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ANALYTICAL SOLUTIONS OF HYPERBOLIC TELEGRAPH EQUATION BY THE LAPLACE DECOMPOSITION METHOD

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Abstract

In the present study, we propose the Laplace decomposition method for the analytical solution of second-order one-dimensional linear hyperbolic telegraph equation which was formulated by using Ohm's law. Four examples have been illustrated to check the efficiency, accuracy and convergence of this method. The obtained results ensure that LDM is an easy, accurate and reliable mathematical technique for solving a wide class of partial differential equations in various fields.

1. Introduction

Linear and nonlinear partial differential equations have a wide range of applications in the field of science and engineering such as fluid dynamics, condense matter, plasma physics, quantum field theory, particle physics, chemical kinematics, biology, solid-state physics, aerospace, cosmic-ray transport and so on. Therefore, solutions of these differential equations are essential for perceives different physical phenomena. Hyperbolic partial differential equations are used for modeling many important phenomena's

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such as the vibrations of structures (e.g. machines, buildings, and beams) also the basis for fundamental equations of atomic physics and frequently used in signal analysis for transmission and propagation of electrical signals in a cable transmission line as well as in wave phenomena [1].

Telegraph equation is a hyperbolic partial differential equation that is mostly used in modeling radio frequencies, random walk theory, electromagnetic waves, voltage and current on transmission lines, oceanic diffusion etc. It has features of both diffusion and wave motion. Telegraph equation is used for modeling mixture between diffusion and wave propagation by establishing a term that accounts for effect of finite velocity on standard heat or mass transport equation [2]. Telegraph equation is more appropriate than ordinary diffusion equation in modeling reaction diffusion for such branches of sciences and also a better model for describing certain fluid flow problem involving suspensions as compare to heat equation [3]. Telegraph equation is come across in the study of pulsating blood flow in arteries and in one-dimensional random motion of bugs along a hedge [4]. It also appears in propagation of acoustic waves in Darcy-type porous media [5]. For the transport of energetic charged particles in turbulent magnetic fields such as low-energy cosmic rays in the solar wind, the telegraph equation is the better alternative to diffusion equation [6]. In recent years many researchers paid attention to the analysis and implementation of stable methods for analytic and numerical solution of telegraph equation. Akbar Mohebbi and Mehdi Dehghan apply a compact finite difference approximation of fourth-order for discretizing spatial derivative of telegraph equation and collocation method for time component [7]. R. K. Mohanty, Jain and Arora have developed an unconditionally stable alternating direction implicit method for the linear hyperbolic equation in two and three space dimensions. [8, 9]. S. Yousefi applied Legendre multiwavelet galerkin method for solving the telegraph equation [10]. Radu Cascaval and coworkers' study fractional telegraph equation for better understanding of anomalous diffusion process observed in blood flow experiments [11]. For the study of telegraph equation, interpolating scaling functions used by M. Lakestani and B. Saray [12], M. Datar and K. Takale used variational iteration method [13], NHPM is adopted by J. Biazar and M. Eslami, [14], dual reciprocity boundary integral equation method was applied by M. Dehghan and A. Ghesmati [15] and so on. In this paper, we employed the Laplace decomposition method [16, 17].

The outline of this paper is as follows, Introduction is given in session 1, in session 2 LDM is explained, applications are given in session 3 and conclusion is drawn in session 4.

2. Laplace Decomposition Method [16]

Consider general linear partial differential equation

$$\mathcal{L}u(x, t) + \mathcal{R}(x, t) = g(x, t), \; i = 1, 2, \ldots, n \tag{1}$$

with initial conditions

$$u(0, t) = f_1(t) \text{ and } u_x(0, t) = f_2(t).$$

Where $\mathcal{L} = \frac{\partial^2}{\partial x^2}$, $\mathcal{R}$ is remaining linear operator and $g(x, t)$ is source term. Taking Laplace transform of (1) w. r. t.

$$s^2 u(s, t) - su(0, t) - u_x(0, t) = \mathfrak{L}_x[g(x, t)] - \mathfrak{L}_x[\Re u(x, t)]$$

$$u(s, t) = \frac{f_1(t)}{s} + \frac{f_2(t)}{s^2} + \frac{1}{s^2}\mathfrak{L}_x[g(x, t)] - \frac{1}{s^2}\mathfrak{L}_x[\mathcal{R}u(x, t)].$$

