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RADIO HERONIAN MEAN GRACEFUL LABELING ON DEGREE SPLITTING OF GRAPHS

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Abstract

A mapping $g : V(G) \to N$ is a radio heronian mean labeling if for any two distinct vertices s and t of G, $d(s, t) + \left\lceil \frac{g(s) + g(t) + \sqrt{g(s)g(t)}}{3} \right\rceil \geq 1 + D$, where D is the diameter of G. The radio heronian mean number of g, $rhmn\,(g)$, is the maximum number assigned to any vertex of G. The radio heronian mean number of G, $rhmn\,(G)$, is the minimum value of $rhmn\,(g)$ taken overall radio heronian mean labeling g of G. If $rhmn\,(G) = |V(G)|$, we call such graphs as radio heronian mean graceful graphs. In this paper, we investigate the radio heronian mean graceful labeling on degree splitting of star, bistar, triangular snake, quadrilateral snake and irregular triangular snake graphs.

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1. Introduction

Throughout this paper we consider the graphs are simple, finite and undirected. Let vertex and edge sets of G be denoted by $V(G)$ and $E(G)$ respectively. A labeling of a graph G is an assignment of integers to the vertices, or edges, or both subject to certain conditions. Radio labeling or multilevel distance labeling is motivated by the channel assignment problem for radio transmitters [6]. Ponraj et al. [8] introduced the concept of radio mean labeling of graphs. The lowest span taken overall radio mean labelings of the graph G is the radio mean number of G. The maximum integer than g maps to a vertex of G is the span of a labeling g. S. S. Sandya, et al. [11] introduced heronian mean labeling of graphs. Gallian [4] introduced graceful labeling of graphs. In this sequel, we introduced radio heronian mean graceful labeling. We follow the concept of Harary [5] and Gallian [4] for standard terminology and notations.

2. Definitions

Definition 2.1 [2] [3]**.** Let G be a graph with vertex set $\{T_1, T_2, \ldots, T_i, S\}$, where each T_i is a set of vertices with at least two vertices have the same degree and $S = V - \bigcup T_i$. The degree splitting $DS(G)$ is obtained from G by introducing the new vertices z_i, $1 \leq i \leq k$ and joining z_i to each vertex of $T_i (1 \leq i \leq k)$.

Definition 2.2 [3]**.** A bipartite graph which is complete then such graphs are said to be a star graph and it is denoted by $K_{1,k}$.

Definition 2.3 [2]**.** The bistar graph $B_{k,k}$ is obtained from K_2 by joining k pendent edges to each ends of K_2. The edge of K_2 is known as central edge of $B_{k,k}$ and the vertices of K_2 are known as central vertices of $B_{k,k}$.

Definition 2.4 [1] [8]**.** A triangular snake T_k is obtained from a path $s_1, s_2, s_3, \ldots s_k$ by joining s_i and $B_{k,k}$ to a new vertex t_i for $1 \leq i \leq k-1$.

Definition 2.5 [8]**.** A quadrilateral snake Q_k is obtained from a path $s_1, s_2, s_3, \ldots, s_k$ by joining s_i and s_{i+1} to new vertices t_i and t_i, respectively and adding the edges $t_i t_i'$, $1 \le i \le k-1$. That is every edge of a path is replaced by a cycle C_4.

Definition 2.6 [1]**.** The irregular triangular snake IT_k is the graph obtained from the path $s_1, s_2, \ldots, s_k$ by joining s_i and s_{i+2} to a new vertex t_i for $1 \le i \le k-2$.

3. Main Result

Result 3.1. $DS(K_{1,k})$ is radio heronian mean graceful for $k \ge 2$.

Proof. Let s be a centre vertex and $s_1, s_2, \ldots, s_k$ be the vertices which are joined to s. Introduce a new vertex t and join t to s_i, $1 \le i \le k$. The resultant graph is $DS(K_{1,k})$ whose edge set is $E(DS(K_{1,k}))$ $= \{ss_i, ts_i / 1 \le i \le k\}$ and $E(DS(K_{1,k})) = 2$.

Define a function $g : DS(K_{1,k}) \to N$ by

$g(s) = 1;$

$g(t) = 2;$

$g(s_i) = i + 2, 1 \le i \le k;$

The condition for radio heronian mean labeling is verified as follows.

case a

Consider the set of vertices (s, s_i), $1 \le i \le k$

$$d(s, s_i) + \left\lceil \frac{g(s) + g(s_i) + \sqrt{g(s)g(s_i)}}{3} \right\rceil \ge 1 + \left\lceil \frac{i + 3 + \sqrt{i+2}}{3} \right\rceil \ge 3 = 1 + D(DS(K_{1,k}))$$

case b.

