 2. If three sides of the triangle ΔPQR are timelike lines, it is found similarly.

Example 3.3. Given three points $P_1 = (0, 0)$, $P_2 = (3, 1)$, $P_3 = (6, -1)$, these points can be considered the vertices of a triangle which sides are spacelike lines. Using Theorem 3.2, we get inscribed hyperbola (Figure 3.3).

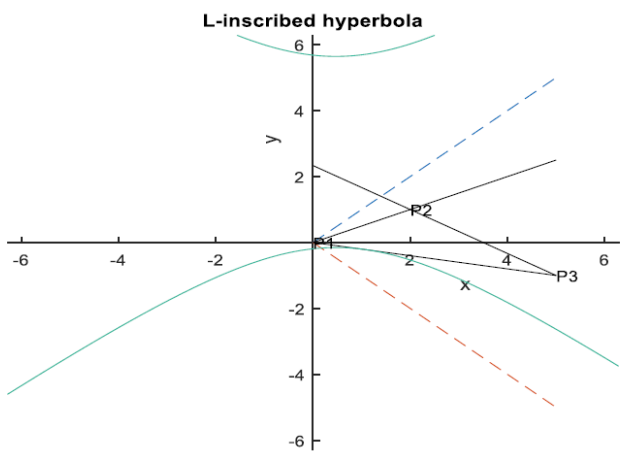

Figure 3.3. Inscribed hyperbola a triangle which sides are spacelike lines in the Lorentz-Minkowski plane.

Example 3.4. Given three points $P_1 = (0, 0)$, $P_2 = (1, 3)$, $P_3 = (-2, 7)$, these points can be considered the vertices of a triangle which sides are timelike lines. Using Theorem 3.2, we get inscribed hyperbola (Figure 3.4).

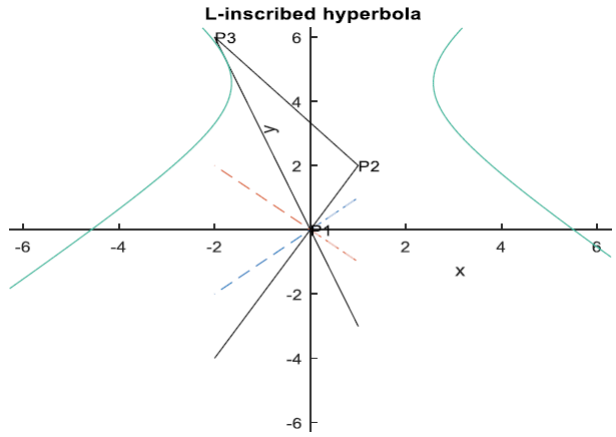

Figure 3.4. Inscribed hyperbola of a triangle which sides are time like lines in the Lorentz-Minkowski plane.

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