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INDEPENDENT DOMINATION NUMBER IN CYCLE NECKLACE GRAPHOF LENGTH m WITH A PATH OF LENGTH $n-1$

N. SENTHURPRIYA[1] and S. MEENAKSHI[2]

[1]Research Scholar
[2]Associate Professor
Department of Mathematics
Vels Institute of Science,
Technology and Advanced Studies
Chennai 600 117, India
E-mail: pspriyasaha@gmail.com
meenakshikarthikeyan@yahoo.co.in

Abstract

Let G be a graph with Vertex set V and edge set E, let S be the subset of V such that every vertex in V which is not in S must contain at least one neighbour in S. If S is both an independent and dominating set of a graph G, then S is said to be an independent dominating set of graph G. The domination number of G is denoted by $\gamma(G)$, the minimum cardinality of dominating set of G. The independent domination number is denoted by $i(G)$, the minimum cardinality of an independent dominating set in G. In this paper, we obtain independent domination number for a cycle necklace of length m with a path length n.

Introduction

The earliest ideas of dominating sets, it would seem, date back to the origin of the game of chess in India over 400 years ago, in which one studies sets of chess pieces which cover or dominate various opposing piece or various squares of the chessboard.

In more recent times the eight Queens and Five Queens problems rekindled interest in dominating concepts example, In the books of Ahrens in

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[18]. Finally with the publications of the books by Berge in [2] and ore in [9] the topic of domination was given formal mathematical definition but by 1972 relatively little work had been done on this topic until Cockayne and Hedetniemi [3, 4] began to study it, ultimately published a 1975 survey of the results that had been obtained by that time and the independent domination number and the notation $i(G)$ were introduced. This seems to have brought the subject sufficient into focus to set research on a much wider scale into motion. Independent dominating sets in regular graphs and in cubic graphs in particular, are well studied in Goddard et al. [17], Kostichka [1] and Lam et al. [10]. Favaron [8] initiated the quest of finding sharp upper bounds for independent domination number in general graphs, as functions of n and δ. This work was extended by Haviland [6]. Cockayne et al. [5] obtained the upper bound for the product of the independent domination numbers of a graph and its complement while Shiu et al. [16] found the upper bounds for the independent domination number of triangle-free graphs and also characterized the extremal graphs achieving these upper bounds. Allan and Laskar [11] discussed the graphs with equal domination and independent domination numbers while Southey and Henning [7] considered the ratio of the independent domination number versus the domination number in a cubic graph and also characterized the graphs achieving this ratio of $4/3$.

We denote the path on n vertices as P_n, the cycle on n vertices as C_n.

Necklace graph:

In the technical combinatorial sense, an a-ary necklace of length n is a string of n characters each of 'a' possible types. Since Necklace graph Ne_n is not a path we have $\dim(Ne_n) \geq 2$ for any $n \geq 1$.

Definition 1 [12]**.** For $v \in V(G)$, the open neighbourhood of v, denoted as $N_G(v)$, is the set of vertices adjacent with v; and the closed neighbourhood of v, denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighbourhood of S is defined as $N_G[S] = \bigcup_{v \in S} N_G(v)$ and the closed neighbourhood of S is defined as $N_G[S] = N_G(S) \cup S$. For brevity, we denote $N_G(S)$ by $N(S)$ and $N_G[S]$ by $N[S]$.

Definition 2 [12]**.** For a graph $G(V, E)$, $S \subseteq V$ is a dominating set of G if every vertex in V/S has at least one neighbour in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

Definition 3 [14]**.** An independent set in a graph G is a set of pairwise nonadjacent vertices of G. A set S of vertices in a graph G is called an independent dominating set if S is both an independent set and a dominating set of G. The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set in G.