Consider the set of vertices (s, t)

$$d(s, t) + \left\lceil \frac{g(s) + g(s_i) + \sqrt{g(s)g(s_i)}}{3} \right\rceil \ge 2 + \left\lceil \frac{3 + \sqrt{2}}{3} \right\rceil \ge 3$$

Case c. Consider the set of vertices (t, s_i), $1 \leq i \leq k$

$$d(t, s_i) + \left\lceil \frac{g(s) + g(s_i) + \sqrt{g(s)g(s_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{i + 4 + \sqrt{(2)(i+2)}}{3} \right\rceil \geq 3$$

case d

Consider the set of vertices (s_i, s_j), $i \neq j$, $1 \leq i, j \leq k$

$$d(s_i, s_i) + \left\lceil \frac{g(s_i) + g(s_j) + \sqrt{g(s_i)g(s_j)}}{3} \right\rceil \geq 1 + \left\lceil \frac{i + j + 4 + \sqrt{(i+2)(j+2)}}{3} \right\rceil \geq 3.$$

Hence for all set of vertices radio heronian mean condition is satisfied. Thus, g is a valid radio heronian mean labeling of $(DS(K_{1,k}))$. Therefore, $rhmn(DS(K_{1,k})) \leq rhmn(g) = k + 2$. Since g is injective, $rhmn(DS(K_{1,k})) \geq k + 2$ for all radio heronian mean labeling g and hence $rhmn(DS(K_{1,k})) = k + 2$ for $k \geq 2$. Clearly, $|V(DS(K_{1,k}))| = k + 2$. Thus, $rhmn(DS(K_{1,k})) = |V(DS(K_{1,k}))|$. Hence the degree splitting of star graph $DS(K_{1,k})$ is radio heronian mean graceful for $k \geq 2$.

Illustration 3.1. The following is the illustration of $DS(K_{1,k})$.

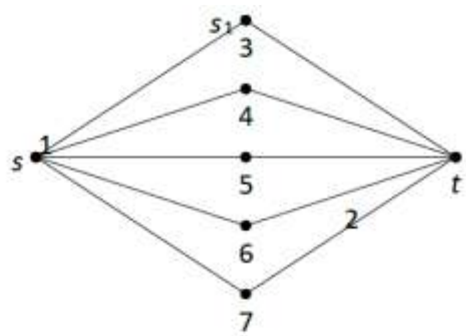

Figure 1.

Result 3.2. $DS(B_{k,k})$ is radio heronian mean graceful for $k \geq 2$.

Proof. Let s and t be central vertices of $B_{k,k}$ and the corresponding pendent vertices by $s_i, t_i, 1 \leq i \leq k$. Introduce two new vertices s' and t'.

Join s' to s, t and t' to $s_i, t_i 1 \le i \le k$. The resultant graph is $DS(B_{k,k})$ whose edge set is $E(DS(B_{k,k})) = \{\{ss_i, tt_i, s_i, t't_i / 1 \le i \le k\} \cup \{st, s's, s't\}\}$ and $D(DS(B_{k,k})) = 3$.

Define a function $g : V(DS(B_{k,k})) \to N$ by

$g(s) = 2k + 3;$

$g(t) = 2k + 4;$

$g(s_i) = i + 2,\ 1 \le i \le k;$

$g(s_t) = k + i + 2,\ 1 \le i \le k;$

$g(s') = 1;$

$g(t') = 2;$

The condition for radio heronian mean labeling is verified as follows.

case a

Consider the set of vertices (s, t)

$$d(s, t) + \left\lceil \frac{g(s) + g(t) + \sqrt{g(s)g(t)}}{3} \right\rceil \ge 1 + \left\lceil \frac{4k + 7 + \sqrt{(2k+3)(2k+4)}}{3} \right\rceil$$

$$\ge 4 = 1 + D(DS(B_{k,k}))$$

case b

Consider the set of vertices (s, s')

$$d(s, s') + \left\lceil \frac{g(s) + g(s') + \sqrt{g(s)g(s')}}{3} \right\rceil \ge 1 + \left\lceil \frac{2k + 4 + \sqrt{2k+3}}{3} \right\rceil \ge 4$$

case c

Consider the set of vertices (s, t')

$$d(s, t') + \left\lceil \frac{g(s) + g(t') + \sqrt{g(s)g(t')}}{3} \right\rceil \ge 2 + \left\lceil \frac{2k + 5 + \sqrt{(2)(2k+3)}}{3} \right\rceil \ge 4$$