Taking inverse Laplace transform

$$u(x, t) = f_1(t) + xf_2(t) + \mathfrak{L}_x^{-1}\left\{\frac{1}{s^2}\mathfrak{L}_x[g(x, t)]\right\} - \mathfrak{L}_x^{-1}\left\{\frac{1}{s^2}\mathfrak{L}_x[\mathcal{R}u(x, t)]\right\}. \tag{2}$$

Consider solution of the equation (1) is in series form

$$u(x, t) = \sum_{i=0}^{\infty} u_n(x, t). \tag{3}$$

From equation (2) to (3)

$$\sum_{j=0}^{\infty} u_n(x, t) = f_1(t) + xf_2(t) + \mathfrak{L}_x^{-1}\left\{\frac{1}{s^2}\mathfrak{L}_x[g(x, t)]\right\} - \mathfrak{L}_x^{-1}\left\{\frac{1}{s^2}\mathfrak{L}_x[\mathcal{R}\sum_{i=0}^{\infty} u_n(x, t)]\right\}. \tag{4}$$

Where

$$u_0(x,\, t) = f_1(t) + xf_2(t) + \mathfrak{L}_x^{-1}\left\{\frac{1}{s^2}\,\mathfrak{L}_x[g(x,\, t)]\right\},$$

$$u_{n+1}(x, t) = -\mathfrak{L}_x^{-1}\left\{\frac{1}{s^2}\mathfrak{L}_x\left[\mathcal{R}\sum_{i=0}^{\infty} u_n(x, t)\right]\right\}, n = 0, 1, 2, 3, \dots \tag{4}$$

Based on the LDM solution of equation (1) is

$$u(x,\, t) = \lim_{n\to\infty} \phi_n(x,\, t), \tag{5}$$

where $\phi_n(x,\, t) = \sum_{k=0}^{n} u_k(x,\, t)$.

3. Applications

Example 1. Consider non homogeneous hyperbolic telegraph equation [18]

$$u_{tt} + u_t + u = u_{xx} + (2 - 2t + t^2)\,(x - x^2)\,e^{-t} + 2t^2e^{-t}, \tag{6}$$

with initial conditions

$$u(0,\, t) = 0 \;\text{ and }\; u_x(0,\, t) = t^2e^{-t}.$$

By rearranging the terms of equation (6) we get,

$$u_{xx} = -(x - x^2)\,(2 - 2t + t^2)\,e^{-1} - 2t^2e^{-t} + u_{tt} + u_t + u. \tag{7}$$

Taking Laplace transform of equation (6) w. r. t. x

$$s^2u(s,\, t) - su(0,\, t) - u_x(0,\, t)$$

$$= \left(\frac{2}{s^3} - \frac{1}{s^2}\right) e^{-t}(2 - 2t + t^2) - 2e^{-t}t^2\,\frac{1}{s} + L_x[u_{tt} + u_t + u],$$

$$u(s, t) = \left(\frac{2}{s^5} - \frac{1}{s^4}\right) e^{-1}(2 - 2t + t^2) - 2e^{-t}t^2\,\frac{1}{s^3} + t^2e^{-t}\,\frac{1}{s^2} + \frac{1}{s^2}\,L_x[u_{tt} + u_t + u].$$

Taking inverse Laplace transform with respect to x

$$u(x,\, t) = \left(2\,\frac{x^4}{4!} - \frac{x^3}{3!}\right) e^{-t}(2 - 2t + t^2) - e^{-t}t^2x^2 + e^{-t}t^2x$$

$$+ L_x^{-1}\left[\frac{1}{s^2}\, L_x[u_{tt} + u_t + u\right], \tag{8}$$

LDM algorithm gives following recurrence relation:

$$u_0(x,\, t) = \left(2\,\frac{x^4}{4!} - \frac{x^3}{3!}\right) e^{-t}(2 - 2t + t^2) + e^{-t}t^2\left(x - 2\,\frac{x^2}{2!}\right),$$

$$u_{n+1}(x,\, t) = L_x^{-1}\left[\frac{1}{s^2}\, L_x\, [\sum\nolimits_{i=1}^{\infty} u_{n_{tt}} + \sum\nolimits_{i=1}^{\infty} u_{n_t} + \sum\nolimits_{i=1}^{\infty} u_n]\right],\ n = 0,\, 1,\, 2,\, \ldots \tag{9}$$