Proposition [15]**.** *For the path and cycle* $i(P_n) = i(C_n) = \left\lceil \frac{n}{3} \right\rceil$. *In P-Cycle Necklace there are m vertices with* K_m *copies in G. where* K_m *is the cycle graph with length m and* $v_1, v_2, \ldots, v_n$ *are the vertices of the path of length* $n-1$. *Each* K_m *are the components of G and is denoted by* H_i *(subgraphs of G) the subgraph of G contains the minimum number of vertices* $n = 3$. *As the cycle has the minimum number of vertices* $n = 3$.

Independent Domination Number in Cycle Necklace graph of length m with a Path of length $n-1$

Theorem. *Let G be a P-Cycle Necklace graph* $PCN(P_{n-1}; C_m, m = 3, 4, 5, \ldots)$ *then*

$$i(G) = \begin{cases} n\left\lceil \frac{m}{3} \right\rceil, & \text{for } m \bmod 3 \equiv 2 \\ \left\lceil \frac{nm}{3} \right\rceil, & \text{otherwise} \end{cases}$$

Proof.

Case (i)

To prove $i(G) = n\left\lceil \frac{m}{3} \right\rceil$, for $m \bmod 3 \equiv 2$. Label the vertices of $PCN(P_{n-1}; C_m, m = 3, 4, 5 \ldots)$ as follows: Label the vertices of the path P_{n-1} of G as $v_1, v_2, \ldots, v_n$ (hence n is the number of vertices in the path P_{n-1})

and the cycle necklace of G from left to right as H_i and in each cycle we get $\frac{m}{3}$ vertices.

Therefore, we get n times $\left\lceil \frac{m}{3} \right\rceil$ vertices of G in S.

Hence $i(G) = n\left\lceil \frac{m}{3} \right\rceil$, for m mod $3 \equiv 2$

Example, for figure 1, $PCN(P_1; C_5)$

C_5 mod $3 \equiv 2$ (Here $n = 2$, $m = 5$)

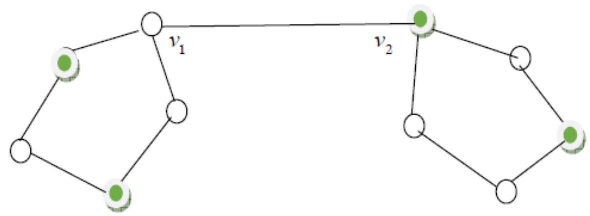

Figure 1.

In figure 1 $i(G) = n\left\lceil \frac{m}{3} \right\rceil$, for m mod $3 \equiv 2$

$$i(G) = 2\left\lceil \frac{5}{3} \right\rceil = 4.$$

The green colored vertices represent the independent domination set.

Example, for figure 2, $PCN(P_4; C_5)$

C_5 mod $3 \equiv 2$ (Here $n = 5$, $m = 5$)

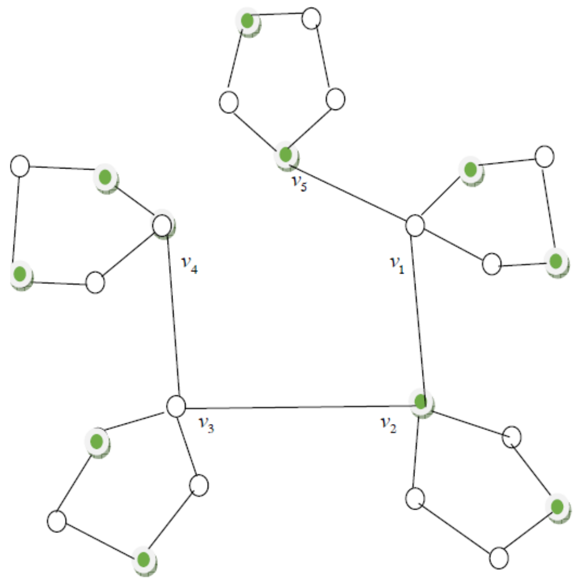

Figure 2.

In figure $i(G) = n\left\lceil \frac{m}{3} \right\rceil$, for $m \bmod 3 \equiv 2$

$$i(G) = 5\left\lceil \frac{5}{3} \right\rceil = 10.$$

The green colored vertices represent the independent domination set. For figure 3, $PCN(P_3;\, C_8)$.