case d

Consider the set of vertices (s, s_i), $1 \leq i \leq k$

$$d(s, s_i) + \left\lceil \frac{g(s) + g(s_i) + \sqrt{g(s)g(s_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{2k + i + 5 + \sqrt{(2k+3)(i, 2)}}{3} \right\rceil \geq 4$$

case e

Consider the set of vertices (s, t_i), $1 \leq i \leq k$

$$d(s, t_i) + \left\lceil \frac{g(s) + g(t_i) + \sqrt{g(s)g(t_i)}}{3} \right\rceil \geq 2$$

$$+ \left\lceil \frac{3k + i + 5 + \sqrt{(2k+3)(k+i+2)}}{3} \right\rceil \geq 4$$

case f

Consider the set of vertices (t, s')

$$d(t, s') + \left\lceil \frac{g(s) + g(s') + \sqrt{g(s)g(s')}}{3} \right\rceil \geq 1 + \left\lceil \frac{2k + 5 + \sqrt{2k+4}}{3} \right\rceil \geq 4$$

case g

Consider the set of vertices (t, t')

$$d(t, t') + \left\lceil \frac{g(t) + g(t') + \sqrt{g(t)g(t')}}{3} \right\rceil \geq 2 + \left\lceil \frac{2k + 6 + \sqrt{(2)(2k+4)}}{3} \right\rceil \geq 4$$

case h

Consider the set of vertices (t, s_i), $1 \leq i \leq k$

$$d(t, s_i) + \left\lceil \frac{g(t) + g(s_i) + \sqrt{g(t)g(s_i)}}{3} \right\rceil \geq 2$$

$$+ \left\lceil \frac{2k + i + 6 + \sqrt{(2k+4)(i+2)}}{3} \right\rceil \geq 4$$

case i

Consider the set of vertices (t, t_i), $1 \leq i \leq k$

$$d(t, t_i) + \left\lceil \frac{g(t) + g(t_i) + \sqrt{g(t)g(t_i)}}{3} \right\rceil \geq 1$$

$$+ \left\lceil \frac{3k + i + 6 + \sqrt{(2k+4)(k+i+2)}}{3} \right\rceil \geq 4$$

case j

Consider the set of vertices (s_i, s_j), $i \neq j$, $1 \leq i, j \leq k$

$$d(s_i, t_j) + \left\lceil \frac{g(s_i) + g(t_j) + \sqrt{g(s_i)g(t_j)}}{3} \right\rceil \geq 2$$

$$+ \left\lceil \frac{k + j + 4 + \sqrt{(i+2)(j+2)}}{3} \right\rceil \geq 4$$

case k

Consider the set of vertices (s_i, t_j), $1 \leq i, j \leq k$

$$d(s_i, t_j) + \left\lceil \frac{g(s_i) + g(t_j) + \sqrt{g(s_i)g(t_j)}}{3} \right\rceil \geq 2$$

$$+ \left\lceil \frac{k + i + j + 4 + \sqrt{(i+2)(k+j+2)}}{3} \right\rceil \geq 4$$

case l

Consider the set of vertices (t_i, t_j), $i \neq j$, $1 \leq i, j \leq k$

$$d(t_i, t_j) + \left\lceil \frac{g(t_i) + g(t_j) + \sqrt{g(t_i)g(t_j)}}{3} \right\rceil \geq 2$$

$$+ \left\lceil \frac{2k + i + j + 4 + \sqrt{(k+i+2)(k+j+2)}}{3} \right\rceil \geq 4$$

case m

Consider the set of vertices (s', s_i), $1 \leq i \leq k$

$$d(s', s_i) + \left\lceil \frac{g(s') + g(s_i) + \sqrt{g(s')g(s_i)}}{3} \right\rceil \geq 2 + \left\lceil \frac{i + 3 + \sqrt{1 + 2}}{3} \right\rceil \geq 4$$

case n

Consider the set of vertices (s', t_i), $1 \leq i \leq k$

$$d(s', t_i) + \left\lceil \frac{g(s') + g(t_i) + \sqrt{g(s')g(t_i)}}{3} \right\rceil \geq 2 + \left\lceil \frac{k + i + 3 + \sqrt{k + i + 2}}{3} \right\rceil \geq 4$$

case o

Consider the set of vertices (t', s_i), $1 \leq i \leq k$

$$d(t', s_i) + \left\lceil \frac{g(t') + g(s_i) + \sqrt{g(t')g(s_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{i + 4 + \sqrt{(2)(i + 2)}}{3} \right\rceil \geq 4$$

case p

Consider the set of vertices (t', t_i), $1 \leq i \leq k$

$$d(t', t_i) + \left\lceil \frac{g(t') + g(t_i) + \sqrt{g(t')g(t_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{k + i + 4 + \sqrt{(2)(k + i + 2)}}{3} \right\rceil \geq 4$$

case q

Consider the set of vertices (t', s')

$$d(t', s') + \left\lceil \frac{g(t') + g(s') + \sqrt{g(t')g(s')}}{3} \right\rceil \geq 3 + \left\lceil \frac{3 + \sqrt{2}}{3} \right\rceil \geq 4$$