By using above relation, few components of the series are:

$$u_1(x,\, t) = L_x^{-1}\left[\frac{1}{s^2}\, L_x[\sum_{i=1}^{\infty} u_{0_{tt}} + \sum_{i=1}^{\infty} u_{0_t} + \sum_{i=1}^{\infty} u_0]\right]$$

$$= L_x^{-1}\left[\frac{1}{s^2}\, L_x\left[\left(2\,\frac{x^4}{4!} - \frac{x^3}{3!}\right) e^{-t}(6 - 4t + t^2) + \left(x - 2\,\frac{x^2}{2!}\right) e^{-t}(2 - 2t + t^2)\right]\right],$$

$$u_1(x,\, t) = \left(2\,\frac{x^6}{6!} - \frac{x^5}{5!}\right) e^{-t}(6 - 4t + t^2) + \left(\frac{x^3}{3!} - 2\,\frac{x^4}{4!}\right) e^{-t}(2 - 2t + t^2),$$

$$u_2(x,\, t) = L_x^{-1}\left[\frac{1}{S^2}\, L_x\left[\sum_{i=1}^{\infty} u_{1_{tt}} + \sum_{i=1}^{\infty} u_{1_t} + \sum_{i=1}^{\infty} u_1\right]\right]$$

$$u_2(x,\, t) = \left(2\,\frac{x^8}{8!} - \frac{x^7}{7!}\right) e^{-t}(12 - 6t + t^2) + \left(\frac{x^5}{5!} - 2\,\frac{x^6}{6!}\right) e^{-t}(6 - 4t + t^2)$$

$$u_3(x,\, t) = \left(2\,\frac{x^{10}}{10!} - \frac{x^9}{9!}\right) e^{-t}(20 - 8t + t^2) + \left(\frac{x^7}{7!} - 2\,\frac{x^8}{8!}\right) e^{-t}(12 - 6t + t^2)$$

$$u_4(x,\, t) = \left(2\,\frac{x^{12}}{12!} - \frac{x^{11}}{11!}\right) e^{-t}(30 - 10t + t^2) + \left(\frac{x^9}{9!} - 2\,\frac{x^{10}}{10!}\right) e^{-t}(20 - 8t + t^2)$$

$$u_5(x, t) = \left(2\frac{x^{14}}{14!} - \frac{x^{13}}{13!}\right) e^{-t}(42 - 12t + t^2) + \left(\frac{x^{11}}{11!} - 2\frac{x^{12}}{12!}\right) e^{-t}(30 - 10t + t^2),$$

and so on. In the same manner we can compute the remaining components of the series. In this case noise terms are obtained, therefore terms with opposite signs get cancelled. Hence solution of equation (6) is

$$u(x, t) = u_0 + u_1 + u_2 + u_3 + u_4 + \ldots$$

$$u(x, t) = e^{-t}t^2(x - x^2) \tag{10}$$

which is exact solution of equation (6)

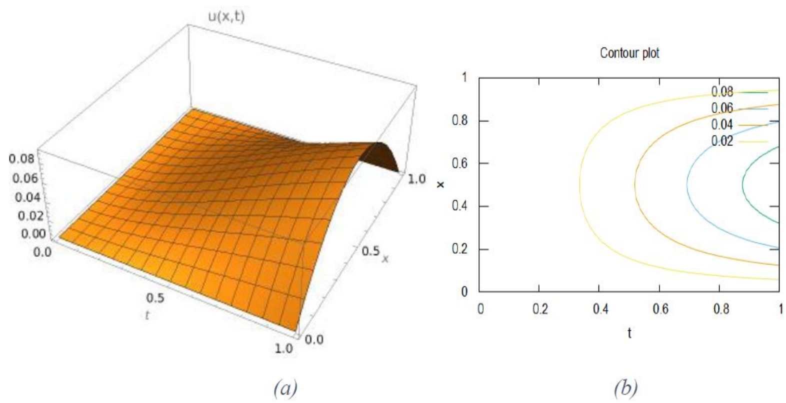


Figure 1. (a) 3D visualization (b) contour plot of exact solution of equation (6) for $0 \le x \le 1$ and $0 \le t \le 1$.