$C_8 \bmod 3 \equiv 2$ (Here $n = 4$, $m = 8$)

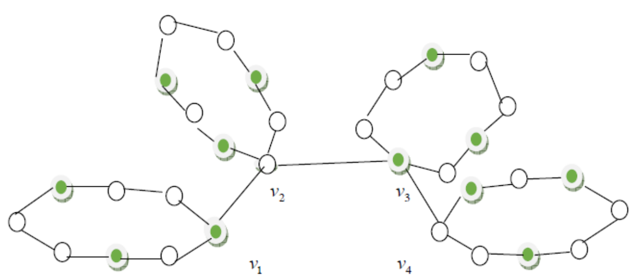

Figure 3.

In figure 3 $i(G) = n\left\lceil \frac{m}{3} \right\rceil$, for $m \bmod 3 \equiv 2$

$$i(G) = 4\left\lceil \frac{5}{3} \right\rceil = 12.$$

The green colored vertices represent the independent domination set. For figure 4, $PCN(P_5; C_{11})$ C_{11} mod $3 \equiv 2$ (Here $n = 6$, $m = 11$)

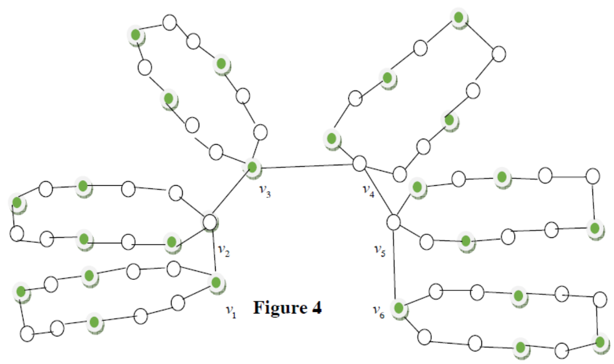

Figure 4

In figure 4 $i(G) = n\left\lceil \frac{m}{3} \right\rceil$, for $m \bmod 3 \equiv 2$

$$i(G) = 6\left\lceil \frac{m}{3} \right\rceil = 24.$$

The green colored vertices represent the independent domination set.

Case (ii)

To prove $i(G) = \left\lceil \frac{nm}{3} \right\rceil$, otherwise

Label the vertices of $PCN(P_{n-1}; C_m, m = 3, 4, 5, \ldots)$ as follows:

Label the vertices of the path P_{n-1} of G as $v_1, v_2, \ldots, v_n$ (hence n is the number of vertices in the path P_{n-1}) and the cycle necklace of G from left to right as H_i. Hence the total number of vertices in G will be n times of m (which is nm), we get $\left\lceil \frac{nm}{3} \right\rceil$ vertices of G in S.

Hence $i(G) = \left\lceil \frac{nm}{3} \right\rceil$, otherwise

Example, for figure 5, $PCN(P_2;\ C_3)$. Here $n = 3$, $m = 3$

$$i(G) = \left\lceil \frac{3 * 3}{3} \right\rceil = 3.$$

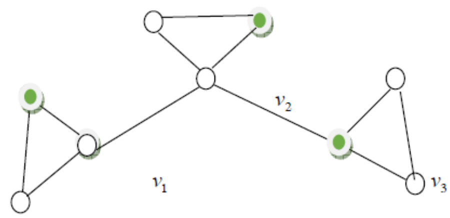

Figure 5.