Hence for all set of vertices radio heronian mean condition is satisfied. Thus, s is a valid radio heronian mean labeling of $DS(B_{k,k})$. Therefore, $rhmn(DS(B_{k,k})) \leq rhmn(g) = 2k + 4$. Since g is injective, $rhmn(DS(B_{k,k})) = 2k + 4$ for all radio heronian mean labeling g and hence $rhmn(DS(B_{k,k})) = 2k + 4$ for $k \geq 2$. Clearly, $|V(DS(B_{k,k}))| = 2k + 4$.

Thus, $rhmn(DS(B_{k,k})) = |V(DS(B_{k,k}))|$. Hence the degree splitting of bistar graph $DS(B_{k,k})$ is radio heronian mean graceful for $k \geq 2$.

Illustration 3.2. The following is the illustration of $DS(B_{4,4})$.

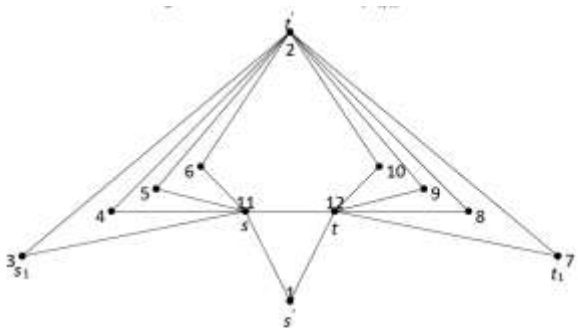

Figure 2.

Result 3.3. $DS(T_k)$ is radio heronian mean graceful for $k \geq 4$.

Proof. Let $s_i, 1 \leq i \leq k$ be the vertices of the path P_k. Join s_i and s_{i+1} to new vertex $t_i, 1 \leq i \leq k-1$. The resultant graph is T_k. Introduce two new vertices s and t. Join s to $s_i, 2 \leq i \leq k-1$ and t to $s_1, s_k, t_i, 1 \leq i \leq k-1$. The resultant graph is $DS(T_k)$ whose edge set is $E(DS(T_k))$ $= \{\{s_i s_{i+1}, s_i t_i, t_i s_{i+1}, t_i / 1 \leq i \leq k-1\} \cup \{s_1 t, s_k t\} \cup \{st_i / 2 \leq i \leq k-1\}\}$ and $D(DS(T_k)) = 3$.

Define a function $g : V(DS(T_k)) \to N$ by

$g(s) = 1;$

$g(t) = 2;$

$g(s_i) = k + i + 1, 1 \leq i \leq k;$

$g(t_i) = i + 2, 1 \leq i \leq k-1;$

The condition for radio heronian mean labeling is verified as follows.

case a

Consider the set of vertices (s, t)

$$d(s, t)+\left\lceil \frac{g(s)+g(t)+\sqrt{g(s)g(t)}}{3} \right\rceil \geq 3+\left\lceil \frac{3+\sqrt{2}}{3} \right\rceil \geq 4 = 1 + D(DS(T_k))$$

case b

Consider the set of vertices (s, s_i), $1 \leq i \leq k$

$$d(s, s_i)+\left\lceil \frac{g(s)+g(s_i)+\sqrt{g(s)g(s_i)}}{3} \right\rceil \geq 1+\left\lceil \frac{k+i+2+\sqrt{k+i+1}}{3} \right\rceil \geq 4$$

case c

Consider the set of vertices (s, t_i), $1 \leq i \leq k-1$

$$d(s, t_i)+\left\lceil \frac{g(s)+g(t_i)+\sqrt{g(s)g(t_i)}}{3} \right\rceil \geq 2+\left\lceil \frac{i+3+\sqrt{i+2}}{3} \right\rceil \geq 4$$

case d

Consider the set of vertices (t, s_i), $1 \leq i \leq k$

$$d(t, s_i)+\left\lceil \frac{g(s)+g(s_i)+\sqrt{g(t)g(s_i)}}{3} \right\rceil \geq 1+\left\lceil \frac{k+i+3+\sqrt{(2)(k+i+1)}}{3} \right\rceil \geq 4$$