Example 2. Consider non homogeneous telegraph equation [19]

$$u_{tt} + 8u_t + 4u = u_{xx} - 2e^{-t}\sin x, \tag{11}$$

With initial conditions

$$u(0, t) = 0 \text{ and } u_x(0, t) = e^{-t}.$$

Proceeding as before, few components of the series (3) are

$$u_0(x, t) = e^{-t}(3x - 2\sin x),$$

$$u_1(x, t) = -3e^{-t}\left(3\frac{x^3}{3!} - 2x + 2\sin x\right)$$

$$u_2(x, t) = 9e^{-t}\left(3\frac{x^5}{5!} - 2\frac{x^3}{3!} + 2x - 2\sin x\right)$$

$$u_3(x, t) = -27e^{-t}\left(3\frac{x^7}{7!} - 2\frac{x^5}{5!} + 2\frac{x^3}{3!} - 2x + 2\sin x\right)$$

$$u_4(x, t) = 81e^{-t}\left(3\frac{x^9}{9!} - 2\frac{x^7}{7!} + 2\frac{x^5}{5!} - 2\frac{x^3}{3!} + 2x - 2\sin x\right)$$

$$u_5(x, t) = -243e^{-t}\left(3\frac{x^{11}}{11!} - 2\frac{x^9}{9!} + 2\frac{x^7}{7!} - 2\frac{x^5}{5!} + 2\frac{x^3}{3!} - 2x - 2\sin x\right)$$

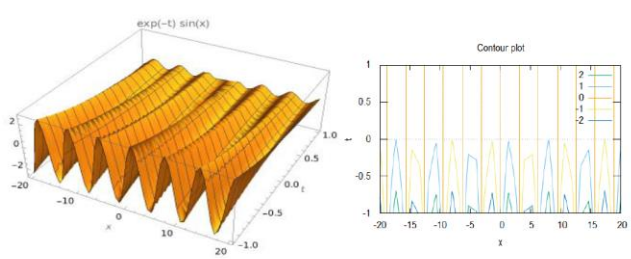
and so on. Let n-term approximate solution for $n = 5, 10$ is

$$\phi_5(x, t) = e^{-t}\left[729\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}\right) - 728\sin x\right],$$

$$\phi_{10}(x, t) = e^{-t}\left[59049\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots + \frac{x^{21}}{21!}\right) - 59048\sin x\right].$$

Then exact solution of equation (12) is

$$u(x, t) = \lim_{n\to\infty} \phi_n(x, t) = e^{-t}\sin x. \tag{12}$$



(a) (b)

Figure 2. (a) Periodic solution (b) contour plot of solution of equation (11) for $-20 \le x \le 20$ and $-1 \le t \le 1$.

Example 3. Consider homogeneous telegraph equation [13]

$$u_{xx} = u_{tt} + 2u_t + u \tag{13}$$

With initial conditions

$$u(0,\, t) = e^{-t} \text{ and } u_x(0,\, t) = 1$$

By using LDM few components of the series are

$$u_0(x,\, t) = e^{-t} + x$$

$$u_1(x,\, t) = \frac{x^3}{3!}$$

$$u_2(x,\, t) = \frac{x^5}{5!}$$

$$u_3(x,\, t) = \frac{x^7}{7!}$$

$$u_4(x,\, t) = \frac{x^9}{9!}$$

By the same procedure rest of the components of the series are easily calculated and n-term approximate solution is:

$$\phi_n(x,\, t) = e^{-t} + x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots + \frac{x^{2n+1}}{(2n+1)!}.$$

Exact solution of Equation (13) is

$$u(x,\, t) = \lim_{n \to \infty} \phi_n(x,\, t) = e^{-t} + \sinh(x). \tag{14}$$

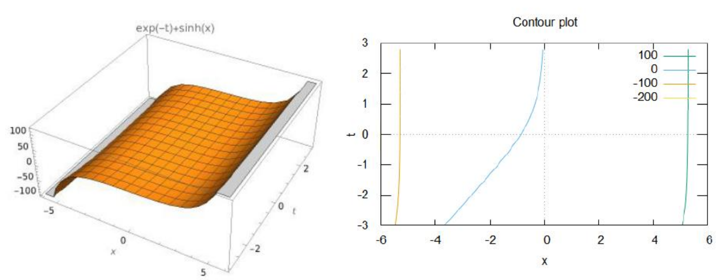


Figure 3. 3D visualization (b) contour plot of solution of equation (13) for $-6 \le x \le 6$ and $-3 \le t \le 3$.