In figure 5 the green colored vertices represent the independent domination set. For figure 6, $PCN(P_3;\ C_4)$. Here $n = 4$, $m = 4$.

$$i(G) = \left\lceil \frac{4 * 4}{3} \right\rceil = 6.$$

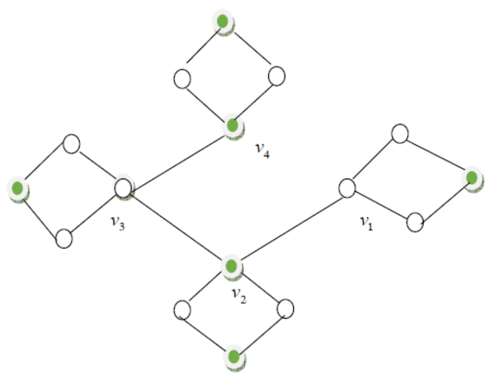

Figure 6.

In figure 6, the green colored vertices represent the independent domination set. For figure 7, $PCN(P_4;\ C_6)$. Here $n = 5$, $m = 6$

$$i(G) = \left\lceil \frac{5 * 6}{3} \right\rceil = 10.$$

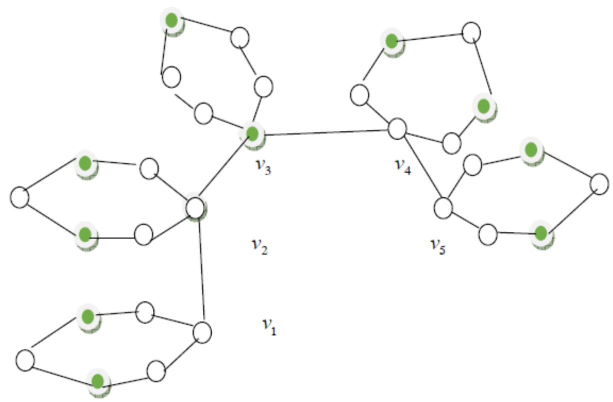

Figure 7.

In figure 7, the green colored vertices represent the independent domination set.

Generalized Table for Independent Domination Number in Cycle Necklace Graph of Length *m* with a Path of Length $n-1$

Length of the path P_{n-1} with n vertices	Length of the Cycle C_n	$i(G)$
n	3	$\left\lceil \frac{nm}{3} \right\rceil$
n	4	$\left\lceil \frac{nm}{3} \right\rceil$
n	5	$n\left\lceil \frac{nm}{3} \right\rceil$, for $m \bmod 3 \equiv 2$
n	6	$\left\lceil \frac{nm}{3} \right\rceil$
n	7	$\left\lceil \frac{nm}{3} \right\rceil$
n	8	$n\left\lceil \frac{m}{3} \right\rceil$, for $m \bmod 3 \equiv 2$
n	9	$\left\lceil \frac{nm}{3} \right\rceil$

n	10	$\left\lceil \frac{nm}{3} \right\rceil$
n	11	$n\left\lceil \frac{m}{3} \right\rceil$, for $m \bmod 3 \equiv 2$
n	12	$\left\lceil \frac{nm}{3} \right\rceil$
n	13	$\left\lceil \frac{nm}{3} \right\rceil$
n	14	$n\left\lceil \frac{nm}{3} \right\rceil$, for $m \bmod 3 \equiv 2$
n	15	$\left\lceil \frac{nm}{3} \right\rceil$
n	16	$\left\lceil \frac{nm}{3} \right\rceil$
n	17	$n\left\lceil \frac{m}{3} \right\rceil$, for $m \bmod 3 \equiv 2$
n	18	$\left\lceil \frac{nm}{3} \right\rceil$
n	19	$\left\lceil \frac{nm}{3} \right\rceil$
n	20	$n\left\lceil \frac{m}{3} \right\rceil$, for $m \bmod 3 \equiv 2$
n	.	.
n	.	.
n	.	.
n	m	$n\left\lceil \frac{m}{3} \right\rceil$, for $m \bmod 3 \equiv 2$ $n\left\lceil \frac{nm}{3} \right\rceil$, otherwise

Conclusion

In this paper, we obtain the generalized Independent Domination Number in Cycle Necklace graph of length m with a Path of length $n-1$ Further, we can extend this as a chain cycle necklace graph and to obtain the independent domination number for the same graph.

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