case e

Consider the set of vertices (t, t_i), $1 \leq i \leq k-1$

$$d(t, t_i)+\left\lceil \frac{g(t)+g(t_i)+\sqrt{g(t)g(t_i)}}{3} \right\rceil \geq 1+\left\lceil \frac{i+4+\sqrt{(2)(i+2)}}{3} \right\rceil \geq 4$$

case f

Consider the set of vertices (s_i, s_j), $i \neq j$, $1 \leq i, j \leq k$

$$d(s_i, s_j)+\left\lceil \frac{g(s_i)+g(s_i)+\sqrt{g(s_i)g(s_j)}}{3} \right\rceil \geq 1$$

$$+\left\lceil \frac{2k+i+j-2+\sqrt{(k+i+1)(k+j+1)}}{3} \right\rceil \geq 4$$

case g

Consider the set of vertices (t_i, t_j), $i \neq j$, $1 \leq i, j \leq k-1$

$$d(t_i, t_j) + \left\lceil \frac{g(t_i) + g(t_i) + \sqrt{g(t_i) g(t_j)}}{3} \right\rceil \geq 2$$

$$+ \left\lceil \frac{i + j + 4 + \sqrt{(i+2)(j+1)}}{3} \right\rceil \geq 4$$

case h

Consider the set of vertices (s_i, t_j), $1 \leq i \leq k$, $1 \leq j \leq k-1$

$$d(s_i, t_j) + \left\lceil \frac{g(s_i) + g(t_j) + \sqrt{g(s_i) g(t_j)}}{3} \right\rceil \geq 1$$

$$+ \left\lceil \frac{k + i + j + 3 + \sqrt{(k+i+1)(j+2)}}{3} \right\rceil \geq 4$$

Hence for all set of vertices radio heronian mean condition is satisfied for all pairs of vertices. Thus g is a valid radio heronian mean labeling of $DS(T_k)$. Therefore, $rhmn(DS(T_k)) \leq rhmn(g) = 2k+1$. Since g is injective, $rhmn(DS(T_k)) \geq 2k+1$ for all radio heronian mean labeling g and hence $rhmn(DS(T_k)) \geq 2k+1$ for $k \geq 4$ Clearly, $|V(DS(T_k))| = 2k+1$. Thus, $rhmn(DS(T_k)) = |V(DS(T_k))|$. Hence the degree splitting of triangular snake graph $DS(T_k)$ is radio heronian mean graceful for $k \geq 4$.

Illustration 3.3. The following is the illustration of $DS(T_5)$.

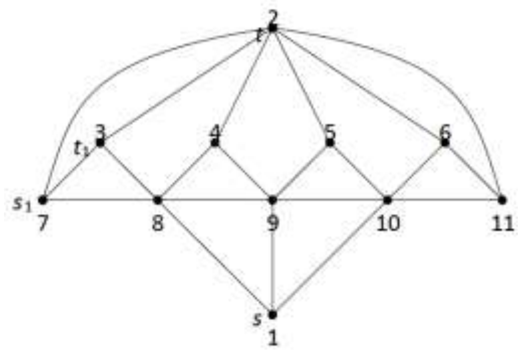

Figure 3.

Result 3.4. $DS(Q_k)$ is radio heronian mean graceful for $k \geq 4$.

Proof. Let s_i, $1 \leq i \leq k$ be the vertices of the path P_k. For $1 \leq i \leq k-1$, add vertices t_i and t'_i and joined with s_i and s_{i+1} respectively. Next join the vertices t_i and t'_i, $1 \leq i \leq k-1$. The resultant graph is Q_k. Introduce two new vertices s and t. Join s to s_i, $2 \leq i \leq k-1$ and t to s_i, $2 \leq i \leq k-1$. The resultant graph is $DS(Q_k)$ whose edge set is $E(DS(Q_k))$ $= \{\{s_i s_{i+1}, s_i t_i, t_i\, s_{i+1}, t\, t_i, tt'_i, t_i t'_i\, t'_i s_{i+1} / 1 \leq i \leq k-1\} \cup \{s_1 t, s_k t\}$ $\cup \{sst_i / 2 \leq i \leq k-1\}\}$ and $D(DS(Q_k)) = 3$.