Example 4. Consider non homogeneous telegraph equation with different initial conditions [20]

$$u_{tt} + 2u_t + u = u_{xx} - 2\sin(t)\sin(x) + \cos(t)\sin(x), \tag{15}$$

with initial conditions

$$u(0, t) = 0 \text{ and } u_x(0, t) = \cos(t).$$

Proceeding as before, the first some components of the series are

$$u_0(x, t) = 2\sin(t)(x - \sin(x)) + \cos(t)\sin(x)$$

$$u_1(x, t) = 4\cos(t)\left(\frac{x^3}{3!} - x + \sin(x)\right) - 2\sin(t)(x - \sin(x))$$

$$u_2(x, t) = -8\sin(t)\left(\frac{x^5}{5!} - \frac{x^3}{3!} + x - \sin(x)\right) - 4\cos(t)\left(\frac{x^3}{3!} - x + \sin(x)\right)$$

$$u_3(x, t) = -16\cos(t)\left(\frac{x^7}{7!} - \frac{x^5}{5!} + \frac{x^3}{3!} - x + \sin(x)\right) + 8\sin(t)\left(\frac{x^5}{5!} - \frac{x^3}{3!} + x - \sin(x)\right)$$

$$u_4(x, t) = 32\sin(t)\left(\frac{x^9}{9!} - \frac{x^7}{7!} + \frac{x^5}{5!} - \frac{x^3}{3!} + x - \sin(x)\right)$$

$$+ 16\cos(t)\left(\frac{x^7}{7!} - \frac{x^5}{5!} + \frac{x^3}{3!} - x + \sin(x)\right)$$

$$u_5(x, t) = 64\cos(t)\left(\frac{x^{11}}{11!} - \frac{x^9}{9!} + \frac{x^7}{7!} - \frac{x^5}{5!} + \frac{x^3}{3!} - x + \sin(x)\right)$$

$$- 32\sin(t)\left(\frac{x^9}{9!} - \frac{x^7}{7!} + \frac{x^5}{5!} - \frac{x^3}{3!} + x - \sin(x)\right)$$

and so on. In this case also noise terms occur, by cancelling the terms with opposite sign we get exact solution of equation (15) as follows:

$$u(x, t) = \sin(x)\cos(t). \tag{16}$$

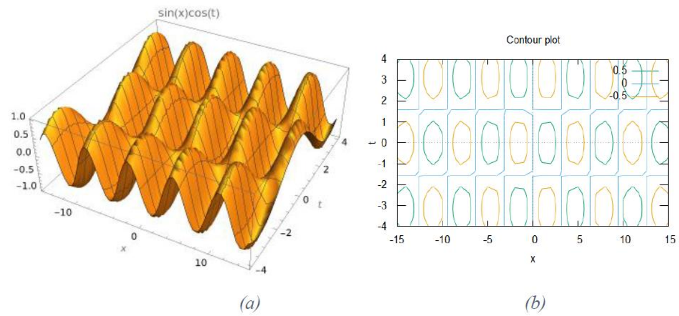


Figure 4. (a) Periodic solution (b) contour plot of solution of equation (15) for $-15 \leq x \leq 15, -4 \leq t \leq 4$.

4. Conclusion

LDM has been successfully implemented to obtain the solutions of telegraph equation. Noise terms are occurring in examples 1 and 4, by cancelling the terms with the opposite signs, we get the exact solution and for the remaining examples, we get solutions in series form, which converge to the exact solution. 3D visualization and contour plots of solutions are shown in Figure 1-4. The obtained results guarantee that LDM is accurate, rapidly convergent, and easy to apply to linear as well as nonlinear partial and ordinary differential equations in the field of science and engineering.

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