Define a function $g : V(DS(Q_k)) \to N$ by

$g(s) = 1;$

$g(t) = 2;$

$g(s_i) = 2k + i, 1 \leq i \leq k;$

$g(t_i) = i + 2, 1 \leq i \leq k - 1;$

$g(t'_i) = k + i + 1, 1 \leq i \leq k - 1;$

Now we verify the radio heronian mean condition for g.

case a

Consider the set of vertices (s, t)

$$d(s, t) + \left\lceil \frac{g(s) + g(t) + \sqrt{g(s)g(t)}}{3} \right\rceil \geq 3 + \left\lceil \frac{3 + \sqrt{2}}{3} \right\rceil \geq 4 = 1 + D(DS(Q_k))$$

case b

Consider the set of vertices $(s, s_i), 1 \leq i \leq k$

$$d(s, s_i) + \left\lceil \frac{g(s) + g(s_i) + \sqrt{g(s)\, g(s_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{2k + i + 1 + \sqrt{2k + i}}{3} \right\rceil \geq 4$$

case c

Consider the set of vertices $(s, t_i), 1 \leq i \leq k - 1$

$$d(s, t_i) + \left\lceil \frac{g(s) + g(t_i) + \sqrt{g(s)\, g(t_i)}}{3} \right\rceil \geq 2 + \left\lceil \frac{i + 3 + \sqrt{i + 2}}{3} \right\rceil \geq 4$$

case d

Consider the set of vertices $(s, t_i'), 1 \leq i \leq k - 1$

$$d(s, t_i') + \left\lceil \frac{g(s) + g(t_i') + \sqrt{g(s)\, g(t_i')}}{3} \right\rceil \geq 2 + \left\lceil \frac{k + i + 2 + \sqrt{k + i + 1}}{3} \right\rceil \geq 4$$

case e

Consider the set of vertices $(t, s_i), 1 \leq i \leq k$

$$d(t, s_i) + \left\lceil \frac{g(t) + g(s_i) + \sqrt{g(t)\, g(s_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{2k + i + 2 + \sqrt{(2)\,(2k + i)}}{3} \right\rceil \geq 4$$

case f

Consider the set of vertices $(t, t_i), 1 \leq i \leq k - 1$

$$d(t, t_i) + \left\lceil \frac{g(t) + g(t_i) + \sqrt{g(t)\, g(t_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{i + 4 + \sqrt{(2)\,(i + 2)}}{3} \right\rceil \geq 4$$

case g

Consider the set of vertices $(t, t_i'), 1 \leq i \leq k - 1$

$$d(t, t_i') + \left\lceil \frac{g(t) + g(t_i') + \sqrt{g(t)\, g(t_i')}}{3} \right\rceil \geq 1 + \left\lceil \frac{k + i + 3 + \sqrt{(2)(k + i + 1)}}{3} \right\rceil \geq 4$$

case h

Consider the set of vertices $(s_i, s_j), i \neq j, 1 \leq i, j \leq k$

$$d(s_i, s_j) + \left\lceil \frac{g(s_i) + g(s_j) + \sqrt{g(s_i)\, g(s_j)}}{3} \right\rceil \geq 1 + \left\lceil \frac{4k + i + j + \sqrt{(2k + i)(2k + j)}}{3} \right\rceil \geq 4$$

case i

Consider the set of vertices $(t_i, t_j), i \neq j, 1 \leq i, j \leq k - 1$

$$d(t_i, t_j) + \left\lceil \frac{g(t_i) + g(t_j) + \sqrt{g(t_i)\, g(t_j)}}{3} \right\rceil \geq 2 + \left\lceil \frac{i + j + 4 + \sqrt{(i + 2)(j + 2)}}{3} \right\rceil \geq 4$$

case j

Consider the set of vertices $(s_i, t_j), 1 \leq i \leq k, 1 \leq j \leq k - 1$

$$d(s_i, t_j) + \left\lceil \frac{g(s_i) + g(t_j) + \sqrt{g(s_i)\, g(t_j)}}{3} \right\rceil \geq 1 + \left\lceil \frac{2k + i + j + 2 + \sqrt{(2k + i)(j + 2)}}{3} \right\rceil \geq 4$$

case k

Consider the set of vertices $(t_i, t_j'), 1 \leq i, j \leq k - 1$

$$d(t_i, t_j') + \left\lceil \frac{g(t_i) + g(t_j') + \sqrt{g(t_i) g(t_j')}}{3} \right\rceil \geq 1 + \left\lceil \frac{k + i + j + 3 + \sqrt{(i + 2)(k + j + 1)}}{3} \right\rceil \geq 4$$

case l

Consider the set of vertices $(t_i', t_j'), i \neq j, 1 \leq i, j \leq k - 1$

$$d(t_i', t_j') + \left\lceil \frac{g(t_i') + g(t_j') + \sqrt{g(t_i') g(t_j')}}{3} \right\rceil \geq 2 + \left\lceil \frac{2k + i + j + 2 + \sqrt{(k + i + 1)(k + j + 1)}}{3} \right\rceil \geq 4$$

case m

Consider the set of vertices $(s_i, t_j'), 1 \leq i \leq k, 1 \leq j \leq k - 1$

$$d(s_i, t'_j) + \left\lceil \frac{g(s_i) + g(t'_j) + \sqrt{g(s_i)\,g(t'_j)}}{3} \right\rceil \geq 2 + \left\lceil \frac{3k + i + j + 1 + \sqrt{(2k+i)(k+j+1)}}{3} \right\rceil \geq 4$$

Hence for all set of vertices radio heronian mean condition is satisfied for all pairs of vertices. Thus g is a valid radio heronian mean labeling of $DS(Q_k)$. Therefore, $rhmn(DS(Q_k)) \leq rhmn(g) = 3k$. Since g is injective, $rhmn(DS(Q_k)) \geq 3k$ for all radio heronian mean labeling g and hence $rhmn(DS(Q_k)) = 3k$ for $k \geq 4$. Clearly, $|V(DS(Q_k))| = 3k$. Thus, $rhmn(DS(Q_k)) = |V(DS(Q_k))|$. Hence the degree splitting of triangular snake graph $DS(Q_k)$ is radio heronian mean graceful for $k \geq 4$.

Illustration 3.4. The following is the illustration of $DS(Q_5)$.

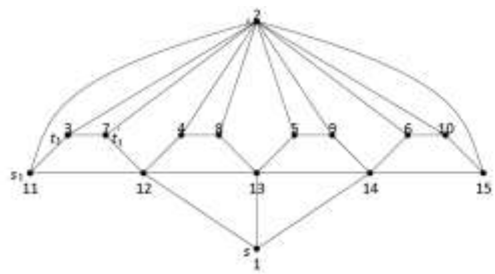

Figure 4.

Result 3.5. $DS(IT_k)$ is radio heronian mean graceful for $k \geq 6$.

Proof. Let $s_1, s_2, \ldots, s_k$ be the vertices of path P_k. Join s_i and s_{i+2} to new vertex t_i, $1 \leq i \leq k-2$. The resultant graph is IT_k. Introduce three new vertices s, s' and t. Join s to $s_2, s_{n-1}, s'_i, 3 \leq i \leq k-2$ and t to $s_1, s_k, t_i, 1 \leq i \leq k-2$ respectively. The resultant graph is $DS(IT_k)$ whose edge set is $E(DS(IT_k)) = \{\{s_i s_{i+1}/1 \leq i \leq k-1\} \cup \{s_i t_i, tt_i/1 \leq i \leq k-2\} \cup \{s' s_i/3 \leq i \leq k-2\} \cup \{ss_2, ss_{k-1}, s_1 t, s_k t\}\}$ and $D(DS(IT_k)) = 3$.

Define a function $g : V(DS(IT_k)) \to N$ by

$g(s) = 1;$

$g(s') = 2;$

$g(t) = 3;$

$g(s_i) = k + i + 1, 1 \leq i \leq k;$

$g(t_i) = i + 3, 1 \leq i \leq k - 2;$

The condition for radio heronian mean labeling is verified as follows.

case a

Consider the set of vertices (s, s')

$$d(s, s') + \left\lceil \frac{g(s) + g(s') + \sqrt{g(s)\, g(s')}}{3} \right\rceil \geq 3 + \left\lceil \frac{3 + \sqrt{2}}{3} \right\rceil \geq 4 = 1 + D(DS(IT_k))$$

case b

Consider the set of vertices (s, t)

$$d(s, t) + \left\lceil \frac{g(s) + g(t) + \sqrt{g(s)\, g(t)}}{3} \right\rceil \geq 3 + \left\lceil \frac{4 + \sqrt{3}}{3} \right\rceil \geq 4$$

case c

Consider the set of vertices $(s, s_i), 1 \leq i \leq k$

$$d(s, s_i) + \left\lceil \frac{g(s) + g(s_i) + \sqrt{g(s)\, g(s_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{k + i + 2 + \sqrt{k + i + 1}}{3} \right\rceil \geq 4$$

case d

Consider the set of vertices $(s, t_i), 1 \leq i \leq k - 2$

$$d(s, t_i) + \left\lceil \frac{g(s) + g(t_i) + \sqrt{g(s)\, g(t_i)}}{3} \right\rceil \geq 2 + \left\lceil \frac{i + 4 + \sqrt{i + 3}}{3} \right\rceil \geq 4$$

case e

Consider the set of vertices (s', t)

$$d(s', t) + \left\lceil \frac{g(s') + g(t) + \sqrt{g(s')\, g(t)}}{3} \right\rceil \geq 3 + \left\lceil \frac{5 + \sqrt{6}}{3} \right\rceil \geq 4$$

case f

Consider the set of vertices (s', s_i), $1 \leq i \leq k$

$$d(s', s_i) + \left\lceil \frac{g(s') + g(s_i) + \sqrt{g(s')\, g(s_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{k + i + 3 + \sqrt{(2)(k + i + 1)}}{3} \right\rceil \geq 4$$

case g

Consider the set of vertices (s', t_i), $1 \leq i \leq k-2$

$$d(s', t_i) + \left\lceil \frac{g(s') + g(t_i) + \sqrt{g(s')\, g(t_i)}}{3} \right\rceil \geq 2 + \left\lceil \frac{i + 5 + \sqrt{(2)(i + 3)}}{3} \right\rceil \geq 4$$

case h

Consider the set of vertices (t, s_i), $1 \leq i \leq k$

$$d(t, s_i) + \left\lceil \frac{g(t) + g(s_i) + \sqrt{g(t)\, g(s_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{k + i + 4 + \sqrt{(3)(k + i + 1)}}{3} \right\rceil \geq 4$$

case i

Consider the set of vertices (t, t_i), $1 \leq i \leq k-2$

$$d(t, t_i) + \left\lceil \frac{g(t) + g(t_i) + \sqrt{g(t)\, g(t_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{i + 6 + \sqrt{(3)(i + 3)}}{3} \right\rceil \geq 4$$

case j

Consider the set of vertices (s_i, s_j), $i \neq j$, $1 \leq i, j \leq k$

$$d(s_i, s_j) + \left\lceil \frac{g(s_i) + g(s_j) + \sqrt{g(s_i) g(s_j)}}{3} \right\rceil \geq 1 + \left\lceil \frac{2k + i + j + 2 + \sqrt{(k + i + 1)(k + i + 1)}}{3} \right\rceil \geq 4$$

case k

Consider the set of vertices (t_i, t_j), $i \neq j$, $1 \leq i, j \leq k-2$

$$d(t_i, t_j) + \left\lceil \frac{g(t_i) + g(t_j) + \sqrt{g(t_i)\, g(t_j)}}{3} \right\rceil \geq 1 + \left\lceil \frac{i + j + 6 + \sqrt{(i + 3)(j + 3)}}{3} \right\rceil \geq 4$$

case l

Consider the set of vertices $(s_i, t_j), 1 \leq i \leq k, 1 \leq j \leq k-2$

$$d(s_i, t_j) + \left\lceil \frac{g(s_i) + g(t_j) + \sqrt{g(s_i)\, g(t_j)}}{3} \right\rceil \geq 1 + \left\lceil \frac{k + i + j + 4 + \sqrt{(k + i + 1)(j + 3)}}{3} \right\rceil \geq 4$$

Hence for all set of vertices radio heronian mean condition is satisfied. Thus, g is a valid radio heronian mean labeling of $DS(IT_k)$. Therefore, $rhmn(DS(IT_k)) \leq rhmn(g) = 2k+1$. Since g is injective, $rhmn(DS(IT_k)) \geq 2k+1$ for all radio heronian mean labeling g and hence $rhmn(DS(IT_k)) \geq 2k+1$ for $k \geq 6$. Clearly, $|V(DS(IT_k))| = 2k+1$. Thus, $rhmn(DS(IT_k)) = |V(DS(IT_k))|$. Hence the degree splitting of irregular triangular snake graph $DS(IT_k)$ is radio heronian mean graceful for $k \geq 6$.

Illustration 3.5. The following is the illustration of $DS(IT_7)$.

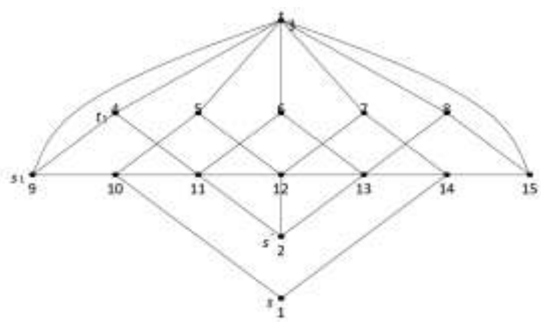

Figure 5.

Conclusion

In this paper, we investigate degree splitting of star, bistar, triangular snake, quadrilateral snake and irregular triangular snake are Radio heronian mean graceful.

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